Optimal Costs of a Two-dimensional Warranty Servicing Strategy with an Imperfect Repair Option

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Abstract

A warranty policy for a product should balance the interests of both producer and consumer. Consumer protection is typically provided by a guarantee of replacement or some form of repair of the product failing within a promised warranty period, while an approach to provide a corresponding protection for the manufacturer is to limit the maximum usage allowed under warranty. Such warranty policies are two-dimensional, and the warranty expires at the end of the promised warranty period or upon reaching the maximum usage allowed, whichever occurs sooner. From a manufacturer’s point of view, reducing warranty costs is an issue of great interest. In this paper, we look at two different servicing strategies for a two-dimensional warranty scheme involving minimal and imperfect repairs. Our work demonstrates the modeling and analysis of costs under these servicing strategies and compare their performance to other strategies that have been investigated in the literature.

Keywords: two-dimensional warranty, warranty servicing cost, imperfect repairs.

1 Introduction

Modern manufacturing is characterized by speedily developing technology, exposure to the global market, fierce competition, well-informed and demanding consumers. These factors have posed serious challenges to the manufacturers and policy makers across the globe jockeying for competitive advantage. In the purchase decisions, consumers typically compare the characteristics of different products of competing brands. When these comparable products are similar or nearly identical, it becomes very difficult to choose a particular brand solely based on product related characteristics such as model specifications and other features, such as financing offered by the manufacturer, price etc. In this case, provision of post-sales support adds to the product’s appeal, and is thus a useful marketing tool.

Such support is provided by the manufacturer generally in the form of repair/replace warranty, maintenance servicing or money-back guarantee. When a new product is manufactured,

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each generation becomes more complex than previous ones and the consumers are not sure
about the product reliability. In this case an attractive warranty servicing scheme signals
higher product quality and provides greater assurance to customers in the sense that the man-
ufacturer will provide some remedial action (repair/replace/money-back) to compensate for the
failure of the item during a preassigned time period.

Blischkes [2] was the first review paper on warranties and it dealt with mathematical models
for warranty cost analysis. The three-part review paper (Product Warranty Management -
I, II, III; Blischke and Murthy [3], Murthy and Blischke [16],[17]) proposed a taxonomy for
new product warranties and discussed various issues. Murthy and Djamaludin [18] review the
literature over the period 1990 to 2002. Biedenweg [1] showed that the optimal strategy is to
replace with a new item at any failure occurring up to a certain time measured from the initial
purchase and then repair all other failures that occur during the remainder of the warranty
period. This technique of splitting the warranty period into distinct intervals for replacement
and repair is also used by Nguyen and Murthy [21], in their work any item failures occurring
during the second part of the warranty period are rectified using a stock of used items. Nguyen
and Murthy [22] extended Biedenweg’s [1] model by adding a third interval where failed items
are either replaced or repaired and a new warranty is given at each failure. The first warranty
servicing model involving minimal repair and assuming constant repair and replacement costs is
that of Nguyen [22]. In a later paper, and with the same assumptions as Nguyen [22], Jack and
Van der Duyn Schouten [9] investigated the structure of the manufacturer’s optimal servicing
strategy over a warranty period [0,W], using a dynamic programming model. The optimal
strategy of Jack and Van der Duyn Schouten [9] yields the smallest expected warranty servicing
cost, but the computation of the control limit policy can be tedious and involve considerable
computational effort. The strategy also requires continuous monitoring of the item’s age by
the manufacturer which is not very practical. The new strategy proposed by Jack et. al. [10]
again involves splitting the warranty period [0,W] into three distinct intervals for carrying out
repairs and replacements. A maximum of one replacement is allowed and there is no need to
monitor the item’s age.

On the other hand a two-dimensional warranty is characterized by a region in a two-
dimensional plane. Different shapes for the region characterize different policies and many differ-
ent shapes have been proposed (see Blischke and Murthy [[3],[4]] and Singpurwalla and Wilson
[[23],[24]]). The expected warranty costs for a variety of policies can be found in Moskowitz
and Chun [[15] and [5], Chapter 13], Singpurwalla and Wilson [24], Blischke and Murthy [[4],
Chapter 8], Murthy et al. [19] and Chun and Tang [7]. Kim and Rao [13] deal with the cost
analysis based on a bivariate exponential distribution. Jack et.al. [11] considered the three
distinct intervals and extended the warranty servicing strategy to the 2-D case. In this article
following the idea of Jack et. al. [11] we consider a 2-D warranty servicing strategies which are
sensitive to the rate of usage, that will lead us to a proposed new warranty servicing strategy
(sec. 2.2), its formulation, analysis, results and conclusion (sec. 2.3-2.5).

2 Usage Rate Based Servicing Strategies

In the case of two-dimensional warranties, there are effects of both age and usage on the product
degradation and failure needs to be modeled accordingly. The usage can be the output (e.g.,
copies produced for a photocopier), distance traveled (e.g., kilometers for an automobile) and
the number of times or hours the product has been used (e.g., take offs and landings or the
total hours flown for an aircraft).
The modeling approach assumes that the usage rate $Y$ varies from customer to customer but is constant for a given customer. Therefore $Y$ is a random variable that can be modeled using a density function $g(y)$. Conditional on $Y = y$, the total usage $u$ at age $x$ is given by

$$u = yx, \quad 0 \leq u < \infty$$  \hspace{1cm} (1)

Given usage rate $y$, the conditional hazard (failure rate) function $h_y(x) \equiv h(x; y) \geq 0$ is assumed to be increasing in item’s age $x$ and usage rate $y$. Failures over time are modeled by a counting process. If failed items are replaced (by new ones), then this counting process is a renewal process associated with the conditional distribution $F_y(x)$, which can be derived from $h_y(x)$. If failed items are repaired then the counting process is characterized by a conditional intensity function $\lambda_y(x)$, which is an increasing function of $x$ and $y$. If all repairs are minimal [9] and repair times are negligible, then $\lambda_y(x) = h_y(x)$.

### 2.1 Modeling failures

We consider a repairable item sold with a two dimensional non-renewing free replacement warranty of period $W$ and maximum usage level $U$, that requires the manufacturer to either repair or replace the item when it fails. Failure occurs if warranty exceeds time $W$ or total usage exceeds $U$. We make the following additional assumptions:

1. $h(x; \alpha(y))$ is an increasing function of age $x$ and usage rate $y$.
2. No preventive maintenance is carried out on the item during the warranty period, either by the manufacturer or by the consumer.
3. All item failures are detected immediately and result in immediate claims by the consumer.
4. All claims are valid and must be rectified by the manufacturer immediately through repairs.
5. Repair and replacement times are small relative to the mean time between item failures and therefore can be ignored.

A product can be considered as a system containing several interconnected components. When the components are statistically independent the reliability of the product is a function of the individual component reliabilities. During the design stage, decisions are made about component reliabilities in order to ensure that the product has the desired reliability at some nominal usage rate $y_0$. When the usage rate differs from this nominal value used in the design, the reliabilities of some of the components can be affected and this in turn affects the product reliability. As the usage rate increases, the rate of degradation increases and this, in turn, accelerates the time to failure. As a result, the product reliability decreases (increases) as the usage rate increases (decreases).

#### 2.1.1 Modeling first failure

The effect of usage rate on degradation can be modeled by "Accelerated Failure Time model" (AFT) ([20],[6]). If $T_0(T_y$ respectively) denotes the time to first failure under usage rate $y_0 (y)$, then the standard AFT model postulates,

$$\frac{T_y}{T_0} = \left(\frac{y_0}{y}\right)^\gamma,$$  \hspace{1cm} (2)
where $\gamma \in [1, \infty)$ is the so called acceleration parameter.

Note for usage rates $y$ more (less) than the nominal usage rate $y_0$, the resulting actual time $T_y$ to failure is a fraction (multiple) of the nominal failure time $T_0$. Let

$$\mathcal{F} = \{F(x; \alpha) : \alpha \in A\}$$

be a scale parameter family indexed by a scale parameter $\alpha \in A \subset (0, \infty)$ for some indexed set $A$. If the cumulative distribution function (cdf) of $T_0$ is $F(x; \alpha_0) \in \mathcal{F}$; then, by (2), the cdf of AFT $T_y$ is

$$F(x; \alpha(y)) = F_0((\frac{y}{y_0})^\gamma x; \alpha_0)$$

i.e., cdf of $T_y$ is the same as that for $T_0$ but with scale parameter given by

$$\alpha(y) = (\frac{y_0}{y})^\gamma \alpha_0 \quad \text{where} \quad \gamma \geq 1.$$

The hazard and the cumulative hazard functions associated with $F(x, \alpha(y))$ are given by

$$h(x; \alpha(y)) = \frac{f(x; \alpha(y))}{F(x; \alpha(y))}$$

and

$$H(x; \alpha(y)) = \int_0^x h(u, \alpha(y)) du$$

where $f(x; \alpha(y))$ is the associated density function.

2.1.2 Modeling subsequent failures

Subsequent failures depend on the type of action taken to rectify a failed item. For a non-repairable product the only option is to replace the failed item by a new one. In the case of a repairable product, the subsequent failures depend on the type of repair carried out. If it is a minimal repair, reliability of the product after repair is same as that just before failure. If it is an imperfect repair [8], reliability after pair is improved but is inferior to that of a new item.

Here we confine our attention to minimal repair and assume that repair times are negligible (relative to the mean time between failures) and so can be ignored. Failures over time occur according to a non-homogeneous Poisson process (NHPP) with intensity function having the same form as the hazard function for time to first failure. Thus, if the product has usage rate $y$, the failure intensity function is

$$\lambda_y(x) = h(x; \alpha(y))$$

where $h(x; \alpha(y))$ is the hazard function given by (6).

2.1.3 Warranty policy and coverage

The product is sold with a two-dimensional warranty with warranty region the rectangle $[0, W) \times [0, U)$, where $W$ is the time limit and $U$ the usage limit. The warranty expires at the first instance when the age of the item reaches $W$ or its usage reaches $U$, whichever occurs first.
Clearly if the usage rate $y$ is at most $U/W$ then the warranty expires at age $W$ and an estimate of the total usage is $yW$. When $y$ is greater than $U/W$, the warranty expires at age $U/y$ when the usage limit $U$ is reached. If $W_y$ denotes the warranty expiry time when the usage rate is $y$, then

$$W_y = \begin{cases} W, & y \leq U/W; \\ U/y, & y > U/W. \end{cases}$$

(9)

2.1.4 Servicing strategies for 2-D warranties

Jack et.al. [23] have considered a 2-D warranty servicing strategy using minimal repairs, except for the first failure to be ‘rectified’ (i.e., ‘repaired’) by a replacement. Such a strategy can be described via three disjoint intervals $[0, K_y)$, $[K_y, L_y)$ and $[L_y, W_y)$ with $0 < K_y < L_y < W_y$, along the age (time) scale where failures in the initial interval $[0, K_y)$ when the item is relatively new undergo only minimal repair; the first failure in the middle interval $[K_y, L_y)$ rectified by a replacement and all subsequent failures therein, as well as in the interval $[L_y, W_y)$ when the item is relatively old getting only quick fixes via minimal repairs. Such a strategy minimizes what is known to be near-optimal among 1-D warranty policies (viz., Jack et.al.[10], Jiang et.al.[12]).

In the 1-D replacement/repair warranty (FRW, F-free, to the consumer) policies, Yun et.al. [25] have investigated the impact of allowing ‘imperfect repair’ (IR) as a mode of rectifying the first failure in the middle interval $[K_y, L_y)$ to restore the unit to a working condition. They describe the degree of ‘imperfect repair’ via a parameter $\delta \in [0, 1]$ with $\delta = 0(1$, respectively being equivalent to minimal repair (replacement), and assume that it is possible to restore a failed equipment with any chosen degree ($\delta$) of repair.

2.2 A new servicing strategy for 2-D warranties

For 2-D warranties, alternatives to ‘minimal repair’ in the middle interval $[K_y, L_y)$ in Jack et.al. [11] approach is restricted to replacements (i.e., ‘perfect repairs’) only. We propose and investigate a new strategy. Our current work described here, is an attempt to extend the model and analysis of 2-D warranties by allowing imperfect repairs defined by $(\delta)$ as expressed by Yun et.al. [25]. For a given usage rate $y$ the value of the parameters $K_y$ and $L_y$ are selected to minimize the expected warranty servicing cost. If $K_y^*$ and $L_y^*$ denote the optimal values then, as $y$ varies the set of points $(K_y^*, L_y^*)$ define a closed curve as indicated in Fig. 1(Jack et.al. [11]) below. Let $\Gamma$ denote the region enclosed by this curve.

**Our new servicing strategy:**

For items sold with 2-D warranties, the first failure in the region $\Gamma$ is rectified with an imperfect repair and all other failures are repaired minimally.

The region $\Gamma$ depends on the type of model used for item failures and on the cost of each minimal repair and the imperfect repair. Let $C_r$ denote the cost of minimal repair and $C_i(\delta_y(x), x)$ ($> C_r$) denote the cost of imperfect repair conditioned on $y$. Here given usage rate $y$, $\delta_y(x) \in [0, 1]$ is a function of age $x$, denoting the conditional proportional reduction factor in the hazard rate after failure at age $x$. We will consider two different strategies:

i) if $\delta_y(x)$ is a function of both age($x$) and usage($y$).

ii) if $\delta_y(x)(= \delta_y)$ is a function of usage rate($y$) only.
2.2.1 Model formulation

If the item is minimally repaired at each failure then this type of rectification action has a negligible impact on its reliability. The hazard rate for item lifetime after a minimal repair is the same as that just before failure. If repair times are small relative to the mean time between failures (so that minimal repairs can be treated as being instantaneous) then item failures over time follow a non-homogeneous Poisson process (NHPP) with intensity function \( \lambda_y(x) = h(x; \alpha(y)) \). The intensity function is also referred to as the rate of occurrence of failures (ROCOF).

In contrast, an imperfect repair improves the items operating condition and the hazard rate of item lifetime after a repair is smaller. This can be modeled as follows. For a given usage rate \( y \), if the failure occurs at age \( x_i \) the conditional hazard rate just before failure is \( h(x_i-; \alpha(y)) \) and after repair it is given by

\[
h(x_i+; \alpha(y)) = h(x_i-; \alpha(y)) - \delta_y(x_i)(h(x_i-; \alpha(y)) - h(0; \alpha(y)))
\]

where \( \delta_y(x_i) \) can take values in the interval \([0, 1]\). The reduction in the hazard rate is a linear function of \( \delta_y(x_i) \). \( \delta_y(x_i) \) is a decision variable with a higher value indicating a greater improvement in the reliability after repair.

Imperfect repairs are also assumed to be instantaneous. The hazard rates are increasing functions of time (reflecting the degradation effect of age). Under minimal repair the effect on the system hazard rate is negligible. Under imperfect repair, the reliability after repair is improved but is inferior to that of a new item. This implies that the cost of an imperfect repair is greater than that of a minimal repair and this cost increases as the level of hazard rate reduction increases.

2.3 Analysis of servicing strategy

In this section the conditional expected warranty servicing cost \( J(K_y, L_y, \Delta_y(K_y, L_y)) \) for a given usage rate \( y \) is derived as a function of parameters \( K_y, L_y \) (subject to the constraints \( 0 \leq K_y \leq L_y \leq W_y \)) and function \( \Delta_y(K_y, L_y) \equiv \{ \delta_y(x); K_y \leq x \leq L_y \} \).
2.3.1 Conditional expected warranty cost

For a given usage rate $y$, let $T_{1y}$ denote the time of the first failure under usage rate $y$ after age $K_y$. The conditional distribution function, density function and survival function for $T_{1y}$ is given by

$$F_1(t; \alpha(y)) = \frac{F(t; \alpha(y)) - F(K_y; \alpha(y))}{F(K_y; \alpha(y))} \quad f_1(t; \alpha(y)) = \frac{f(t; \alpha(y))}{F(K_y; \alpha(y))}$$

and

$$F_1(t; \alpha(y)) = \frac{F(t; \alpha(y))}{F(K_y; \alpha(y))}, \quad t \geq K_y$$

respectively. We need to consider two cases:

1. $K_y \leq T_1 = x \leq L_y$

2. $T_1 = x > L_y$

In each case, all failures over $(0, K_y)$ are minimally repaired, so the failures occur according to an NHPP process with conditional intensity function $\lambda_y(x) = h(x; \alpha(y))$ and the conditional expected warranty servicing cost for this interval is given by

$$C_r \int_0^{K_y} h(x; \alpha(y)) \, dx$$

The conditional expected cost, conditional on $K_y \leq T_1 = t_1 \leq L_y$, is obtained as follows. The first failure in $[K_y, L_y]$ occurs at age $t_1$ and is imperfectly repaired. All failures over the remaining interval $(t_1, W_y]$ are minimally repaired. As a result, the failures over this interval occur according to an NHPP with conditional intensity function:

$$\lambda_y(x) = h(x; \alpha(y)) - \delta_y(t_1)(h(t_1; \alpha(y)) - h(0; \alpha(y))), \quad t_1 \leq x \leq W_y. \quad (10)$$

The expected cost function of servicing failures over $(t_1, W_y]$ is given by

$$C_r \int_{t_1}^{W_y} [h(x; \alpha(y)) - \delta_y(t_1)(h(t_1; \alpha(y)) - h(0; \alpha(y)))] \, dx.$$  \quad (11)

As a result, the conditional expected warranty cost for usage rate $y$ and $K_y \leq T_1 = t_1 \leq L_y$ is given by

$$J(K_y, L_y, \Delta_y(K_y, L_y)|K_y \leq x \leq L_y) =$$

$$C_r \int_0^{K_y} h(x; \alpha(y)) \, dx + C_r \int_{t_1}^{W_y} [h(x; \alpha(y)) - \delta_y(t_1)(h(t_1; \alpha(y)) - h(0; \alpha(y)))] \, dx + C_i(\delta(t_1), t_1). \quad (12)$$

The conditional expected cost, conditional on $T_1 = t_1 > L_y$, is obtained as follows. Note that there is no failure in $[K_y, L_y]$ and failures over the remaining interval $(L_y, W_y]$ occur according to an NHPP with intensity function

$$\lambda_y(x) = h(x; \alpha(y)), \quad L_y \leq x \leq W_y$$
As a result, the conditional expected warranty cost is given by
\[
J(K_y, L_y, \Delta_y(K_y, L_y)|x > L_y) = C_r \int_{0}^{K_y} h(x; \alpha(y)) dx + C_r \int_{L_y}^{W_y} h(x; \alpha(y)) dx. \tag{13}
\]

For a given usage rate \(y\) the expected warranty cost is obtained by unconditioning on \(T_i\) and is given by
\[
J(K_y, L_y, \delta_y(K_y, L_y)) =
\]
\[
J(K_y, L_y, \delta_y(x)|x > L_y)\overline{F}_1(L_y; \alpha(y)) + \int_{K_y}^{L_y} J(K_y, L_y, \delta_y(x)|K_y \leq x \leq L_y)f_1(x; \alpha(y)) dx. \tag{14}
\]

Define \(H(t; \alpha(y)) = \int_{0}^{t} h(u; \alpha(y)) du\).
Then (14) can be rewritten as
\[
J(K_y, L_y, \Delta_y(K_y, L_y)) = \Psi(K_y, L_y) + \Phi(\Delta(K_y, L_y), K_y, L_y). \tag{15}
\]

where
\[
\Psi(K_y, L_y) = C_r(H(K_y; \alpha(y)) - [H(W_y; \alpha(y)) - H(L_y; \alpha(y))] \frac{\overline{F}(L_y; \alpha(y))}{\overline{F}(K_y; \alpha(y))} + \int_{K_y}^{L_y} [H(W_y; \alpha(y)) - H(x; \alpha(y))] \frac{f(x; \alpha(y))}{\overline{F}(K_y; \alpha(y))} dx
\]
and
\[
\Phi(\Delta_y(K_y, L_y), K_y, L_y) = \int_{K_y}^{L_y} \left[C_i(\delta_y(x), x) - C_r \delta_y(x)\{h(x; \alpha(y)) - h(0; \alpha(y))\}(W_y - x)\right] \frac{f(x; \alpha(y))}{\overline{F}(K_y; \alpha(y))} dx
\]

### 2.3.2 Optimization problem for Strategy 1

The optimization problem is given by
\[
\min_{K_y, L_y, \Delta_y(K_y, L_y)} J(K_y, L_y, \Delta_y(K_y, L_y)) = \Psi(K_y, L_y) + \Phi(\Delta(K_y, L_y), K_y, L_y). \tag{16}
\]

Note that this involves selecting optimally the two parameters \(K_y\) and \(L_y\) for a given \(y\) (subject to the constraints \(0 \leq K_y \leq L_y \leq W_y\) and the function \(\Delta_y(K_y, L_y) = \{\delta_y(x): K_y \leq x \leq L_y\}\) (subject to the constraints \(0 \leq \delta_y(x) \leq 1\)).

Let \(K_y^*\) and \(L_y^*\) denote the optimal solution. We obtain this using a two-stage approach. In stage 1, for a fixed \(K_y^*\) and \(L_y^*\), we obtain the optimal \(\Delta_y^*(K_y, L_y)\) that minimizes
\[
J(K_y, L_y, \Delta_y(K_y, L_y)).
\]
Then, in stage 2, we obtain the optimal \((K_y^*, L_y^*)\) by minimizing
\[
J(K_y, L_y, \Delta_y^*(K_y, L_y)).
\]

**Stage 1:** To determine \(\Delta_y^*(K_y, L_y)\) we need to focus on \(\Phi(\Delta_y(K_y, L_y), K_y, L_y)\) given by (6) and this can be rewritten as
\[
\Phi(\Delta_y(K_y, L_y), K_y, L_y) = \int_{K_y}^{L_y} \left[C_i(\delta_y(x), x) - \delta_y(x)\xi_y(x)\right] \frac{f(x; \alpha(y))}{\overline{F}(K_y; \alpha(y))} dx \tag{17}
\]
where

\[ \xi_y(x) = C_r \{ h(x; \alpha(y)) - h(0; \alpha(y)) \} (W_y - x) \].

Assume the baseline survival distribution \( F_0 \) of the product’s lifetime is such that \( \xi_y(x) \) is concave in the item’s age \( x \). This postulate is satisfied by many parametric lifetime models that are increasingly degrading with age, such as the IFR Weibulls.

We need to determine the optimal form for \( \delta_y(x) \) for every point \( x \) along the time axis. The optimal \( \delta_y(x) \) must result in \( C_i(\delta_y(x), x) - \delta_y(x) \xi_y(x) \) being a minimum for each \( x \in [K_y, L_y] \). As result, \( \delta_y^*(x) \) can be obtained by examining:

\[ v(z_y, x) = [C_i(z_y, x) - \xi_y(x) z_y] \]

for each \( x \in [K_y, L_y] \). For a fixed \( x \), \( C_i(z_y, x) \) is an increasing function of \( z_y \) as shown in Fig. 2. \( \xi_y(x) z_y \), the second term in \( v(z_y, x) \), is linear in \( z_y \) and so is a straight line when plotted as a function of \( z_y \), as shown in Fig. 2.

We need to consider the following two cases.

**Case (1):** The line \( \xi_y(x) z_y \) lies below the curve \( C_i(z_y, x) \). This corresponds to (a) in Fig. 2. In this case, \( \delta_y^*(x) = 0 \). This is because the cost of any imperfect repair with \( \delta_y^*(x) > 0 \) is not worth the reduction in the expected warranty servicing cost when compared with only minimal repair \( \delta_y^*(x) = 0 \).

**Case (2):** The straight line \( \xi_y(x) z_y \) and the curve \( C_i(z_y, x) \) intersect. This corresponds to (b) in Fig. 2. and in this case we have \( \delta_y^*(x) > 0 \). Since \( 0 \leq \delta_y^*(x) \leq 1 \) then either \( \delta_y^*(x) = 1 \) (the boundary solution) or \( 0 < \delta_y^*(x) < 1 \) (an interior point solution). In the latter case, the optimal value is obtained from the usual first order condition. This yields \( \delta_y^*(x) = z_y^* \) for a given \( y \) with \( z_y^* \) given by

\[ \delta C_i(z_y, x) \delta z_y = \xi_y(x) \].

\[ (19) \]
Let the straight line $\kappa z_y$ be a tangent to the curve $C_i(z_y, x)$ at $z_y = \tilde{z}$. This is shown by (c) in Fig. 2. $\kappa$ and $\tilde{z}$ are obtained by solving the simultaneous equations given below:

$$C_i(\tilde{z}, x) = \kappa \tilde{z} \quad \text{and} \quad \frac{\delta C_i(z_y, x)}{\delta z_y} |_{z_y = \tilde{z}} = \kappa. \quad (20)$$

where $\xi_y(x)$ is a concave function as shown in Fig. 3. with $\xi_y(0) = 0$ and $\xi_y(W_y) = 0$. Define

$$\xi_{y\text{(max)}} = \max_{0 \leq x \leq W_y} \xi_y(x). \quad (21)$$

**Proposition:** If $\xi_{y\text{(max)}} < \kappa$ then $\delta^*_y(x) = 0$ for all x. If $\xi_{y\text{(max)}} > \kappa$ then $\delta^*_y(x) > 0$ for $0 \leq \tau_{1y} \leq x \leq \tau_{2y} \leq W_y$ where $\tau_{1y}$ and $\tau_{2y}$ are the solutions of the equation $\xi_y(x) = \kappa$. For x outside the interval $[\tau_{1y}, \tau_{2y}]$, $\delta^*_y(x) = 0$.

Note: This implies that $\delta^*_y(x)$ has a shape as shown in Fig. 4., and note that $\delta^*_y(x)$ does not depend on $K_y$ and $L_y$.

**Stage 2:** Let $\Delta^*_y(K_y, L_y) \equiv \{ \delta^*_y(x) : 0 \leq x \leq W_y \}$ which is obtained from Stage 1. $K_y^*$ and $L_y^*$, the optimal values for $K_y$ and $L_y$, are obtained by solving the following minimization problem

$$\min_{K_y, L_y} J(K_y, L_y, \Delta^*_y(K_y, L_y)) = \Psi(K_y, L_y) + \Phi(\Delta^*_y(K_y, L_y), K_y, L_y).$$
subject to the constraint $0 \leq K_y \leq L_y \leq W_y$. These can be obtained from the usual first-order conditions:

$$\frac{\partial}{\partial K_y} J(K_y, L_y, \Delta_y(K_y, L_y)) = 0 \quad \text{and} \quad \frac{\partial}{\partial L_y} J(K_y, L_y, \Delta_y(K_y, L_y)) = 0 \quad (22)$$

if they lie inside the interval $[0, W_y]$. It is not possible to derive any analytical results from these conditions and the optimal values need to be obtained using a computational approach.

### 2.3.3 Optimization problem for Strategy 2

The optimization problem is given by

$$\min_{K_y, L_y, \delta_y} J(K_y, L_y, \delta_y) = \Psi(K_y, L_y) + \Phi(\delta_y, K_y, L_y). \quad (23)$$

where $\Psi(K_y, L_y)$ is same as (16) and

$$\Phi(\delta_y, K_y, L_y) = \int_{K_y}^{L_y} [C_i(\delta_y) - C_r \delta_y \{h(x; \alpha(y)) - h(0; \alpha(y))\}] (W_y - x) \frac{f(x; \alpha(y))}{F(K_y; \alpha(y))} dx. \quad (24)$$

Here the cost of imperfect repair $C_i(\delta_y)$ will not depend on the age at failure. This problem involves selecting optimally three parameters $\delta_y (0 \leq \delta_y \leq 1)$, $K_y$ and $L_y$ ($0 \leq K_y \leq L_y \leq W_y$) for a given $y$.

We use the two-stage approach. In stage 1, given $y$, we fix $K_y$ and $L_y$ and obtain the optimal $\delta^*_y(K_y, L_y)$ that minimizes $J(K_y, L_y, \delta_y)$. Then in stage 2, we obtain the optimal $(K^*_y, L^*_y)$ by minimizing $J(K_y, L_y, \delta^*_y)$.

**Stage 1**: $\delta^*_y(K_y, L_y)$ is obtained by solving the following optimization problem:

$$\min_{\delta_y \mid K_y, L_y} \phi(\delta_y, K_y, L_y) = \phi_1(K_y, L_y)C_i(\delta_y) + \phi_2(K_y, L_y)\delta_y. \quad (25)$$

where

$$\phi_1(K_y, L_y) = \int_{K_y}^{L_y} f(x; \alpha(y)) dx,$$

and

$$\phi_2(K_y, L_y) = C_r \int_{K_y}^{L_y} \{h(x; \alpha(y)) - h(0; \alpha(y))\} (W_y - x) f(x; \alpha(y)) dx.$$

$\delta^*_y$ can either be an interior point or one of the end-points of the interval $[0,1]$. If $\delta^*_y$ is an interior point then it is obtained from the first order condition:

$$\frac{\partial}{\partial \delta} \phi(\delta_y, K_y, L_y) = 0, \quad \text{or} \quad \phi_1(K_y, L_y) \frac{\partial}{\partial \delta} C_i(\delta_y) = \phi_2(K_y, L_y). \quad (26)$$

Here the optimal $\delta^*_y$ will be a function of $K_y$ and $L_y$. 

**Stage 2**: $K^*_y$ and $L^*_y$ is obtained from the following optimization problem:

$$\min_{K_y, L_y} J(K_y, L_y, \delta^*_y) = \Psi(K_y, L_y) + \Phi(\delta^*_y, K_y, L_y) \quad (27)$$

subject to the constraint $0 \leq K_y \leq L_y \leq W_y$. We need to use computational approach to obtain these optimal values. The optimal reduction when an imperfect repair is carried out is given by $\delta^*_y(K^*_y, L^*_y)$. 

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2.4 Special case: Weibull failure distribution

The distribution function for the time to first failure under the nominal usage rate \( y_0 \) denoted by \( T_0 \) is a Weibull distribution with scale parameter \( \alpha_0 > 0 \) and shape parameter \( \beta > 1 \), so

\[
F(x; \alpha_0) = 1 - \exp \left( -\frac{x}{\alpha_0} \right)^\beta \quad \bar{F}(x; \alpha_0) = \exp \left( -\frac{x}{\alpha_0} \right)^\beta.
\]

Therefore, using the A.F.T. formulation the following functions can be derived for \( T_y \), the time to first failure under the usage rate \( y \):

C.d.f.: \( F(x; \alpha(y)) = 1 - \exp \left( -\frac{x}{\alpha(y)} \right)^\beta = 1 - \exp \left( \frac{y}{y_0} \frac{x}{\alpha_0} \right)^\beta. \)

Survival function: \( \bar{F}(x; \alpha(y)) = \exp \left( -\frac{x}{\alpha(y)} \right)^\beta = \exp \left( \frac{y}{y_0} \frac{x}{\alpha_0} \right)^\beta. \)

Hazard function: \( h(x; \alpha(y)) = \beta \left( \frac{y}{y_0} \right)^\gamma \left( \frac{x^{\beta-1}}{\alpha_0^\beta} \right), \quad (\beta > 1). \)

Cumulative hazard function: \( H(x; \alpha(y)) = \left( \frac{y}{y_0} \right)^\gamma \left( \frac{x^\beta}{\alpha_0^\beta} \right). \)

2.4.1 Strategy 1:

Let \( C_0 \) denote the cost of repair that achieves 100% reduction in the system hazard rate, \( C_r \) \((< C_0)\) denote the cost of minimal repair. Then the cost of imperfect repair for usage rate \( y \) is given by the expression

\[
C_i(z_y, x) = C_r + (C_0 - C_r)z_y^p, \quad (p > 1),
\]

where \( z_y \) is the proportional reduction factor in the failure rate under fixed usage rate \( y \). Thus \( z_y \in [0, 1] \) is a decision variable, with a greater value indicating a greater improvement in the reliability of the item after repair.

From (19), we have

\[
\xi_y(x) = C_r \{ h(x; \alpha(y)) - h(0; \alpha(y)) \} (W_y - x) = C_r \beta \left( \frac{y}{y_0} \right)^\gamma \left( \frac{x^{\beta-1}}{\alpha_0^\beta} \right) (W_y - x). \]

\[
\frac{\partial}{\partial x} \xi_y(x) = 0 \quad \text{gives the maximum at age} \quad x = \frac{\beta - 1}{\beta} W_y,
\]

since \( \xi_y(x) \) is concave in \( x \) for each \( y \). The maximum value of \( \xi_y(x) \) is

\[
\xi_{y(\text{max})} = \max_{0 \leq x \leq W_y} \xi_y(x) = \frac{C_r (\frac{y}{y_0})^\gamma}{\alpha_0^\beta} \left( \frac{\beta - 1}{\beta} \right)^{\beta-1} W_y^\beta.
\]
Clearly, $\xi_{y(\text{max})} > 0$ for all $y$.
From (20) we have:
\[
\tilde{z} = \left(\frac{C_r}{(C_0 - C_r)(p - 1)}\right)^{\frac{1}{p}} \quad \text{and} \quad \kappa = (C_0 - C_r)p\left(\frac{C_r}{(C_0 - C_r)(p - 1)}\right)^{\frac{p - 1}{p}}.
\tag{32}
\]
For each $y$, $\tau_{1y}$ and $\tau_{2y}$ are the solutions of the equation:
\[
C_r \beta \left(\frac{y}{y_0}\right)^{\gamma \beta} \left(\frac{x^{p-1}}{\alpha_0^{\beta}}\right)(W_y - x) - (C_0 - C_r)p\left(\frac{C_r}{(C_0 - C_r)(p - 1)}\right)^{\frac{p - 1}{p}} = 0.
\tag{33}
\]
The optimum $\delta_y^*(x)$ for strategy 1 is
\[
\delta_y^*(x) = \left\{\left(\frac{C_r \beta}{(C_0 - C_r)p}\right)\left(\frac{y}{y_0}\right)^{\gamma \beta} \left(\frac{x^{p-1}}{\alpha_0^{\beta}}\right)\right\}(W_y - x), \quad \text{for} \quad 0 < \tau_{1y} < x < \tau_{2y} < W_y
\tag{34}
\]
Here $\delta_y^*(x)$ does not depend on the values of $K_y$ and $L_y$. As mentioned previously, we need to calculate the values of $K_y^*$ and $L_y^*$ using computational methods.

2.4.2 Strategy 2:
Given $y$, the cost function
\[
C_i(\delta) = C_r + (C_0 - C_r)\delta^p, \quad \text{therefore} \quad \frac{\partial}{\partial \delta} C_i(\delta) = (C_0 - C_r)p\delta^{p-1}.
\]
Using the first order condition
\[
\frac{\partial}{\partial \delta} \phi(\delta, K_y, L_y) = 0.
\]
i.e.,
\[
\frac{\partial}{\partial \delta} \left(\frac{\phi_1(K_y, L_y)C_i(\delta) - \phi_2(K_y, L_y)\delta}{F(K_y, \alpha(y))}\right) = 0
\]
Thus we get,
\[
\delta^*(K_y, L_y) = \left(\frac{\phi_2(K_y, L_y)}{(C_0 - C_r)p\phi_1(K_y, L_y)}\right)^{\frac{1}{p-1}},
\tag{35}
\]
where $\phi_1(K_y, L_y)$ and $\phi_2(K_y, L_y)$ are integral equations given in (25). Unlike strategy 1, here the optimum reduction proportion depends on $K_y$ and $L_y$.  

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2.5 Numerical Example: Strategy 1 and 2 for \( C_0 = 2 \) and \( \beta = 2 \)

We normalise costs so that the cost of minimal repair, \( C^r = 1 \) and consider a range of values for the cost of replacement(perfect repair), i.e., \( C_0 \) varying from 2 to 10. We assume the nominal values warranty period, \( W = 2 \), total usage limit, \( U = 2 \), weibull scale(baseline) parameter, \( \alpha_0 = 1 \), weibull shape parameter, \( \beta = (2, 3) \), nominal usage rate, \( y_0 = 1 \), the AFT model parameter, \( \gamma = 2 \) and imperfect cost function parameter, \( p = 4 \).

Table 1: Optimal servicing strategies and expected servicing costs for \( C_0 = 2 \) and \( \beta = 2 \)

<table>
<thead>
<tr>
<th>( y )</th>
<th>( W_y )</th>
<th>( K_y )</th>
<th>( L_y )</th>
<th>( \text{Cost1} )</th>
<th>( \delta_y )</th>
<th>( \text{Cost2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1000</td>
<td>2.0000</td>
<td>1.9004</td>
<td>1.9000</td>
<td>0.0004</td>
<td>1.2500</td>
<td>1.3100</td>
</tr>
<tr>
<td>0.3000</td>
<td>2.0000</td>
<td>1.9000</td>
<td>1.9000</td>
<td>0.0324</td>
<td>0.2990</td>
<td>1.8900</td>
</tr>
<tr>
<td>0.5000</td>
<td>2.0000</td>
<td>1.8000</td>
<td>1.8000</td>
<td>0.2500</td>
<td>0.2080</td>
<td>1.9200</td>
</tr>
<tr>
<td>0.7000</td>
<td>2.0000</td>
<td>1.4301</td>
<td>1.9860</td>
<td>0.9602</td>
<td>0.2380</td>
<td>1.9200</td>
</tr>
<tr>
<td>0.9000</td>
<td>2.0000</td>
<td>0.8048</td>
<td>1.977</td>
<td>2.1880</td>
<td>0.5400</td>
<td>1.8800</td>
</tr>
<tr>
<td>1.0000</td>
<td>2.0000</td>
<td>0.636</td>
<td>1.768</td>
<td>3.1950</td>
<td>0.6470</td>
<td>1.8470</td>
</tr>
<tr>
<td>1.2000</td>
<td>1.6600</td>
<td>0.590</td>
<td>1.4502</td>
<td>4.0250</td>
<td>0.6057</td>
<td>1.5300</td>
</tr>
<tr>
<td>1.4000</td>
<td>1.4200</td>
<td>0.5329</td>
<td>1.1929</td>
<td>5.1022</td>
<td>0.5690</td>
<td>1.3900</td>
</tr>
<tr>
<td>1.6000</td>
<td>1.2500</td>
<td>0.5029</td>
<td>0.9951</td>
<td>6.1899</td>
<td>0.4790</td>
<td>1.0100</td>
</tr>
<tr>
<td>1.8000</td>
<td>1.1100</td>
<td>0.473</td>
<td>0.9080</td>
<td>7.6135</td>
<td>0.4790</td>
<td>1.0100</td>
</tr>
<tr>
<td>2.0000</td>
<td>1.0000</td>
<td>0.447</td>
<td>0.7719</td>
<td>8.7756</td>
<td>0.4430</td>
<td>0.8740</td>
</tr>
<tr>
<td>2.5000</td>
<td>0.8000</td>
<td>0.3777</td>
<td>0.5542</td>
<td>13.2400</td>
<td>0.3690</td>
<td>0.6490</td>
</tr>
<tr>
<td>3.0000</td>
<td>0.6700</td>
<td>0.3199</td>
<td>0.4130</td>
<td>18.7374</td>
<td>0.3150</td>
<td>0.4910</td>
</tr>
<tr>
<td>3.5000</td>
<td>0.5700</td>
<td>0.3045</td>
<td>0.3420</td>
<td>21.6691</td>
<td>0.2750</td>
<td>0.4190</td>
</tr>
<tr>
<td>4.0000</td>
<td>0.5000</td>
<td>0.2717</td>
<td>0.3206</td>
<td>33.2484</td>
<td>0.2430</td>
<td>0.3420</td>
</tr>
<tr>
<td>4.5000</td>
<td>0.4400</td>
<td>0.2437</td>
<td>0.2650</td>
<td>35.8719</td>
<td>0.2190</td>
<td>0.2609</td>
</tr>
<tr>
<td>5.0000</td>
<td>0.4000</td>
<td>0.2068</td>
<td>0.2467</td>
<td>60.1243</td>
<td>0.1980</td>
<td>0.2423</td>
</tr>
</tbody>
</table>
Figure 5: (Strategies 1 and 2) Region $\Gamma$ for $C_0 = 2$ and $\beta = 2$

The two axes are total usage ($u$) level and age ($x$) respectively. The maximum usage limit ($U$) is $2 \times 10000$ miles and the warranty period ($W$) is 2 years. It can be seen that the region $\Gamma$ obtained from numerical computation is similar to Fig. 1.
As $C_0$ increases the value of $\delta_y^*$ decreases given $y$. Intuitively this makes sense because if the cost of replacement($C_0$) increases, the cost of imperfect repair ($C_i(\delta_y, x)$ which is a function of $C_r$, $C_0$ and $\delta_y$ increases and can be controlled by reducing the value of $\delta_y$.

This mesh plot shows the behavior of $\delta_y$ corresponding to each pair of ($C_0, y$) for a fixed $\beta$. Clearly $\delta_y$ is increasing in $y$ and decreasing in $C_0$ (explained in fig. 6).
Table 2: Cost Comparison for $C_0 = 2$ and $\beta = 2$.

<table>
<thead>
<tr>
<th>$y$</th>
<th>Cost 1</th>
<th>Cost 2</th>
<th>Cost 3</th>
<th>Cost 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9000</td>
<td>2.1880 (11.11)</td>
<td>2.1361 (13.22)</td>
<td>2.4614</td>
<td>2.6244</td>
</tr>
<tr>
<td>1.0000</td>
<td>3.1950 (1.12)</td>
<td>3.0130 (6.75)</td>
<td>3.2312</td>
<td>4.0000</td>
</tr>
<tr>
<td>1.2000</td>
<td>4.0250 (1.78)</td>
<td>4.0210 (1.88)</td>
<td>4.0980</td>
<td>5.7600</td>
</tr>
<tr>
<td>1.4000</td>
<td>5.1022 (0.02)</td>
<td>5.1010 (0.04)</td>
<td>5.1032</td>
<td>7.8400</td>
</tr>
<tr>
<td>1.6000</td>
<td>6.1899 (1.42)</td>
<td>6.1290 (2.39)</td>
<td>6.2790</td>
<td>10.2400</td>
</tr>
<tr>
<td>1.8000</td>
<td>7.6135 (0.03)</td>
<td>7.3985 (2.85)</td>
<td>7.6157</td>
<td>12.9600</td>
</tr>
<tr>
<td>2.0000</td>
<td>8.7756 (3.77)</td>
<td>8.5730 (5.99)</td>
<td>9.1197</td>
<td>16.0000</td>
</tr>
<tr>
<td>2.5000</td>
<td>13.2400 (4.05)</td>
<td>11.4020 (17.37)</td>
<td>13.7987</td>
<td>25.0000</td>
</tr>
<tr>
<td>3.0000</td>
<td>18.7374 (5.32)</td>
<td>13.8050 (30.24)</td>
<td>19.7906</td>
<td>36.0000</td>
</tr>
<tr>
<td>3.5000</td>
<td>21.6691 (21.69)</td>
<td>15.4690 (44.10)</td>
<td>27.6721</td>
<td>49.0000</td>
</tr>
<tr>
<td>4.0000</td>
<td>33.2484 (11.21)</td>
<td>16.0890 (57.03)</td>
<td>37.4444</td>
<td>64.0000</td>
</tr>
<tr>
<td>4.5000</td>
<td>35.8719 (25.67)</td>
<td>15.3930 (68.10)</td>
<td>48.2577</td>
<td>81.0000</td>
</tr>
<tr>
<td>5.0000</td>
<td>60.1243 (02.51)</td>
<td>13.0792 (78.79)</td>
<td>61.6739</td>
<td>100.0000</td>
</tr>
</tbody>
</table>

Figures in brackets are % cost savings relative to Jack et.al.[11]. This shows that our new servicing strategy using an imperfect repair is saving a portion of the total expected cost over the warranty period $W$, compared to the repair-replacement strategy of Jack et.al.[11].

References


