Project 2: Tracking a Point on a Rolling Wheel

- ***Preliminaries to the Rolling Wheel Problem***

We will now apply the material of this chapter to the problem of finding the *x* and *y* coordinates of a specified point on a wheel after it has rolled a given distance. After a introductory illustrative example, the other examples in this section will require use of the trigonometric identities for $sin(a + b)$ and $cos(a + b)$ which.we will introduce.

Illustrative Example 1: A wheel of radius 2" rolls 15π" to the left. If the initial coordinates of a point on the wheel relative to the center of the wheel are $P(2, 0)$. (The center is the origin of a Cartesian coordinate system.). Find the final coordinates of this point relative to the center of the wheel after it rolls 15π" to the left.

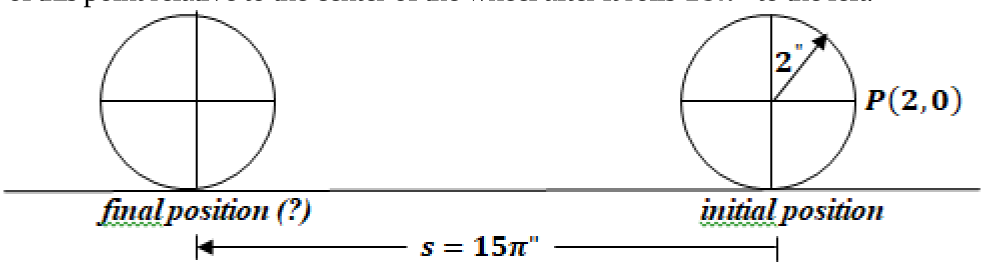

Fig. 1

Solution: In order to find the final position of *P* we first determine the angle θ through which the spoke *OP* rotates as the wheel rolls 15π".

Using the formula $\boldsymbol{s = r\theta}$ for the distance *s* the center of a rolling wheel moves when a spoke on this wheel rotates through an angle of $\boldsymbol{\theta}$ radians, we have:

$$s = r\theta$$
$$15\pi = 2\theta$$
$$\theta = \frac{15\pi}{2}\,rad$$

Because the wheel is rolling counterclockwise, θ is a positive angle. Also, the default units for θ is radians when using this formula.

To locate the position of the spoke *OP* after it has rotated $\theta = \frac{15\pi}{2}\,rad$, we rewrite

$$\frac{15\pi}{2} = \left(7 + \frac{1}{2}\right)\pi = 7\pi + \frac{\pi}{2}$$

Starting at the positive *x* axis, a revolution of 7π radians about the origin puts the spoke on the negative *x* axis (*as would* $\pi, 3\pi, or\ any\ odd\ multiple\ of\ \pi$) and revolving the spoke another $\frac{\pi}{2}\,rad$ leaves it on the negative ***y* axis**, having an angle of inclination equal to $\frac{3\pi}{2}\,rad$.

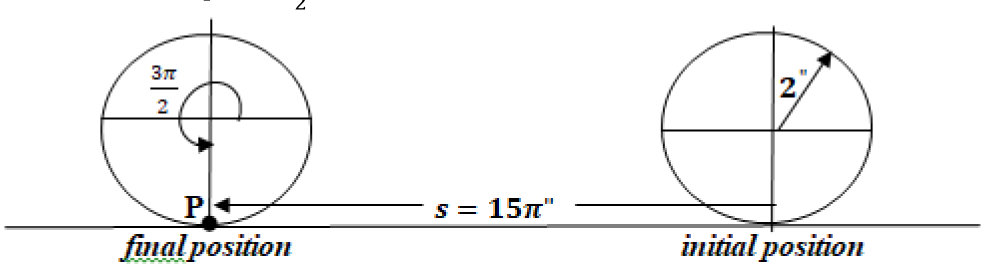

Fig. 2

It is clear that the final coordinates of P are $(0, -2)$ which we can read directly from the diagram above. However, the other more complicated examples in this project will require the use of the formulas below to find the coordinates of a point *P*.

$$\boldsymbol{x = r\cos\alpha\ \ and\ y = r\sin\alpha}$$

Therefore, we will use these formulas to find the coordinates of point P In our example as practice. In our example here, $\boldsymbol{r} = \boldsymbol{2}$ and $\boldsymbol{\alpha} = \frac{3\pi}{2}$ so we have:

$$x = 2\cos\frac{3\pi}{2} \; and \; y = 2\sin\frac{3\pi}{2}$$

Since $\boldsymbol{\alpha} = \frac{3\pi}{2}$ does not fit into a reference triangle, we evaluate its sine and cosine values with curve sketches labeling the appropriate 5 key points.

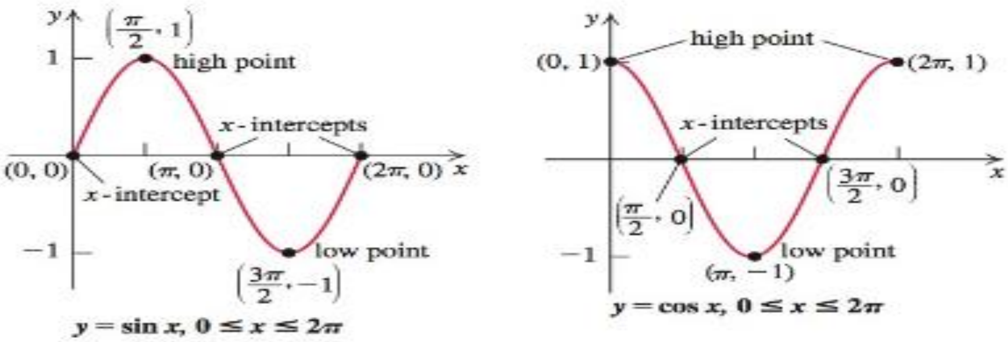

Fig. 3 – 5 key points of the sine and cosine waves

From the diagrams above, $\boldsymbol{sin}\frac{3\pi}{2} = -1$ ***and*** $\boldsymbol{cos}\frac{3\pi}{2} = 0.$
Thus the coordinates of P are:

$$x = 2\cos\frac{3\pi}{2} = 2(0) = 0 \qquad \text{and} \qquad y = 2\sin\frac{3\pi}{2} = 2(-1) = -2$$

Giving us the final coordinates of P as $(0, -2)$, confirming the same result we got from reading the diagram directly..

<u>Illustrative Example 2:</u> A point on a wheel of radius 3" has coordinates $(1, \sqrt{8})$ relative to the center of the wheel. (The center of the wheel is the origin of the Cartesian coordinate system.) Find the coordinates of this point relative to the center of the wheel after it has rolled 23π" to the left.

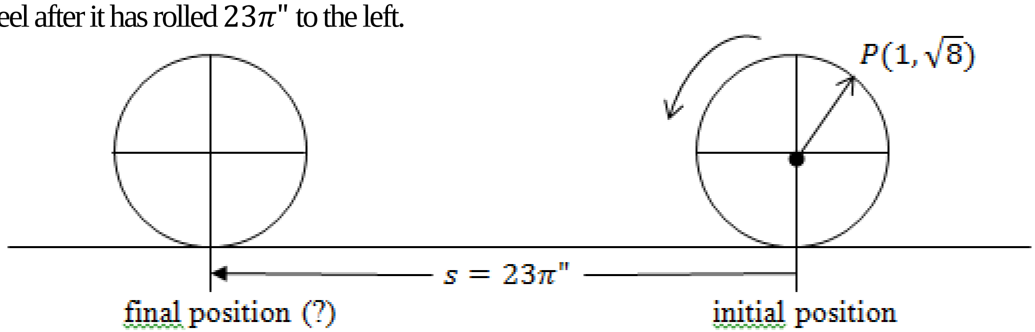

Fig. 4

Solution: a) Finding the angle through which the wheel rotates:

We first find the angle through which the spoke joining the center O to the point P with initial coordinates $(1, \sqrt{8})$ has rotated. Since for a rolling wheel $\boldsymbol{s} = \boldsymbol{r\theta}$ where $\boldsymbol{s}$ is the distance the center has moved, and $\boldsymbol{\theta}$ (***the angle of rotation***) is the angle in radians that a spike on the wheel turns through, we have:

$$s = r\theta$$
$$23\pi = 3\theta$$
$$\theta = \frac{23\pi}{3} \, rad$$

Note: since the wheel is rotating counterclockwise as it rolls to the left, θ is a positive angle.

We call the original angle of inclination of the line *OP* (***the angle OP makes with the positive x axis***) α_1.

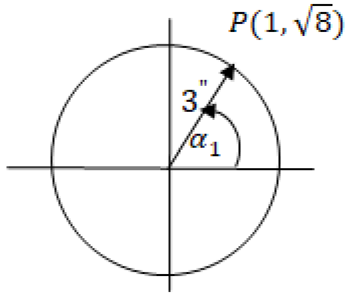

Fig. 5

b) Finding the sine and cosine of the initial angle of inclination α_1 of *OP* :

Although we do not know α_1, we can easily find the values of $\boldsymbol{sin\ \alpha_1}$ and $\boldsymbol{cos\ \alpha_1}$ from the quadrant diagram.

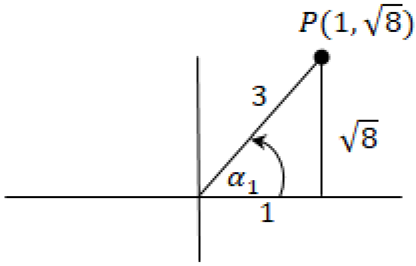

Fig. 6

Since the coordinates of *P* were given, we know the sides of the triangle because the x coordinate 1, is the horizontal distance from the origin and the y coordinate $\sqrt{8}$ is the vertical distance from the origin. Thus:

$$\boldsymbol{sin\ \alpha_1 = \frac{\sqrt{8}}{3}} \quad \text{and} \quad \boldsymbol{cos\ \alpha_1 = \frac{1}{3}}.$$

c) Finding the sine and cosine of α, the final angle of inclination of *OP* :

The final angle of inclination $\boldsymbol{\alpha}$ of the line *OP* is then $\boldsymbol{\alpha = \alpha_1 + \theta = \alpha_1 + \frac{23\pi}{3}}$ since the spoke ***OP*** rotated through an angle of $\theta = \frac{23\pi}{3}$ as we just determined using the formula $\boldsymbol{s = r\theta}$.

Knowing the distance of a point ***P*** from the origin and an angle of inclination of the line from ***P*** to the origin permits us to get the coordinates of ***P*** using:

$$\boldsymbol{x = rcos\ \alpha} \quad \text{and} \quad \boldsymbol{y = rsin\ \alpha}$$

In our problem we have

$$\boldsymbol{r = 3}$$
$$\boldsymbol{\alpha = \alpha_1 + \theta = \alpha_1 + \frac{23\pi}{3}},$$

so the <u>final</u> <u>coordinates</u> of *P* are:

$$\boldsymbol{x = r\cos\alpha = 3\cos\alpha = 3\cos\left(\alpha_1 + \frac{23\pi}{3}\right)}$$
$$\boldsymbol{y = r\sin\alpha = 3\sin\alpha = 3\sin\left(\alpha_1 + \frac{23\pi}{3}\right)}$$

- **Expansion of $cos(a+b)$ and $sin(a+\ b)$**

We will now evaluate the expressions $\mathbf{cos}\left(\boldsymbol{\alpha_1}+\frac{\mathbf{23\pi}}{\mathbf{3}}\right)$ and $\mathbf{sin}\left(\boldsymbol{\alpha_1}+\frac{\mathbf{23\pi}}{\mathbf{3}}\right)$. To do so we require two important trigonometric identities:

$$\boldsymbol{cos(a+b)} = \mathbf{cos}\,\boldsymbol{a}\,\mathbf{cos}\,\boldsymbol{b} - \mathbf{sin}\,\boldsymbol{a}\,\mathbf{sin}\,\boldsymbol{b}$$
$$\boldsymbol{sin(a+b)} = \mathbf{sin}\,\boldsymbol{a}\,\mathbf{cos}\,\boldsymbol{b} + \ \mathbf{cos}\,\boldsymbol{a}\,\mathbf{sin}\,\boldsymbol{b}$$

A way to remember that the sign in $sin(a+b)$ agrees with the + and in $cos(a+b)$ is the $-$ is to say that the "c" in cos stands for <u>change</u> and the "s" in sin stand for <u>same</u>.

These identities will be used extensively in the next section, but this is a good place to introduce them.

Note: $\boldsymbol{cos(a+b)} \neq \mathbf{cos}\,\boldsymbol{a} + \mathbf{cos}\,\boldsymbol{b}$

Applying these trigonometric expansion identities to the above equations for the final coordinates of *P:*

$$\boldsymbol{x} = \mathbf{3\,cos}\left(\boldsymbol{\alpha_1}+\frac{\mathbf{23\pi}}{\mathbf{3}}\right) = \mathbf{3}\left(\mathbf{cos}\,\boldsymbol{\alpha_1}\,\mathbf{cos}\frac{\mathbf{23\pi}}{\mathbf{3}} - \mathbf{sin}\,\boldsymbol{\alpha_1}\,\mathbf{sin}\frac{\mathbf{23\pi}}{\mathbf{3}}\right)$$
$$\boldsymbol{y} = \mathbf{3\,sin}\left(\boldsymbol{\alpha_1}+\frac{\mathbf{23\pi}}{\mathbf{3}}\right) = \mathbf{3}\left(\mathbf{sin}\,\boldsymbol{\alpha_1}\,\mathbf{cos}\frac{\mathbf{23\pi}}{\mathbf{3}} + \mathbf{cos}\,\boldsymbol{\alpha_1}\,\mathbf{sin}\frac{\mathbf{23\pi}}{\mathbf{3}}\right)$$

The brackets are important because they remind us that the 3 must multiply <u>both</u> terms in the expansion.

At the beginning of this exercise, we found that because we knew $(1,\sqrt{8})$ were the original coordinates of the point ***P***, we were able to determine that:

$$\boldsymbol{sin\ \alpha_1} = \frac{\sqrt{\mathbf{8}}}{\mathbf{3}} \text{ and } \boldsymbol{cos\ \alpha_1} = \frac{\mathbf{1}}{\mathbf{3}}$$

You may want to re-read part b) of this illustrative example where we found $sin\,\alpha_1$ and $cos\,\alpha_1$.

Thus to determine the final coordinates of ***P*** from the preceding expansion formulas, we need only to determine $\mathbf{cos}\frac{\mathbf{23\pi}}{\mathbf{3}}$ and $\boldsymbol{sin}\frac{\mathbf{23\pi}}{\mathbf{3}}$. To this end note:

$$\frac{\mathbf{23\pi}}{\mathbf{3}} = \left(\mathbf{7}+\frac{\mathbf{2}}{\mathbf{3}}\right)\boldsymbol{\pi} = \mathbf{7}\boldsymbol{\pi} + \frac{\mathbf{2\pi}}{\mathbf{3}}.$$

Thus the reference ray for $\frac{\mathbf{23\pi}}{\mathbf{3}}$ is determined by rotating a ray starting at the positive *x* axis through ***7π radians*** leaving us at the negative *x* axis and then adding another rotation of $\frac{23\pi}{3}$ *radians* (which is 120°). This leaves the reference ray in the fourth quadrant as shown below:

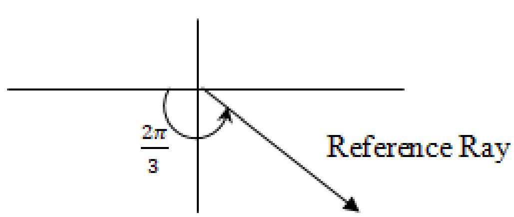

Fig. 7

The reference angle and triangle are shown below:

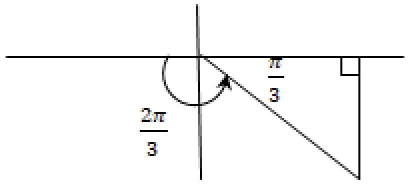

Fig. 8

and inserting the signs of the sides from their position in the quadrant diagram, and the numerical values from our knowledge of a $30 - 60 - 90$ right triangle gives:

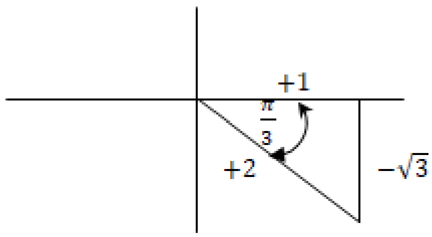

Fig. 9 - The reference triangle for angle of $\frac{2\pi}{3}$ radians

Thus: $\boldsymbol{sin\frac{23\pi}{3}} = -\frac{\sqrt{3}}{2}$ and $\boldsymbol{cos\frac{23\pi}{3}} = \frac{1}{2}$.

Substituting those values and the fact that $\boldsymbol{sin\ \alpha_1} = \frac{\sqrt{8}}{3}$ and $\boldsymbol{cos\ \alpha_1} = \frac{1}{3}$ into the expansion formulas, we can now find the final coordinates for *P*.

$$x = 3\cos\left(\alpha_1 + \frac{23\pi}{3}\right) = 3\left(\cos\alpha_1 \cos\frac{23\pi}{3} - \sin\alpha_1 \sin\frac{23\pi}{3}\right)$$
$$= 3\left[\left(\frac{1}{3}\right)\left(\frac{1}{2}\right) - \left(\frac{\sqrt{8}}{3}\right)\left(-\frac{\sqrt{3}}{2}\right)\right]$$
$$= 3\left[\frac{1}{6} + \frac{\sqrt{24}}{6}\right]$$

Exponents and therefore square roots distribute over factors, so $\sqrt{8}\sqrt{3} = \sqrt{24}$. Exponents do not distribute over terms.

$$= 3\left[\frac{1+\sqrt{24}}{6}\right]$$
$$= \frac{1+\sqrt{24}}{2}$$
$$= \frac{1+2\sqrt{6}}{2}$$

***Note:* $\sqrt{24} = \sqrt{4(6)} = 2\sqrt{6}$**

$$y = 3\sin\left(\alpha_1 + \frac{23\pi}{3}\right) = 3\left(\sin\alpha_1 \cos\frac{23\pi}{3} + \cos\alpha_1 \sin\frac{23\pi}{3}\right)$$
$$= 3\left[\left(\frac{\sqrt{8}}{3}\right)\left(\frac{1}{2}\right) + \frac{1}{3}\left(-\frac{\sqrt{3}}{2}\right)\right]$$
$$= 3\left[\frac{\sqrt{8}}{6} - \frac{\sqrt{3}}{6}\right]$$
$$= 3\left[\frac{\sqrt{8} - \sqrt{3}}{6}\right]$$
$$= \frac{\sqrt{8} - \sqrt{3}}{2}$$
$$= \frac{2\sqrt{2} - \sqrt{3}}{2}$$

Note:* $\sqrt{8} - \sqrt{3} \neq \sqrt{5}$ *because exponents and therefore square roots do not distribute over terms.

Thus the final coordinates of ***P*** with respect to the origin are: $\left(\frac{1+2\sqrt{6}}{2}, \frac{2\sqrt{2} - \sqrt{3}}{2}\right)$.

Homework Exercises:

In the exercises to follow we are given the coordinates of a point *P* on a wheel in its initial position. We are also given the radius of the wheel and the distance ***s*** that it rolls to the left. In each case, use the method illustrated in the text to find the final coordinates of *P*, relative to the center of the wheel.

	radius of wheel	**initial coordinates of P**	**distance wheel rolls**	
1.	$5''$	$(3,4)$	$(15\pi)''$	
2.	$4''$	$(2,2\sqrt{3})$	$(15\pi)''$	
3.	$4''$	$(3,\sqrt{7})$	$(13\pi)''$	
4.	$4''$	$(-\sqrt{7},3)$	$(9\pi)''$	
5.	$6''$	$(2\sqrt{5},4)$	$(19\pi)''$	
6.	$6''$	$(-4,2\sqrt{5})$	$(14\pi)''$	
7.	$3''$	$(-1,-1)$	$(7\pi)''$	**** See note below***
8.	$3''$	$(-1,1)$	$(5\pi)''$	**** See note below***

*** Note**: In problems 7 and 8, the point *P* is an interior point of the wheel; it is not on the rim. This means that the value of r in the formulas to find the coordinates of *P:*

$$\boldsymbol{x = r\cos\alpha \;\; and \; y = r\sin\alpha}$$

and the value of r in the formula $\boldsymbol{s = r\theta}$ to find the distance the wheel rolls, are different.