Modeling with Bivariate Geometric Distributions

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Abstract

We study systems with several components which are subject to different types of failures. Examples of such systems include twin engines of an airplane or the paired organs in a human body. We find that such a system, using conditional arguments, can be characterized as multivariate geometric distributions. We prove that the characterizations of the geometric model can be achieved using conditional probabilities, conditional failure rates, or probability generating function. These new models are fitted to real-life data using the Method of Moments Estimators, Maximum Likelihood Estimators, and Bayes Estimators. The last two estimators are obtained by solving score equations. We also compare two Methods of Moments Estimators in each of the several bivariate geometric models to evaluate their performance using the bias vector and variance-covariance matrix. This comparison is done through a Monte-Carlo simulation for increasing sample sizes. The Chi-square goodness-of-fit tests are used to evaluate model performance.

Keywords: bivariate geometric distribution, conditional failure rate, Bayes estimation, maximum likelihood estimation

Mathematics Subject Classification 2010: 62E10, 62F10
1. Introduction

A variety of bivariate models have been proposed in statistics to represent life-time data. Freund (1961) constructed his model as a bivariate extension of two exponential distributions. Marshall and Olkin (1985) studied a family of bivariate distributions generated by the bivariate Bernoulli distributions. Nair and Nair (1988) studied the characterizations of the bivariate exponential and geometric distributions. Basu and Dhar (1995) proposed a bivariate geometric model (BGD (B&D)) which is analog to the bivariate distribution of Marshall and Olkin (1967). Dhar (1998) derived a new bivariate geometric model (BGD (F)) which is a discrete analog to Freund’s model.

In this article, we first study the bivariate fatal shock model derived by Basu and Dhar (1995). However, this model is a reparameterized version of the bivariate geometric model of Hawkes (1972) and in contrast the BGD (B&D) (1995) random vector takes values in the set of cross-product of positive integers with itself. The other bivariate geometric model we study here is the BGD (F) which deserves more exploration. Thus, this research derives several characterizations of this and the other models.

Some of these characterizations are through conditional distributions. The one for BGD (F), is studied in Section 3, while the characterization of the BGD (B&D), has been done by Sreehari (2005) through Hawkes’ model. Cox (1972) introduced conditional failure rate (CFR) in the area of reliability. Sun and Basu (1995) derived the characterization result based on this CFR for the BGD (B&D). Sreehari (2005) used a revised version of conditional failure rate to derive the characterization theorem for the BGD (B&D) through Hawkes’ (1972) model. This research derives the characterization for the BGD (F) using CFR from Sreehari (2005) in Section 4. In Section 5, the joint probability generating function (p.g.f) of BGD (F) random variables \((X, Y)\) are derived.
and verified using the relationship between joint probability mass function and p.g.f in terms of the repeated partial derivatives (Kocherlakota & Kocherlakota, 1992).

Estimation methods for these bivariate geometric models are developed in Section 6. An application in context of 1995 Winter Olympics data is demonstrated in Section 7. Different random samples of the BGD (B&D) model are generated through Monte Carlo simulation using R programming language. Within each sample size, we calculated two method of moments estimators. The bias, variance, and covariance of the parameters are computed and compared for their performances. We simulated a random sample and fitted models corresponding to maximum likelihood, Bayes and method of moment estimation procedures. In order to evaluate the performance of the different methods of estimation, the Chi-square goodness-of-fit tests are used.

2. Bivariate Geometric Distributions

The following bivariate geometric distributions are recalled in this section in order to derive the characterizations, p.g.f’s, and other results. The survival function for BGD (B&D) is given by:

\[ P(X > x, Y > y) = P(B(x, p_1) = 0, B(y, p_2) = 0, B(x \vee y, p_{12}) = 0) = p_1^x p_2^y p_{12}^{x \vee y}, \]

where \(1 \leq x, y \in \mathbb{Z}^+, 0 < p_1 < 1, 0 < p_2 < 1, \) and \(0 < p_{12} < 1. \) Here, \(x \vee y = \max(x, y)\) and \(\mathbb{Z}^+\) is the set of positive integers.

From the survival function, the bivariate geometric distribution BGD (B&D)
is given by:

\[
P(X = x, Y = y) = \begin{cases} 
  p_x^{x-1}(p_{2p_{12}})^{y-1}q_1(1 - p_{2p_{12}}), & \text{if } x < y, \\
  (p_{1p_2p_{12}})^{x-1}(1 - p_{1p_1p_2} - p_{2p_{12}} + p_{1p_2p_{12}}), & \text{if } x = y, \\
  p_2^{y-1}(p_{1p_2p_{12}})^{x-1}q_2(1 - p_{1p_1p_2}), & \text{if } x > y, 
\end{cases}
\]

(2.1)

where \(1 \leq x, y \in \mathbb{Z}^+\), \(q_i = 1 - p_i\), \(i = 1, 2\).

The BGD (F) is given by:

\[
P(X = m, Y = n) = \begin{cases} 
  \frac{q_1q_4}{p_1p_4} \left[ \frac{p_1p_2}{p_4} \right]^m p_3^n, & \text{if } m < n, m, n = 1, 2, \ldots, \\
  \frac{q_2q_3}{1-p_1p_2} p_{12}^{m-1}, & \text{if } m = n, m, n = 1, 2, \ldots, \\
  \frac{q_2q_3}{p_2p_3} \left[ \frac{p_1p_2}{p_3} \right]^n p_4^m, & \text{if } m > n, m, n = 1, 2, \ldots, 
\end{cases}
\]

(2.2)

where \(0 < p_i < 1, p_i + q_i = 1, i = 1, 2, 3, 4\), \(0 < p_{12} < 1, p_{12} + q_{12} = 1, p_1p_2 < p_3\) and \(p_1p_2 < p_4\). Here, \(p_1\) represents the probability of survival of component 1, and \(p_3\) represents the probability of the survival of component 1 for a unit time, given that it has already survived for some time, at the end of which component 2 has failed; \(p_2\) and \(p_4\) is analogously interpreted for component 2 (Dhar, 2003).

3. Characterizations via Conditionally Distributions

Using the conditional distributions \(g(m|n)\) of \(X\) given \(Y = n\) and \(h(n|m)\) of \(Y\) given \(X = m\) that are obtained from (2.2) and loss of memory property (Dhar, 1998). The following characterization result for bivariate geometric model BGD (F) can be established.

**Theorem 3.1.** Suppose that the conditional distributions \(g(m|n) = P(X = m|Y = n)\) of \(X\) given \(Y = n\) and the conditional distributions \(h(n|m) = P(Y = m|X = n)\) of \(Y\) given \(X = n\) are given by:

\[
\begin{align*}
  g(m|n) &= \frac{q_1q_4}{p_1p_4} \left[ \frac{p_1p_2}{p_4} \right]^m p_3^n, \quad \text{if } m < n, \\
  h(n|m) &= \frac{q_2q_3}{1-p_1p_2} p_{12}^{m-1}, \quad \text{if } m = n, \\
  g(m|n) &= \frac{q_2q_3}{p_2p_3} \left[ \frac{p_1p_2}{p_3} \right]^n p_4^m, \quad \text{if } m > n.
\end{align*}
\]

(2.3)

where \(0 < p_i < 1, p_i + q_i = 1, i = 1, 2, 3, 4\), \(0 < p_{12} < 1, p_{12} + q_{12} = 1, p_1p_2 < p_3\) and \(p_1p_2 < p_4\). Here, \(p_1\) represents the probability of survival of component 1, and \(p_3\) represents the probability of the survival of component 1 for a unit time, given that it has already survived for some time, at the end of which component 2 has failed; \(p_2\) and \(p_4\) is analogously interpreted for component 2 (Dhar, 2003).
Proof: Let $P(X = m) = f_1(m)$ and $P(Y = n) = f_2(n)$. Then the fact

$$P(X = m|Y = n)f_2(n) = P(Y = n|X = m)f_1(m)$$

(3.1)

gives us for $m > n$,

$$\frac{(p_4 - p_1p_2)q_1q_3p_2^n p_1^{m-1}p_4^{n-m-1}}{(p_2 - p_1p_2)(1 - p_4)p_4^{n-1} + (p_4 - p_2)(1 - p_1p_2)(p_1p_2)^{m-1}}f_2(n) =$$

$$\frac{(p_3 - p_1p_2)q_1q_3p_1p_2^n p_3^{m-n-1}}{(p_1 - p_1p_2)(1 - p_3)p_3^{m-1} + (p_3 - p_1)(1 - p_1p_2)(p_1p_2)^{m-1}}f_1(m).$$

In the above equation, isolate $f_2(n)$ and then sum over $n = 1, 2, \ldots$ to get that for $m = 1, 2, \ldots$,

$$f_1(m) = \frac{(p_1 - p_1p_2)(1 - p_3)p_3^{m-1} + (p_3 - p_1)(1 - p_1p_2)(p_1p_2)^{m-1}}{p_3 - p_1p_2}. \quad (3.2)$$

Therefore,

$$P(X = m, Y = n) = P(Y = n|X = m)f_1(m)$$

$$= \frac{q_2q_3}{p_2p_3} \left[ \frac{p_1p_2}{p_3} \right]^n p_3^m, \quad \text{if } m > n.$$
For \( m < n \), by similarly summing equation (3.1) over \( m = 1, 2, \ldots \), we can obtain \( f_2(n) \). Therefore,

\[
P(X = m, Y = n) = P(X = m|Y = n)f_2(n) = \frac{q_1q_4}{p_1p_4} \left[ \frac{p_1p_2}{p_4} \right]^m p_4^n, \quad \text{if } m < n.
\]

For \( m = n \), substitute \( f_1(m) \) as shown in (3.2) into the following

\[
P(X = m, Y = m) = P(Y = m|X = m)f_1(m) = q_1q_2p_2^{m-1} \quad \text{if } m = n.
\]

It can be then observed that the joint distribution of \((X, Y)\) is \( BGD(F) \) with \( p_{12} = p_1p_2 \).

4. Characterizations by Conditional Failure Rate

Several versions of conditional failure rates (CFRs) have been used to characterize different bivariate geometric distributions. One of them defined by Cox (1972) is given by:

\[
r_1(m|n) = \frac{P(M = m, N = n)}{P(M \geq m, N = n)}, \quad \text{for } m > n,
\]

\[
r_2(n|m) = \frac{P(M = m, N = n)}{P(M = m, N \geq n)}, \quad \text{for } n > m, \quad \text{(4.1)}
\]

\[
r(t) = P(\min(M, N) = t)/P(M \geq t, N \geq t),
\]

with \((M, N)\) taking values in the set \( \{0, 1, 2, \ldots\} \times \{0, 1, 2, \ldots\} \), and \( t \in \{0, 1, 2, \ldots\} \).

Notice that \( r(t) \) is the failure rate of \( \min(M, N) \) and \( r_1(m|n) \) is the conditional failure rates of \( M \) given \( N = n \) for \( m > n \). The quantity \( r_2(n|m) \) can be similarly interpreted.

Asha and Nair (1998) considered this CFR functions (4.1) and discussed their roles in characterizing the Hawkes model and extended the domain of \( r_1(m|n) \)
and \( r_2(n|m) \) to the entire region \( \{0, 1, 2, \ldots\} \times \{0, 1, 2, \ldots\} \) which includes the region \( m = n, m, n = 0, 1, 2, \ldots \). Using different constant CFRs with loss of memory property (Dhar, 1998), the specified geometric nature of the BGD (F) density at \((n, n)\), \( n \geq 2 \), as the sufficient conditions, the BGD (F) distribution is derived.

**Theorem 4.1.** Suppose \( X \) and \( Y \) are random variables with probability mass function \( f(m, n) \) satisfying

\[
\begin{align*}
  r_1(m|n) &= q_3, & \text{for } m > n, m, n = 1, 2, \ldots, \\
  r_2(n|m) &= q_4, & \text{for } n > m, m, n = 1, 2, \ldots, \\
  r_1(m|m) &= q_1, & \text{for } m = 1, 2, \ldots, \\
  r_2(n|n) &= q_2, & \text{for } n = 1, 2, \ldots,
\end{align*}
\]

and

\[
f(n, n) = (p_1 p_2)^{n-1} f(1, 1), \quad \text{for } n = 2, 3, \ldots,
\]

where \( 0 < q_i < 1, \quad p_i + q_i = 1, \quad i = 1, 2, 3, 4 \), then the joint distribution of \((X, Y)\) is BGD (F) with \( p_{12} = p_1 \cdot p_2 \).

**Proof:** Using the given \( r_1(m|n) \) and induction on \( k \),

\[
f(n + k, n) = \frac{p_1 q_3}{q_1} p_3^{k-1} f(n, n), \quad k = 1, 2, \ldots, \tag{4.2}
\]

is derived first as follows. Suppose

\[
r_1(m|n) = \begin{cases} 
  q_3, & m > n, m, n = 1, 2, \ldots, \\
  q_1, & m = n, m, n = 1, 2, \ldots
\end{cases} \tag{4.3}
\]

where \( 0 < q_1, q_3 < 1 \). The equality in (4.3) implies that

\[
r_1(n|n) = q_1 = \frac{f(n, n)}{P(X \geq n, Y = n)} = \frac{f(n, n)}{f(n, n) + P(X \geq n + 1, Y = n)},
\]

\[
\frac{1}{q_1} = 1 + \frac{P(X \geq n + 1, Y = n)}{f(n, n)}.
\]

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Thus,

\[ P(X \geq n + 1, Y = n) = \frac{1 - q_1}{q_1} f(n, n). \]

In the region \( m > n \), when \( m = n + 1 \),

\[ r_1(n + 1|n) = q_3 = \frac{f(n + 1, n)}{P(X \geq n + 1, Y = n)}, \]

\[ f(n + 1, n) = q_3 P(X \geq n + 1, Y = n) \]
\[ = q_3 \frac{1 - q_1}{q_1} f(n, n) \]
\[ = \frac{p_1 q_3}{q_1} f(n, n), \]

if \( m = n + 2 \),

\[ r_1(n + 2|n) = q_3 = \frac{f(n + 2, n)}{P(X \geq n + 2, Y = n)}, \]

\[ f(n + 2, n) = q_3 P(X \geq n + 2, Y = n) \]
\[ = q_3 [P(X \geq n + 1, Y = n) - P(X = n + 1, Y = n)] \]
\[ = q_3 [\frac{1 - q_1}{q_1} f(n, n) - f(n + 1, n)] \]
\[ = q_3 [\frac{1 - q_1}{q_1} f(n, n) - q_3 \frac{1 - q_1}{q_1} f(n, n)] \]
\[ = q_3 \frac{1 - q_1}{q_1} f(n, n)(1 - q_3) \]
\[ = \frac{p_1 q_3 p_3}{q_1} f(n, n), \]

Using induction on \( k \), the following equation is true for \( k = 2 \) and \( k = 3 \),

\[ f(n + k - 1, n) = \frac{p_1 q_3}{q_1} p_3^{-2} f(n, n) \quad \text{for } k = 2, 3, \ldots, \]

Now assume the above equation is true for \( k \). It can be shown that the above
equality holds true for \( k + 1, \ k = 1, 2, 3, \ldots \),

\[
f(n + k, n) = q_3 P(X \geq n + k, Y = n)
\]

\[
= q_3 [P(X \geq n + k - 1, Y = n) - P(X = n + k - 1, Y = n)]
\]

\[
= q_3 \left( \frac{f(n + k - 1, n)}{q_3} - f(n + k - 1, n) \right)
\]

\[
= q_3 \left( \frac{1}{q_3} - 1 \right) f(n + k - 1, n)
\]

\[
= q_3 \left( \frac{1 - q_3 p_1 q_3}{q_3} p_3^{k-2} f(n, n) \right)
\]

\[
= \frac{p_1 q_3}{q_1} p_3^{k-1} f(n, n).
\]

Thus, we proved that

\[
f(m, n) = \frac{p_1 q_3}{q_1} p_{3}^{m-n-1} f(n, n), \quad m > n, m, n = 1, 2, \ldots, \quad (4.4)
\]

by induction. Similarly, suppose that

\[
r_2(n|m) = \begin{cases} 
q_4, & n > m, m, n = 1, 2, \ldots, \\
q_2, & m = n, m, n = 1, 2, \ldots,
\end{cases} \quad (4.5)
\]

where \( 0 < q_2, q_4 < 1 \), we can show that

\[
f(m, n) = \frac{p_2 q_4}{q_2} p_{4}^{n-m-1} f(m, m), \quad n > m, m, n = 1, 2, \ldots, \quad (4.6)
\]

In addition, from supposition

\[
f(m, m) = (p_1 p_2)^{m-1} f(1, 1), \quad m = 2, 3, \ldots, \quad (4.7)
\]

Since \( f(m, n) \) must add over \( \mathbb{Z}^+ \times \mathbb{Z}^+ \) up to 1, gives \( f(1, 1) = P(X = 1, Y = 1) = q_1 q_2 \). Substituting the equation (4.7) into equations (4.4) and (4.6), one can see that the joint distribution of \((X, Y)\) is BGD (F) with \( p_{12} = p_1 \cdot p_2 \) as defined in equation (2.2).
5. Characterizations via Probability Generating Function

The joint p.g.f of the paired random variables \((X, Y)\) for the BGD (F) can be derived as follows:

\[
\pi(t_1, t_2) = E[t_1^X t_2^Y] = \sum_{(x,y) \in T} t_1^x t_2^y f(x, y)
\]

\[
= \sum_{x<y} t_1^x t_2^y f(x, y) + \sum_{x>y} t_1^x t_2^y f(x, y) + \sum_{x=y} t_1^x t_2^y f(x, y)
\]

\[
= \frac{q_1 q_4 p_2 t_1 t_2^2}{(1-t_2p_4)(1-t_1 t_2 p_1 p_2)} + \frac{q_2 q_3 p_1^2 t_2}{(1-t_1 p_3)(1-t_1 t_2 p_1 p_2)} + \frac{q_1 q_2 q_1^2 t_1 t_2}{(1-p_1 p_2)(1-t_1 t_2 p_1 p_2)}.
\] (5.1)

Here \(0 < p_i < 1, i = 1, 2, 3, 4, 0 < p_{12} < 1, p_1 p_2 < p_3, p_1 p_2 < p_4, |t_1| \leq \min\{\frac{p_4}{p_1 p_2}, \frac{1}{p_3}\}, |t_2| \leq \min\{\frac{p_3}{p_1 p_2}, \frac{1}{p_4}\} \) and \(|t_1 t_2| < \min\{\frac{1}{p_1 p_2}, \frac{1}{p_1 p_2}\}\).

It is well known that the probability generating function has a one-to-one relationship with the probability mass function. In order to determine the probability mass function \(f(m, n)\) of the BGD (F) as given in equation (2.2), it is required to differentiate \(\pi(t_1, t_2)\) partially with respect to \(t_1, x\) times, and with respect to \(t_2, y\) times at \((0, 0)\). Thus, equation (5.1) is validated using the fact:

\[
f(x, y) = \frac{1}{x! y!} \frac{\partial^{x+y} \pi(t_1, t_2)}{\partial t_1^x \partial t_2^y}|_{t_1=0, t_2=0}.
\]

Conclusively, BGD (F) given in equation (2.2) can be characterized by equation (5.1).

The equation (5.1) can be verified by expressing \(\pi(t_1, t_2) = A(t_1, t_2) + B(t_1, t_2) + C(t_1, t_2)\), where \(A, B,\) and \(C\) are the first three terms on the right hand side of equation (5.1), respectively. Then

\[
\frac{1}{x! y!} \frac{\partial^{x+y} \pi(t_1, t_2)}{\partial t_1^x \partial t_2^y}|_{t_1=0, t_2=0} = \frac{1}{x! y!} \frac{\partial^{x+y} A}{\partial t_1^x \partial t_2^y}|_{t_1=0, t_2=0} + \frac{1}{x! y!} \frac{\partial^{x+y} B}{\partial t_1^x \partial t_2^y}|_{t_1=0, t_2=0} + \frac{1}{x! y!} \frac{\partial^{x+y} C}{\partial t_1^x \partial t_2^y}|_{t_1=0, t_2=0}.
\]
The expressions $A$, $B$, and $C$ can be written as geometric power series. Also, using the fact that the derivatives of the power series can be obtained by term-by-term differentiation. The following expressions can be derived. In the domain $|t_2p_4| < 1$ and $|t_1t_2p_1p_2| < 1$, $A$ can be rewritten as

$$A = \frac{q_1q_4}{p_1p_4} \sum_{m=1}^{\infty} \sum_{j=1}^{\infty} (p_1 p_2)^j p_4^m t_1^j t_2^m.$$

Differentiate $A$ partially with respect to $t_1$, $x$ times, and evaluate it at $t_1 = 0$,

$$\left. \frac{\partial^x A}{\partial t_1^x} \right|_{t_1=0} = \frac{q_1q_4}{p_1p_4} (p_1 p_2)^x x! \sum_{m=1}^{\infty} p_4^m t_2^m.$$

This partial derivative is again differentiated partially with respect to $t_2$, $y$ times and evaluated at $t_1 = 0$, $t_2 = 0$,

$$\left. \frac{\partial^{x+y} A}{\partial t_1^x \partial t_2^y} \right|_{t_1=0, t_2=0} = \begin{cases} \frac{q_1q_4}{p_1p_4} \left[ \frac{p_1 p_2}{p_4} \right]^x p_4^y x! \cdot y!, & \text{if } y > x, \\ 0, & \text{else}. \end{cases}$$

Thus,

$$\frac{1}{x! \cdot y!} \left. \frac{\partial^{x+y} A}{\partial t_1^x \partial t_2^y} \right|_{t_1=0, t_2=0} = \begin{cases} \frac{q_1q_4}{p_1p_4} \left[ \frac{p_1 p_2}{p_4} \right]^x p_4^y, & \text{if } y > x, \\ 0, & \text{else}. \end{cases}$$

(5.2)

Analogously, $B$ can be derived for $|t_1 p_3| < 1$ and $|t_1 t_2 p_1| < 1$.

$$\frac{1}{x! \cdot y!} \left. \frac{\partial^{x+y} B}{\partial t_1^x \partial t_2^y} \right|_{t_1=0, t_2=0} = \begin{cases} \frac{q_2q_3}{p_2p_3} \left[ \frac{p_1 p_2}{p_3} \right]^y p_3^x, & \text{if } y < x, \\ 0, & \text{else}. \end{cases}$$

(5.3)

Also, $C$ can be rewritten for $|t_1 t_2 p_{12}| < 1$ as:

$$C = \frac{q_1q_2q_{12}}{(1 - p_1 p_2) p_{12}} \sum_{m=1}^{\infty} (t_1 t_2 p_{12})^m.$$

Differentiate $C$ partially with respect to $t_1$, $x$ times, and evaluate it at $t_1 = 0$,

$$\left. \frac{\partial^x C}{\partial t_1^x} \right|_{t_1=0} = \frac{q_1q_2q_{12}}{(1 - p_1 p_2) p_{12}} \cdot p_1^x \cdot t_2^x : x!.$$
This partial derivative is again differentiated partially with respect to \( t_2 \), \( y \) times, and evaluated at \( t_2 = 0 \), to give

\[
\frac{\partial^{x+y} C}{\partial t_1^x \partial t_2^y} \bigg|_{t_1=0, t_2=0} = \begin{cases} 
\frac{q_1 q_2 q_{12}}{(1-p_1 p_2) p_{12}} \cdot p_{12}^{x-1} \cdot x!, & \text{if } y = x, \\
0, & \text{elsewhere.}
\end{cases}
\]

Thus,

\[
\frac{1}{x!} \frac{\partial^{x+y} C}{\partial t_1^x \partial t_2^y} \bigg|_{t_1=0, t_2=0} = \begin{cases} 
\frac{q_1 q_2 q_{12}}{1-p_1 p_2} \cdot p_{12}^{x-1}, & \text{if } y = x, \\
0, & \text{elsewhere.}
\end{cases}
\]

(5.4)

The three equations (5.2), (5.3), and (5.4) added together yield the joint probability function of the BGD (F) as given in equation (2.2).

The p.g.f of the paired random variables \((X, Y)\) for the BGD (B&D) is given below:

\[
E[t_1^X t_2^Y] = \sum_{(x,y) \in T} t_1^x t_2^y f(x,y)
\]

\[
= \sum_{x<y} t_1^x t_2^y f(x,y) + \sum_{x>y} t_1^x t_2^y f(x,y) + \sum_{x=y} t_1^x t_2^y f(x,y)
\]

\[
= \frac{t_1 t_2 q_1 (1-p_2 p_3) (t_2 p_1 p_3)}{(1-t_1 t_2 p_1 p_3 p_2 p_3) (1-t_2 p_2 p_3)} + \frac{t_1 t_2 q_2 (1-p_1 p_3) (t_1 p_1 p_3)}{(1-t_1 t_2 p_1 p_2 p_3) (1-t_1 p_1 p_3)}
\]

\[
+ \frac{t_1 t_2 (1-p_1 p_3 - p_2 p_3 + p_1 p_2 p_3)}{(1-t_1 t_2 p_1 p_2 p_3)}.
\]

(5.5)

Here \(0 < p_i < 1, i = 1, 2, 3, p_3 = p_{12}, \mid t_1 \mid < 1/p_1, \mid t_2 \mid < 1/p_2\), and \( \mid t_1 t_2 \mid < 1/p_1 p_2 p_3\). The same p.g.f is obtained from the most natural generalization of the geometric distribution of Hawkes (1972, equation 3). Using an analogous method, we have the following result. Let \((X, Y)\) be a bivariate random vector in the support of \(Z^+ \times Z^+\) and the probability generating function of the pair of the random variables \((X, Y)\) is in the form of equation (5.5). Then \((X, Y)\) has the BGD (B&D) given in (2.1).
6. Estimation Methods

6.1. Estimation for the BGD (B&D)

Suppose we have \( n \) systems under the same condition and the lifetime of each system follows the BGD (B&D) as given in equation (2.1). Then the likelihood function under this condition is:

\[
L(x, y|p_1, p_2, p_{12}) = p_1^a p_2^b p_{12}^c q_1^d q_2^e (1 - p_1 p_{12})^f (1 - p_2 p_{12})^g,
\]

by letting \( a = \sum_{i=1}^{n} x_i - n \), \( b = \sum_{i=1}^{n} y_i - n \), \( c = \sum_{i=1}^{n} I[y_i < x_i] \), \( d = \sum_{i=1}^{n} I[x_i < y_i] \), \( e = \sum_{i=1}^{n} y_i I[x_i < y_i] + \sum_{i=1}^{n} x_i I[y_i \leq x_i] - n \), and \( g = \sum_{i=1}^{n} I[x_i = y_i] \). The maximum likelihood estimators (MLE) are obtained by solving the score equations:

\[
a = \frac{a}{p_1} - \frac{d}{1 - p_1} - \frac{c p_{12}}{1 - p_1 p_{12}} + \frac{g(-p_{12} + p_2 p_{12})}{1 - p_1 p_{12} - p_2 p_{12} + p_1 p_2 p_{12}} = 0,
\]

\[
b = \frac{b}{p_2} - \frac{c}{1 - p_2} - \frac{d p_{12}}{1 - p_1 p_{12}} + \frac{g(-p_{12} + p_1 p_{12})}{1 - p_1 p_{12} - p_2 p_{12} + p_1 p_2 p_{12}} = 0,
\]

\[
e = \frac{e}{p_{12}} - \frac{c p_1}{1 - p_1 p_{12}} - \frac{d p_2}{1 - p_2 p_{12}} + \frac{g(-p_1 - p_2 + p_1 p_2)}{1 - p_1 p_{12} - p_2 p_{12} + p_1 p_2 p_{12}} = 0.
\]

Bayes estimation for bivariate geometric distribution was used in Hare (2009). However, instead of using a bivariate Dirichlet distribution (BDD) as the prior which was applicable in Hare (2009), a uniform prior distribution is chosen here. The uniform prior distribution on \( (p_1, p_2, p_{12}) \) is considered with probability density function:

\[
f(p_1, p_2, p_{12}) = \begin{cases} 
1, & \text{if } 0 < p_1, p_2 < 1, 0 < p_{12} \leq 1, \\
0, & \text{otherwise}. 
\end{cases}
\]

(6.1.5)
The posterior distribution of \((p_1, p_2, p_{12})\) is in the form of
\[
\pi(p_1, p_2, p_{12}|x, y) = \frac{1}{C} p_1^{a} p_2^{b} p_{12}^{c} (1 - p_1 p_{12})^{d} (1 - p_2 p_{12})^{f} \cdot (1 - p_1 p_{12} - p_2 p_{12} + p_1 p_2 p_{12})^{g},
\]
\[(6.1.6)\]
where \(0 < p_1, p_2 < 1\), \(0 < p_{12} \leq 1\), and \(C\) is the constant obtained by integrating the likelihood function (6.1.1) with respect to \(p_1, p_2, p_{12}\) each from zero to one.

Using mean square error with Euclidean norm as the risk function \(E||\hat{p} - p||^2\), where vector \(\hat{p}\) is the estimator of the vector of parameters \(p\), the Bayes estimator of the unknown parameter is simply the conditional mean of the posterior distribution.
\[
p^*_i = E(p_i|x, y) = \int_0^1 p_i f(p_i|x, y) dp_i,
\]
\[(6.1.7)\]
where \(p_i = p_1, p_2, p_{12}\), and \(f(p_i|x, y)\) is the marginal posterior distribution of \(p_1, p_2, p_{12}\), respectively, i.e.,
\[
f(p_i|x, y) = \int_0^1 \int_0^1 \pi(p_1, p_2, p_{12}|x, y) dp_j dp_k,
\]
\[(6.1.8)\]
where \(i, j, k = 1, 2, 12\), and \(i \neq j \neq k\).

Dhar (1998) also used the method of moment (MOM) to estimate the parameters by fitting a discrete bivariate geometric distribution to practical data sets. To evaluate the parameters in the BGD (B&D) using this method, the moments for \(X\), \(Y\) and the product of \(X\) and \(Y\) are derived as:
\[
E(X) = \sum_{x=1}^{\infty} x P(X = x) = (1 - p_1 p_{12})^{-1},
\]
\[
E(Y) = \sum_{y=1}^{\infty} y P(Y = y) = (1 - p_2 p_{12})^{-1},
\]
\[
E(XY) = \frac{(1 - p_1 p_2 p_{12}^2)}{(1 - p_1 p_{12})(1 - p_2 p_{12})(1 - p_1 p_2 p_{12})}.
\]
Then, an equation set can be constructed as:
\[
(1 - p_1 p_{12})^{-1} = \pi,
\]
\[14\]
\[(1 - p_2p_{12})^{-1} = \overline{y},\]
\[(1 - p_1p_2p_{12}^2) = \overline{x},\]
where \(\overline{x} = \sum_{i=1}^{n} x_i/n, \overline{y} = \sum_{i=1}^{n} y_i/n\) and \(\overline{z} = \sum_{i=1}^{n} x_iy_i/n.\)

The equation set can be solved for MOM estimators for \(p_1, p_2\) and \(p_{12}\). Let’s denote this method as MOM1.

\[
\tilde{p}_1 = \frac{\overline{y}(\overline{z} - \overline{y} - y + 1)}{\overline{y}(\overline{y} - 1)} \tag{6.1.9}
\]
\[
\tilde{p}_2 = \frac{\overline{x}(\overline{z} - \overline{x} - x + 1)}{\overline{x}(\overline{x} - 1)} \tag{6.1.10}
\]
\[
\tilde{p}_{12} = \frac{\overline{x}(\overline{y} - 1)(\overline{y} - 1)}{\overline{x} \cdot \overline{y} \cdot (\overline{z} - \overline{x} - \overline{y} + 1)} \tag{6.1.11}
\]

An alternative moment considered here is \(\mathbb{E}[\min(X, Y)]\) instead of \(\mathbb{E}(XY)\), which is found to be \((1 - p_1p_2p_{12})^{-1}\). Then the alternative set of MOM estimators are

\[
\hat{p}_1 = \frac{\overline{y} - \overline{y} \cdot \overline{w}}{\overline{w} - \overline{y} \cdot \overline{w}} \tag{6.1.12}
\]
\[
\hat{p}_2 = \frac{\overline{x} - \overline{x} \cdot \overline{w}}{\overline{w} - \overline{x} \cdot \overline{w}} \tag{6.1.13}
\]
\[
\hat{p}_{12} = \frac{\overline{x} \cdot \overline{y} \cdot \overline{w} - \overline{w} \cdot \overline{x} \cdot \overline{y} + \overline{x} \cdot \overline{y} \cdot \overline{w}}{\overline{x} \cdot \overline{y} \cdot \overline{w} - \overline{x} \cdot \overline{y} \cdot \overline{w}} \tag{6.1.14}
\]
where \(\overline{x} = \sum_{i=1}^{n} x_i/n, \overline{y} = \sum_{i=1}^{n} y_i/n\) and \(\overline{w} = \sum_{i=1}^{n} \min(x_i, y_i)\). Let’s denote this by MOM2.

6.2. Estimation for the BGD (F)

The likelihood function for the BGD (F) under the assumption \(p_{12} = p_1p_2\) in the region \(\{x, y = 1, 2, \cdots\}\) can be written as:

\[
L(x, y|p_1, p_2, p_3, p_4) = p_1^{d + c - 1} q_1^{b + c - 1} q_2^{a + c - 1} q_3^{d - a} q_4^{h - b} q_5^{q} q_6^{p} q_7^{n} q_8^{m}, \tag{6.2.1}
\]
where \(a = \sum_{i=1}^{n} I\{x_i > y_i\}, b = \sum_{i=1}^{n} I\{x_i < y_i\}, c = \sum_{i=1}^{n} I\{x_i = y_i\}, d = \sum_{i=1}^{n} y_i I\{x_i > y_i\}, e = \sum_{i=1}^{n} x_i I\{x_i < y_i\}, \phi = \sum_{i=1}^{n} x_i I\{x_i = y_i\}, g = \sum_{i=1}^{n} x_i I\{x_i > y_i\}, \text{ and } h = \sum_{i=1}^{n} y_i I\{x_i < y_i\}.\)
Under the assumption \( p_{12} = p_1 p_2 \), the same process was executed to derive the MLE of \( p_1, p_2, p_3 \) and \( p_4 \).

\[
\hat{p}_1 = \frac{d + e + \phi - b - c}{d + e + \phi} \quad (6.2.2)
\]

\[
\hat{p}_2 = \frac{d + e + \phi - a - c}{d + e + \phi} \quad (6.2.3)
\]

\[
\hat{p}_3 = \frac{g - a - d}{g - d} \quad (6.2.4)
\]

\[
\hat{p}_4 = \frac{h - b - e}{h - e} \quad (6.2.5)
\]

Likewise, the Bayes estimation was applied on the BGD (F) using the uniform prior distribution on \((p_1, p_2, p_3, p_4)\) with the additional assumption \( p_{12} = p_1 p_2 \).

\[
f(p_1, p_2, p_3, p_4) = \begin{cases} 
18/11, & \text{if } 0 < p_i < 1, p_1 p_2 < p_3, p_1 p_2 < p_4, \\
0, & \text{otherwise.} 
\end{cases} \quad (6.2.6)
\]

The posterior distribution of \((p_1, p_2, p_3, p_4)\) is in the form of

\[
\pi(p_1, p_2, p_3, p_4 \mid x, y) = \frac{1}{C} p_1^{d+e+f-b-c} p_2^{d+e+f-a-c} p_3^{g-d-a} p_4^{h-e-b} q_1^a q_2^b q_3^c q_4^d,
\]

(6.2.7)

where \( 0 < p_i < 1, i = 1, 2, 3, 4, p_1 p_2 < p_3, p_1 p_2 < p_4 \), and \( C \) is the constant obtained by integrating the Likelihood-function (6.2.1) with respect to \( p_1, p_2, p_3, p_4 \) each from zero to one. Use the same risk function, we obtain the Bayes estimators as follows:

\[
p_1^* = \frac{d + e + \phi - b - c + 1}{d + e + \phi + 2} \quad (6.2.8)
\]

\[
p_2^* = \frac{d + e + \phi - a - c + 1}{d + e + \phi + 2} \quad (6.2.9)
\]

\[
p_3^* = \frac{g - d - a + 1}{g - d + 2} \quad (6.2.10)
\]

\[
p_4^* = \frac{h - b - e + 1}{h - e + 2} \quad (6.2.11)
\]
An alternative method of moments was developed using \( p_{12} = p_1 p_2 \), moments \( EX, EY, EX^2 \) and \( EZ \) were considered, where \( Z = \min(X,Y) \).

\[
EX = \frac{p_3 - p_1}{(p_3 - p_1 p_2)(1 - p_1 p_2)} + \frac{p_1 q_2}{(p_4 - p_1 p_2)(1 - p_1)} \tag{6.2.12}
\]

\[
EY = \frac{p_4 - p_2}{(p_4 - p_1 p_2)(1 - p_1 p_2)} + \frac{p_2 q_1}{(p_4 - p_1 p_2)(1 - p_4)} \tag{6.2.13}
\]

\[
EX^2 = \frac{(p_3 - p_1)(1 + p_1 p_2)}{(p_3 - p_1 p_2)(1 - p_1 p_2)^2} + \frac{p_1 q_2(1 + p_4)}{(p_3 - p_1 p_2)(1 - p_3)^2} \tag{6.2.14}
\]

\[
EZ = (1 - p_1 p_2)^{-1} \tag{6.2.15}
\]

Equating the four equations \( EX = \bar{x} \), \( EY = \bar{y} \), \( EX^2 = \bar{x}_2 \) = \( \frac{\sum_{i=1}^{n} x_i^2}{n} \) and \( EZ = \bar{z} = \frac{\sum_{i=1}^{n} \min(x_i, y_i)}{n} \), the method of moments estimators are generated:

\[
p_1 = \frac{-2\bar{x}\bar{z}^2 - 2\bar{z}^2 + \bar{x}\bar{z} + \bar{x} - 2\bar{z}^2 - 2\bar{x}_2\bar{z} + \bar{x}_2}{\bar{x}_2\bar{z} + \bar{x}_2\bar{z} - 2\bar{x}\bar{z}^2} \tag{6.2.16}
\]

\[
p_2 = \frac{\bar{x}_2 + \bar{x} - \bar{x}_2\bar{z} - 3\bar{x}\bar{z} + 2\bar{x}\bar{z}^2}{2\bar{x}\bar{z}^2 - 2\bar{z}^2 + \bar{x}\bar{z} + \bar{x} - 2\bar{z}^2 + \bar{x}_2\bar{z} + \bar{x}_2\bar{z} + \bar{x}_2\bar{z}} \tag{6.2.17}
\]

\[
p_3 = \frac{\bar{x}_2 - \bar{x} + 2\bar{z} - 2\bar{x}\bar{z}}{\bar{x}_2 + \bar{x} - 2\bar{x}\bar{z}} \tag{6.2.18}
\]

\[
p_4 = (\bar{x}_2\bar{y} - \bar{x} - \bar{x}_2 + \bar{x}\bar{y} - 2\bar{x}\bar{z} - 2\bar{x}^2\bar{y} + \bar{x}_2\bar{z}^2 + \bar{x}\bar{z}^2 - 2\bar{x}\bar{z}^3 - 2\bar{y}\bar{z}^2 + 2\bar{x}\bar{z}^2 + 2\bar{x}\bar{y}\bar{z}^2 - \bar{x}_2\bar{y}\bar{z} + \bar{x}\bar{y}\bar{z})/(\bar{x}_2\bar{y} + \bar{x}\bar{y} - \bar{x}_2\bar{z} - 2\bar{z}^2 - \bar{x}_2\bar{z}^2 - \bar{x}\bar{z}^2 + 2\bar{x}\bar{z}^2 - 2\bar{x}\bar{z}^3 - 2\bar{y}\bar{z}^2 + 2\bar{x}\bar{z}^2 - \bar{x}_2\bar{z}^2 + \bar{x}\bar{z}^2)
\]

An alternative method of moments was developed using \( EX, EY, EY^2 \) and \( EZ \), where

\[
EY^2 = \frac{(p_4 - p_2)(1 + p_1 p_2)}{(p_4 - p_1 p_2)(1 - p_1 p_2)^2} + \frac{p_2 q_1(1 + p_4)}{(p_4 - p_1 p_2)(1 - p_4)^2}. \tag{6.2.20}
\]

Then the alternative set of method of moments estimators are:

\[
p_1 = \frac{\bar{y}_2 + \bar{y} - \bar{y}_2 \bar{z} - 3\bar{y}\bar{z} + 2\bar{y}\bar{z}^2}{-2\bar{y}\bar{z}^2 + 2\bar{y}^2 + \bar{y}\bar{z} + \bar{y} - 2\bar{z}^2 - \bar{y}_2\bar{z} + \bar{y}_2} \tag{6.2.21}
\]
\[ p_2 = \frac{-2\bar{y}^2 + 2\bar{y}\bar{z}^2 + \bar{y}\bar{z} + \bar{y} - 2\bar{z}^2 - \bar{y}_2\bar{z} + \bar{y}_2}{\bar{y}_2\bar{z} + \bar{y}\bar{z} - 2\bar{y}\bar{z}^2} \quad (6.2.22) \]

\[ p_3 = (\bar{x}\bar{y}_2 - \bar{y} - \bar{y}_2 + \bar{x}\bar{y} - 2\bar{y}\bar{z} - 2\bar{x}\bar{z}^2 - 2\bar{x}\bar{z}\bar{y}_2 + \bar{y}_2\bar{z}^2 + \bar{y}\bar{z}^2 - 2\bar{y}\bar{z}^3 \]
\[ + 2\bar{y}^2 + 2\bar{z}^2 + 2\bar{x}\bar{y}\bar{z}_2 - \bar{x}\bar{y}_2\bar{z} + \bar{x}\bar{y}\bar{z})/(\bar{x}\bar{y}_2 + \bar{x}\bar{y} - \bar{y}_2\bar{z} - \bar{y}\bar{z} - 2\bar{x}\bar{y}^2 - 2\bar{x}\bar{z}^2 \]
\[ + \bar{y}_2\bar{z}^2 - \bar{y}\bar{z}^2 + 2\bar{y}^2\bar{z} - 2\bar{y}\bar{z}^3 + 2\bar{z}^3 + 2\bar{x}\bar{y}\bar{z}^2 - \bar{x}\bar{y}_2\bar{z} + \bar{x}\bar{y}\bar{z}) \quad (6.2.23) \]

\[ p_4 = \frac{\bar{y}_2 - \bar{y} + 2\bar{z} - 2\bar{y}\bar{z}}{\bar{y}_2 + \bar{y} - 2\bar{y}\bar{z}} \quad (6.2.24) \]

7. Data Analysis with Modeling

7.1. A Real Data Example

The BGD (B&D) is fitted to a real data set from Dhar (2003) for demonstration purposes. This data set consists of scores given by seven judges from seven different countries in the form of a video recording. The score given by each judge is a discrete random variable taking positive integer values and also the midpoints of consecutive integers between zero and ten. The data given in Table 1 displays the scores which have been converted into integer valued random variable. The score corresponding to the dive of Michael Murphy of Australia (item number 3) was not displayed by NBC sports during the recording.

It is reasonable to assume that the maximum scores \((X, Y)\) follow the BGD (B&D). Thus, one can calculate the estimators of \(p_1\), \(p_2\), and \(p_{12}\) using the estimation methods discussed in Section 6.1. The results are summarized in Table 2. These results were obtained using Mathematica. After comparison, we see that the estimators obtained through maximum likelihood estimation, Bayes method, and MOM2 are close, while the MOM1 estimators are slightly off by about 0.06. Furthermore, these estimates helped us to identify the judges from which particular region tend to give higher scores than the other region. By substituting these estimates into \(P(X > Y) = p_1p_{12}q_2/(1 - p_1p_2p_{12})\), for \(x > y\),
Table 1: Scores taken from a video recorded during the summer of 1995 relayed by NBC sports TV, IX World Cup diving competition, Atlanta, Georgia.

<table>
<thead>
<tr>
<th>Item</th>
<th>Diver</th>
<th>X: max score, Asian &amp; Caucasus</th>
<th>Y: max score, West</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Sun Shuwei, China</td>
<td>19</td>
<td>19</td>
</tr>
<tr>
<td>2</td>
<td>David Pichler, USA</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>3</td>
<td>Jan Hempel, Germany</td>
<td>13</td>
<td>14</td>
</tr>
<tr>
<td>4</td>
<td>Roman Volodkuv, Ukraine</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>5</td>
<td>Sergei Kudrevich, Belarus</td>
<td>14</td>
<td>14</td>
</tr>
<tr>
<td>6</td>
<td>Patrick Jeffrey, USA</td>
<td>15</td>
<td>14</td>
</tr>
<tr>
<td>7</td>
<td>Valdimir Timoshinin, Russia</td>
<td>13</td>
<td>16</td>
</tr>
<tr>
<td>8</td>
<td>Dimitry Sautin, Russia</td>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td>9</td>
<td>Xiao Hailiang, China</td>
<td>13</td>
<td>13</td>
</tr>
<tr>
<td>10</td>
<td>Sun Shuwei, China</td>
<td>15</td>
<td>16</td>
</tr>
<tr>
<td>11</td>
<td>David Pichler, USA</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>12</td>
<td>Jan Hempel, Germany</td>
<td>17</td>
<td>18</td>
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<td>13</td>
<td>Roman Volodkuv, Ukraine</td>
<td>16</td>
<td>16</td>
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<td>14</td>
<td>Sergei Kudrevich, Belarus</td>
<td>12</td>
<td>13</td>
</tr>
<tr>
<td>15</td>
<td>Patrick Jeffrey, USA</td>
<td>14</td>
<td>14</td>
</tr>
<tr>
<td>16</td>
<td>Valdimir Timoshinin, Russia</td>
<td>12</td>
<td>13</td>
</tr>
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<td>17</td>
<td>Dimitry Sautin, Russia</td>
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<td>18</td>
</tr>
<tr>
<td>18</td>
<td>Xiao Hailiang, China</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>19</td>
<td>Sun Shuwei, China</td>
<td>18</td>
<td>18</td>
</tr>
</tbody>
</table>

Table 1: Scores taken from a video recorded during the summer of 1995 relayed by NBC sports TV, IX World Cup diving competition, Atlanta, Georgia.

The estimated probabilities are shown in Table 3. Thus, it can be seen that the probability $\hat{P}(X < Y)$ is higher than the probability $\hat{P}(X > Y)$ for each column (estimation method), which shows that judges from West tend to give higher scores than judges from Asia and Caucasus. This conclusion is consistent with the empirical estimates $\hat{P}(X > Y) = 2/19 = 0.1053$ and $\hat{P}(X < Y) = 9/19 = 0.4737$.

7.2. Simulation Results

A Monte Carlo simulation study was performed by generating 500 samples from the BGD (B&D) of sizes $n = 20$, $n = 50$, $n = 100$, using the algorithm
Table 2: Estimated parameters by fitting the BGD (B&D) model to the data set shown in Table 1.

<table>
<thead>
<tr>
<th>Estimator</th>
<th>MLE</th>
<th>Bayes</th>
<th>MOM1</th>
<th>MOM2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{p}_1$</td>
<td>0.961605</td>
<td>0.957713</td>
<td>0.9343997</td>
<td>0.9968594</td>
</tr>
<tr>
<td>$\hat{p}_2$</td>
<td>0.985481</td>
<td>0.979849</td>
<td>0.9365146</td>
<td>0.9991156</td>
</tr>
<tr>
<td>$\hat{p}_{12}$</td>
<td>0.940199</td>
<td>0.939019</td>
<td>0.9934730</td>
<td>0.9312265</td>
</tr>
</tbody>
</table>

Table 3: Comparisons of probabilities reflecting which group tends to give higher scores for data set given in Table 1.

<table>
<thead>
<tr>
<th>Method</th>
<th>MLE</th>
<th>Bayes</th>
<th>MOM1</th>
<th>MOM2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X &gt; Y)$</td>
<td>0.1204</td>
<td>0.1525</td>
<td>0.4511</td>
<td>0.0113</td>
</tr>
<tr>
<td>$P(X &lt; Y)$</td>
<td>0.3263</td>
<td>0.3275</td>
<td>0.4672</td>
<td>0.0403</td>
</tr>
</tbody>
</table>

patterned after the study of Sunil Dhar (1998), which is, using the marginal distribution of $Y$ and the conditional distribution of $X$ given $Y$. Within each sample, the MOM1 estimators and MOM2 estimators of $p_1$, $p_2$ and $p_{12}$ were calculated. Then for 500 simulations, the estimated bias and variances of $\hat{p}$ for the samples from BGD (B&D) using methods of moments when $n=20$, 50, 100 are given in Table 4. To measure and compare the magnitudes of the estimated bias vectors and estimated variance-covariance matrices of $\hat{p}$, the Euclidean norm $\|A\| = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}$ is used and the results are reported in Table 5.

The results in Table 5 show that the Euclidean norms of the estimated bias vectors using MOM2 are less than those computed using MOM1 with respect to different sample sizes. This is also true for the norms of the estimated variance-covariance matrices. Hence, MOM2 (using score equation corresponding to $E[min(X,Y)]$ instead of $E[XY]$) provides more accurate estimations for the parameters than MOM1 (using score equation corresponding to $E[XY]$ instead of $E[min(X,Y)]$). Also, as should be the case, the magnitude of the estimated bias and that of the estimated variance-covariance matrix decrease as sample
Table 4: Estimated bias and variances of $\hat{p}$ for the samples from BGD (B&D) using methods of moments when $n=20$, 50, 100.

<table>
<thead>
<tr>
<th>MOM1</th>
<th>MOM2</th>
</tr>
</thead>
<tbody>
<tr>
<td>True parameters</td>
<td>bias</td>
</tr>
<tr>
<td>$p_1 = 0.95$</td>
<td>-0.011958</td>
</tr>
<tr>
<td>$p_2 = 0.96$</td>
<td>-0.0092005</td>
</tr>
<tr>
<td>$p_{12} = 0.97$</td>
<td>0.0067059</td>
</tr>
<tr>
<td>$p_1 = 0.95$</td>
<td>-0.0035407</td>
</tr>
<tr>
<td>$p_2 = 0.96$</td>
<td>-0.0039660</td>
</tr>
<tr>
<td>$p_{12} = 0.97$</td>
<td>0.0026561</td>
</tr>
<tr>
<td>$p_1 = 0.95$</td>
<td>-0.0020813</td>
</tr>
<tr>
<td>$p_2 = 0.96$</td>
<td>-0.0021517</td>
</tr>
<tr>
<td>$p_{12} = 0.97$</td>
<td>0.0016712</td>
</tr>
</tbody>
</table>

Table 5: Summary of Euclidean norms of the estimated bias vectors (ENEB) and that of the estimated variance-covariance matrices (ENVC).

<table>
<thead>
<tr>
<th>MOM1</th>
<th>MOM2</th>
</tr>
</thead>
<tbody>
<tr>
<td>ENEB</td>
<td>ENVC</td>
</tr>
<tr>
<td>$n=20$</td>
<td>0.016510970</td>
</tr>
<tr>
<td>$n=50$</td>
<td>0.005943112</td>
</tr>
<tr>
<td>$n=100$</td>
<td>0.003428488</td>
</tr>
</tbody>
</table>

A chi-square goodness-of-fit test was performed to assess the performances of different estimation methods using one random sample from BGD (B&D) as given in Table 6. Let $H_0$: the data follows BGD (B&D) distribution; $H_1$: negation $H_0$. The maximum likelihood estimators, Bayes estimators and two method of moment estimators are computed using Mathematica and the results are shown in Table 7. The degree of freedom of a chi-square goodness-of-fit test is one less than the number of classes under a given multinomial distribution. Thus, considering there are three parameters in the BGD (B&D) distribution, it seems plausible to divide the region $Z^+ \times Z^+$ into seven cells:

1. $0 < x \leq 5$, $0 < y \leq 5$,
Table 6: A randomly simulated sample from BGD (B&D).

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>X</th>
<th>Y</th>
<th>X</th>
<th>Y</th>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>41</td>
<td>38</td>
<td>14</td>
<td>6</td>
<td>2</td>
<td>24</td>
<td>9</td>
<td>3</td>
</tr>
<tr>
<td>12</td>
<td>12</td>
<td>1</td>
<td>10</td>
<td>4</td>
<td>2</td>
<td>65</td>
<td>3</td>
</tr>
<tr>
<td>14</td>
<td>21</td>
<td>14</td>
<td>14</td>
<td>50</td>
<td>10</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>11</td>
<td>42</td>
<td>12</td>
<td>11</td>
<td>11</td>
<td>23</td>
<td>81</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>7</td>
<td>7</td>
<td>10</td>
<td>19</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 7: Estimated parameters for the BGD (B&D) for the sample data given in Table 6.

<table>
<thead>
<tr>
<th>Method</th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$p_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MLE</td>
<td>0.9647230</td>
<td>0.9606100</td>
<td>0.9746760</td>
</tr>
<tr>
<td>Bayes</td>
<td>0.9616000</td>
<td>0.9573000</td>
<td>0.9728000</td>
</tr>
<tr>
<td>MOM1</td>
<td>0.9595252</td>
<td>0.9535610</td>
<td>0.9790196</td>
</tr>
<tr>
<td>MOM2</td>
<td>0.9514900</td>
<td>0.9455758</td>
<td>0.9872872</td>
</tr>
</tbody>
</table>

2. $0 < x \leq 5$, $5 < y \leq 15$,
3. $5 < x \leq 15$, $0 < y \leq 5$,
4. $5 < x \leq 20$, $5 < y \leq 15$,
5. $0 < x \leq 20$, $15 < y \leq 25$,
6. $20 < x \leq 65$, $0 < y \leq 15$,
7. otherwise.

The chi-square is computed based on $\chi^2 = \sum_{i=1}^{7} \frac{(o_i - e_i)^2}{e_i}$, where $o_i$ is the number of the observations in region $i$ and $e_i$ is the expected observations in the region $i$. The maximum likelihood estimator, Bayes estimators, and the method of moments estimators shown in Table 7 are assumed to be true values of the $p$'s, then the corresponding $\chi^2$ and $p-values$ are computed and compared with degree of freedom 7-1=6 and 7-1-3=3, respectively. The results are summarized in Table 8.

From Table 8, it is observed that the chi-square goodness-of-fit statistics calculated using Bayes estimation, methods of moments estimation are close.
The result obtained using maximum likelihood estimators is slightly off by about 1.00 in the computed $\chi^2$ test statistic value. All the results are consistent with each other since all the p-values are greater than the alpha value of 0.05. The best fit based on largest $p-value$ here is obtained from MOM2 for both degrees of freedom 6 and 3.

8. Conclusion

In this article, characterizations through conditional probabilities, conditional failure rates and probability generating functions are considered for several bivariate geometric models. Also, estimation methods for these bivariate geometric models are discussed. These methods are applied and compared on some data sets.

Acknowledgements

This research is part of the author’s PhD dissertation. The author is most grateful to the advisor for helpful suggestions and comments.

References


