Stokes-Darcy Boundary Integral Solutions Using Preconditioners

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Abstract

In a system where a free fluid flow is coupled to flow in a porous medium, different PDEs are solved simultaneously in two subdomains. We consider steady Stokes equations in the free region, coupled across a fixed interface to Darcy equations in the porous substrate. In this paper, the numerical solution is obtained using the boundary integral formulation with regularized Green’s function. Higher accuracy is achieved by applying a correction process, which also results in the improvement of the condition number of the linear system. In this work, an appropriate preconditioner based on the singular part of corrections is introduced to improve the convergence of a Krylov subspace method applied to solve the integral formulation.

Key words: Preconditioner, Boundary integral method, Stokes flow, Darcy equations

1. Introduction

We are interested in problems where a domain filled with fluid is separated by an interface from a porous medium filled with the same fluid. Systems like this have many important industrial applications ([4, 21, 23]). We model the free fluid flow by the incompressible Stokes equations, and the flow in the porous medium by Darcy equations. The partial differential equations then have different orders in two subdomains and the coupling conditions at the interface have been investigated in various works ([5, 24, 13, 16, 25]). Among these conditions are the continuity of normal components of velocity and normal stress, and the slip condition for the tangential velocity proposed by Beavers and Joseph [5]. Various numerical methods based on finite elements have been proposed to solve this kind of problem ([6, 10, 11, 13, 14, 18, 19, 20]). In this work, a boundary integral formulation obtained using the free-space Green’s function is used to represent solutions of both equations. The main advantage of this approach is reduction in the dimensionality of the problem. The boundary is represented as a distribution of singularities with strengths to be determined from the boundary and interface conditions. In addition, we apply a regularization-correction procedure to eliminate the singularities in the kernels and enhance the accuracy of the final solution. The resulting linear system is solved by means of a Krylov subspace method (GMRES). This paper is devoted to the improvement of convergence of this method by using an appropriate preconditioner.

To eliminate the singularities that appear in the integral formulation, we regularize the kernels by approximating the delta function with a smooth radially symmetric function. Therefore, a standard quadrature can

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be used to discretize this formulation. Numerical results have demonstrated that this technique results in low accuracy of the solution. As it was done in [27], we use the correction method to reduce the dependence of the numerical error on regularization. These corrections are based on the error due to regularization computed approximately near the singularity point. Precisely, it is defined as the difference between the original integral equations (singular kernels) and the ones with regularized kernels. The resulting integral representations for the fluid quantities are substituted into the boundary and interface conditions. The final integral equation system is then solved using a Krylov Subspace method (GMRES) for the unknown force densities on the boundary. Initial observations showed that correction terms designed primarily to improve the accuracy of solution also significantly reduce the condition numbers of the matrix, therefore providing faster convergence rates using a Krylov subspace method. We present some numerical results illustrating how the correction technique reduces the iteration number of GMRES.

The subject of present investigation concerns the improvement of the Krylov subspace method applied to the coupled Stokes-Darcy integral formulation. Indeed, the resulting linear system is non-symmetric and composed of dense blocks. Although the correction procedure considerably reduces the condition number, this linear system remains ill-conditioned, and its eigenvalues are largely clustered near zero. In addition to these disadvantages, the condition number increases with the number of grid points and convergence depends on the physical parameters of the problem, like fluid viscosity and permeability of the porous medium. The concept of the preconditioner introduced in this paper is based on the observation described above regarding the reduction of condition numbers when corrections are added. The technique consists of first introducing a matrix with a similar block structure as the Stokes-Darcy system. Then, for each block in this initial system we compute, using the singular part given in the corrections, the corresponding block in the new matrix. Its inverse finally defines our preconditioner, which can be solved efficiently since it has a small bandwidth. Indeed, the blocks of this matrix are either diagonal or have only a few non-zero off-diagonal elements. This strategy also suggests the possibility of using this type of preconditioner for problems where the solution can be expressed with similar Green’s functions. We first present numerical results validating our approach for a Stokes flow problem. Then, we show numerical simulations corresponding to the Stokes-Darcy formulation, where significant improvement of GMRES using this preconditioner is demonstrated.

The paper is organized as follows. After the presentation of the model problem in Section 2, we describe the integral formulation and the regularization-correction technique in Sections 3. In Section 4, we introduce the preconditioner, its derivation and final structure. Finally, several numerical experiments validating our approach are presented in Section 5. Section 6 is reserved for conclusions.

2. Problem formulation

In situations where inertia has a negligible effect, the flow is modeled by the linear Stokes equations. In the porous substrate, we use Darcy’s law which is a linear relationship between the driving pressure gradient and the filtration velocity. In two dimensions, we denote the Stokes domain by \( \Omega_S \) and the Darcy one by \( \Omega_D \) (Figure 1). We assume bounded domains with smooth boundaries denoted by \( \partial \Omega_S = \Gamma_S \cup \Sigma \) and \( \partial \Omega_D = \Gamma_D \cup \Sigma \), where \( \Sigma \) is the interface. The steady state equations are therefore given as follows:
In $\Omega_S$:
\[
\begin{cases}
-\nabla p_S + \mu \Delta u_S = 0, \\
\nabla \cdot u_S = 0,
\end{cases}
\]
where $\mu$ is the fluid dynamic viscosity, $p_S$ is the pressure and $u_S = (u_S, v_S)$ is the velocity vector, and

In $\Omega_D$:
\[
\begin{cases}
-\nabla p_D - \mu K^{-1} u_D = 0, \\
\nabla \cdot u_D = 0,
\end{cases}
\]
where $u_D = (u_D, v_D)$ and $p_D$ are respectively the (averaged) fluid velocity and the hydrostatic pressure and $K$ is the permeability of the porous medium. We will assume that the medium is isotropic, $K = kI$. Finally, we assume the following boundary conditions:

\[ u_S = u_0 \quad \text{on } \Gamma_S, \]
\[ p_D = p_0 \quad \text{on } \Gamma_D, \]

and on $\Sigma$

\[ u_S \cdot n_S = -u_D \cdot n_D, \]
\[ p_S - 2\mu n_S \cdot D_S \cdot n_S = p_D, \]
\[ \frac{\partial u_S^{(r)}}{\partial n_S} = \frac{\gamma}{\sqrt{k}} u_S^{(r)}. \]

Here $\gamma$ is a dimensionless slip coefficient that depends on the geometry of the porous medium, $D_S = \frac{1}{2}[\nabla u_S + (\nabla u_S)^T]$ is the Stokes deformation tensor, $u^{(r)}$ is the tangential velocity, and $n_r$ is the unit normal vector that points out of the region $r$, so that $n_D = -n_S$ on $\Sigma$. Equations (5) and (6) represent continuity of normal component of velocity and normal component of normal stress, and (7) is a slip condition of Beavers-Joseph-Saffman ([5, 24]). For detailed discussion of the coupling conditions, refer to [5, 24, 15, 16, 25].

3. Boundary integral formulation and regularization

The linearity of the problems makes the boundary integral method a natural approach to express solutions of various boundary value problems ([22]). The main advantage of this technique is reduction of the dimension of the problem. To derive the integral formulation, the governing equations (1)-(2) are solved where the boundary is represented as a distribution of singular force. Introducing a density $f$ of the force distributed along the Stokes boundary and density $g$ of the force along the Darcy boundary, the equations can be written as

\[ -\nabla p_S + \mu \Delta u_S + \int_{\partial \Omega_S} \delta(x - x(s))f(s)ds = 0, \quad \nabla \cdot u_S = 0, \]
\[ -\nabla p_D - \frac{\mu}{k} u_D + \int_{\partial \Omega_D} \delta(x - x(s))g(s)ds = 0, \quad \nabla \cdot u_D = 0, \]

where $s$ is a boundary parametrization. Using the incompressibility conditions, both the Stokes and Darcy equations can be reduced to Laplace’s equations for pressures. Once calculated, $p_S$ and $p_D$ are substituted into (8) and (9) to compute the velocities $u_S$ and $u_D$. The resulting expressions can be written as ([8, 27])

\[ p_S(x) = \int_{\partial \Omega_S} \nabla G(x - x(s)) \cdot f(s)ds, \]
\[ \mu u_S(x) = \int_{\partial \Omega_S} \{-G(x - x(s))f(s) + \nabla[\nabla B(x - x(s)) \cdot f(s)] \} ds, \]
for Stokes and

\[ p_D(x) = \int_{\partial\Omega_D} \nabla G(x - x(s)) \cdot g(s) ds, \quad (12) \]

\[ \mu u_D(x) = -k \int_{\partial\Omega_D} \nabla [\nabla G(x - x(s))] \cdot g(s) ds, \quad (13) \]

for Darcy quantities, where \( G \) and \( B \) are solutions of \( \Delta G(x) = \delta(x) \) and \( \Delta B(x) = G(x) \) in free space defined by

\[ G(x) = \frac{1}{2\pi} \ln |x|, \quad B(x) = \frac{|x|^2}{8\pi} (\ln |x| - 1), \quad x \in \mathbb{R}^2. \]

The integrands in (10)-(13) have singularities on the boundary. One approach to eliminate those singularities is to use a regularization procedure. Then a standard quadrature can be applied to discretize the integrals to numerical accuracy. However, usual integration is not accurate when the evaluation point is near the boundary, and the integrals become nearly singular. This problem was addressed in the context of computing Stokes flow in a closed domain with higher accuracy in [8], [3].

Moreover, the numerical treatment of these integrals for the nearly singular case was addressed in [2, 8]. A method for computing the single and double layer potentials near the boundary was developed in [2]. It involves first regularizing the functions and discretizing the integrals with a standard quadrature, and then adding corrections for the regularization and discretization errors to achieve higher accuracy.

3.1. Regularization and correction

The main idea of the regularization procedure is to compute the regularized Green’s function from \( \Delta G_\delta(x) = \xi_\delta(x) \), where \( \xi_\delta \) is a smooth function that approximates the delta function and satisfies \( \int \xi_\delta(x) dx = \int \delta(x) dx = 1 \). We use the smoothing function from [27]:

\[ \xi_\delta(x) = \frac{2\delta^4}{\pi(|x|^2 + \delta^2)^3}, \quad (17) \]
where the parameter $\delta$ is chosen according to accuracy constraints. The corresponding regularized Green’s function is

$$G_\delta(x) = \frac{1}{4\pi} \left[ \ln(|x|^2 + \delta^2) - \frac{\delta^2}{|x|^2 + \delta^2} \right],$$

so that $G_\delta \to G$ as $\delta \to 0$. This regularization method for Stokes flow was proposed in [8] and further investigated in applications in [9, 12, 1]. Figure 2 shows the Green’s functions for different values of regularization parameter $\delta$, where $\delta = 0$ is the singular case. As can be seen from the figure, the largest approximation is made near the singularity $|x| = 0$. Away from the singularity we have

$$G_\delta - G = O(\delta^4 |x|^4) \quad \text{for} \quad \delta \ll |x|.$$  \hspace{1cm} (18)

This observation suggests that the approximation can be improved by considering the error near the singularity. The regularization introduces an error in computing (14) of the form

$$ (w_e - w_\delta)(x) = \int_{\partial\Omega} (G - G_\delta)(x - x(s))\phi(s)ds + \int_{\partial\Omega} \nabla(G - G_\delta)(x - x(s)) \cdot n(s)\psi(s)ds. $$  \hspace{1cm} (19)

The leading error term can be identified by expanding the smooth functions $\phi(s)$ and $\psi(s)$ in Taylor series around the point $x(s^*)$ that is closest to $x$:

$$\phi(s) = \phi(s^*) + \phi'(s^*)(s - s^*) + O((s - s^*)^2),$$

$$\psi(s) = \psi(s^*) + \psi'(s^*)(s - s^*) + O((s - s^*)^2),$$

for $s$ near $s^*$, and keeping only the first term in the expansions,

$$ (w_e - w_\delta)^*(x) = \int_{\partial\Omega} (G - G_\delta)(x - x(s))ds \cdot \phi(s^*) + \int_{\partial\Omega} \nabla(G - G_\delta)(x - x(s)) \cdot n(s)ds \cdot \psi(s^*). $$  \hspace{1cm} (20)

These terms are then added as corrections to increase the accuracy, so the fluid quantities are computed as follows:

$$ w(x) = w_\delta(x) + (w_e - w_\delta)^*(x) $$

$$ = \int_{\partial\Omega} G_\delta(x - x(s))\phi(s)ds + \int_{\partial\Omega} \nabla G_\delta(x - x(s)) \cdot n(s)\psi(s)ds $$

$$ + [I^G - I^G_\delta] \phi(s^*) - [1 - I^n_\delta] \psi(s^*), $$  \hspace{1cm} (21)

where
\[ I^{(n)}_\delta(x) = - \int_{\partial \Omega} \nabla G_\delta(x - x(s)) \cdot n(s) ds, \]
\[ I^G(x) = \int_{\partial \Omega} G(x - x(s)) ds, \]
\[ I^G_\delta(x) = \int_{\partial \Omega} G_\delta(x - x(s)) ds, \]
and we have used the identity
\[ \int_{\partial \Omega} \nabla G(x - x(s)) \cdot n(s) ds = -1, \quad x \in \Omega. \]

Equation (21) is valid on the boundary as well, since the jump in the double layer will be taken care of automatically by the correction terms, because
\[ f^{(n)}_\delta \approx \begin{cases} 1, & x \in \Omega, \\ 1/2, & x \in \partial \Omega. \end{cases} \]
The correction will also increase the accuracy of computations when the evaluation point is off the boundary, but near it [27]. This regularization-correction technique is applied to solve the Stokes-Darcy problem (1)-(7).

3.2. Stokes solution

As mentioned, to account for the jump conditions and increase the accuracy for both Stokes and Darcy quantities, the integrals (10)-(13) were modified in [27] to be in the form of potentials (14) by using the following decomposition
\[ f(s) = f^{(n)}(s) n(s) + f^{(\tau)}(s) \tau(s) \]
(22)
of the unknown force distributed along the Stokes boundary, where \( \tau(s) \) and \( n(s) \) are respectively the unit tangential and outward normal vectors at \( s \). Substituting this into (10), the Stokes pressure becomes
\[ p_S(x) = V[f^{(\tau)}(s)](x) + N[f^{(n)}(s)](x), \]
(23)
where the single layer was obtained using integration by parts. Similarly, the Stokes velocity can be written in terms of potentials as ([8, 27])
\[ u_S(x) = V \left[ \frac{f(s)}{2\mu} + \frac{x - x(s)}{2\mu} f^{(\tau)}(s) \right]_s (x) + N \left[ \frac{x - x(s)}{2\mu} f^{(n)}(s) \right]_s (x). \]
(24)
Notice that when \( x \in \partial \Omega_S \), (24) needs to be modified similarly to (16). However, (24) remains continuous since the density of the double layer \( N \) is zero on the boundary (\( x = x(s) \)).

In a coupled Stokes-Darcy problem, the interface conditions (6) and (7) require accurate computation of velocity gradients. Each of the four components (\( \partial u/\partial x, \partial u/\partial y, \partial v/\partial x, \partial v/\partial y \)) can be written in a similar form using the fact that derivatives of potentials can be computed as potentials. To compute \( \frac{\partial u_S}{\partial y}(x) \), for example, the first component in (24) is differentiated using
\[ \frac{\partial}{\partial y} V[\phi](x) = V[(\phi y_s)_s](x) - N[\phi x_s](x), \quad \frac{\partial}{\partial y} N[\psi](x) = V[(\psi y_s)_s](x) + N[\psi x_s](x), \]
with densities \( \phi \) and \( \psi \) given by (24) to obtain the following representation,
\[ \frac{\partial u_S}{\partial y}(x) = V \left[ \frac{1}{2\mu} \left\{ -f_1(s)y_s(s) + \left[ (x - x(s)) f^{(\tau)}(s) \right]_s x_s(s) + \left[ (x - x(s)) f^{(n)}(s) \right]_s y_s(s) \right\} \right]_s (x), \]
\[ + N \left[ \frac{1}{2\mu} \left\{ f_1(s)x_s(s) - \left[ (x - x(s)) f^{(\tau)}(s) \right]_s x_s(s) + \left[ (x - x(s)) f^{(n)}(s) \right]_s y_s(s) \right\} \right]_s (x), \]
(25)
for \( x \in \Omega_S \). If \( x \in \partial \Omega_S \), this expression has to be modified to take into consideration the jump conditions arising from the double layer, see [27, 26] for results and discussion.
We apply the regularization-correction technique (Sec. 3.1) to compute Stokes flow by evaluating the fluid quantities (23)-(25) as in (21). By using integration by parts, the integrals simplify to the following final expressions [27, 26]:

\[ p_S(x) = \int_{\partial \Omega_S} \nabla G_\delta(x - \xi(s)) \cdot f(s) ds - \left[ 1 - I^{(n)}_\delta(x) \right] f^{(n)}(s^*) + \left[ I^G(x) - I^G_\delta(x) \right] f^{(\tau)}(s^*), \]

\[ u_S(x) = \int_{\partial \Omega_S} \left\{ -G_\delta(x - \xi(s)) \frac{f(s)}{2\mu} + \nabla G_\delta(x - \xi(s)) \cdot f(s) \frac{x - \xi(s)}{2\mu} \right\} ds \]

\[ + \left[ I^G(x) - I^G_\delta(x) \right] \Phi(s^*) - \left[ 1 - I^{(n)}_\delta(x) \right] \Psi(s^*), \]

\[ \frac{\partial u_S}{\partial y}(x) = \int_{\partial \Omega_S} \left\{ -\frac{\partial G_\delta}{\partial y}(x - \xi(s)) \frac{f_1(s)}{2\mu} + \left[ \nabla \frac{\partial G_\delta}{\partial y}(x - \xi(s)) \cdot f(s) \frac{x - \xi(s)}{2\mu} \right] + \xi_\delta(x - \xi(s))x'(s) \frac{x - \xi(s)}{2\mu} f^{(n)}(s) \right\} ds \]

\[ + \left[ 1 - I^{(n)}_\delta(x) \right] \left[ \Phi_1(s)x_s - \Psi_1(s)y_s \right]_{s = s^*} \]

\[ + \left[ I^G(x) - I^G_\delta(x) \right] \left[ \Phi(s)y_s + \Psi_1(s)x_s \right]_{s = s^*}, \]

where

\[ \Phi(s) = -\frac{f(s)}{2\mu} + \left[ \frac{x - \xi(s)}{2\mu} f^{(\tau)}(s) \right], \quad \Psi(s) = \frac{x - \xi(s)}{2\mu} f^{(n)}(s) \]

with \( x \in \Omega_S \), and \( \xi_\delta \) is given by (17).

3.3. Darcy solution

Using a decomposition similar to (22) for the force \( g \) along the boundary \( \partial \Omega_D \),

\[ g(s) = g^{(n)}(s)n(s) + g^{(\tau)}(s)\tau(s), \]

the Darcy pressure can be written as

\[ p_D(x) = V \left[ g^{(\tau)}_s(s) \right] + N \left[ g^{(n)}(s) \right]. \]

To derive a similar form for the velocity, potentials in equation (30) are differentiated using the Darcy equation (2) to give [27]

\[ u_D(x) = -\frac{\kappa}{\mu} V \left[ (g^{(\tau)}_s(s)\tau(s) - g^{(n)}_s(s)n(s)) \right]_{s} - \frac{\kappa}{\mu} N \left[ g^{(\tau)}_s(s)n(s) + g^{(n)}_s(s)\tau(s) \right]. \]

This is a general formulation to determine the Darcy flow from two force components on the boundary, \( g^{(\tau)} \) and \( g^{(n)} \). Notice that only one of these components can be determined from the given boundary and interface conditions. To match the unknowns with the conditions, we assume that the Darcy force is normal to the boundary, i.e., \( g^{(\tau)} = 0 \).

The method of regularization with corrections is again used for Darcy quantities (30)-(31) by applying the method of (21). After some algebra that involves simple integration by parts, the following expressions can be obtained [27, 26]:

\[ p_D(x) = \int_{\partial \Omega_D} \nabla G_\delta(x - \xi(s)) \cdot g(s) ds - \left[ 1 - I^{(n)}_\delta(x) \right] g^{(n)}(s^*) + \left[ I^G(x) - I^G_\delta(x) \right] g^{(\tau)}(s^*), \]

\[ u_D(x) = -\frac{k}{\mu} \left\{ \int_{\partial \Omega_D} \left\{ -\xi_\delta(x - \xi(s))g^{(n)}(s)n(s) + \nabla[\nabla G_\delta(x - \xi(s))] \cdot g(s) \right\} ds \right. 

\[ - \left[ 1 - I^{(n)}_\delta(x) \right] \left[ g^{(\tau)}_s(s)n(s) + g^{(n)}_s(s)\tau(s) \right]_{s = s^*} \]

\[ + \left[ I^G(x) - I^G_\delta(x) \right] \left[ g^{(\tau)}_s(s)\tau(s) - g^{(n)}_s(s)n(s) \right]_{s = s^*}, \]
where $x \in \tilde{\Omega}_D$, and as before, $\xi_\delta$ is given by (17).

### 3.4. Coupled system

In the coupled Stokes-Darcy problem, the expressions for both fluid quantities (26)-(28) and (32)-(33) are combined to compute the unknown force densities $f$ and $g$ by satisfying the boundary and interface conditions (3)-(7). For a shorthand notation, we write (26)-(28) and (32)-(33) as

\[
\begin{aligned}
p_S(x) &= K_{\delta c}^p(x) \ast f, \quad u_S(x) = K_{\delta c}^u(x) \ast f, \quad \frac{\partial u_S^r}{\partial n_S} = K_{\delta c}^r(x) \ast f, \quad n_S \cdot D_S \cdot n_S = K_{\delta c}^{\gamma}(x) \ast f, \\
p_D(x) &= H_{\delta c}^p(x) \ast g, \quad u_D(x) = H_{\delta c}^u(x) \ast g,
\end{aligned}
\]

where $f = (f^{(n)}, f^{(r)})^T$, $g = g^{(n)}$, $D_S$ is defined by equation (6), and the subscript $\delta c$ indicates the regularization-correction procedure. Enforcing continuity of normal velocity (5) on the interface gives

\[
(K_{\delta c}^u \cdot n_S)(x) \ast f = - (H_{\delta c}^u \cdot n_D)(x) \ast g, \quad x \in \Sigma.
\]

Other conditions in (3)-(7) are imposed similarly. Then solving the coupled problem involves first solving the following linear system to satisfy the boundary and interface conditions (3)-(7):

\[
(SD) F = b,
\]

where

\[
(SD) = \begin{pmatrix}
K_{\delta c}^u |_{\Gamma_S} & 0 \\
(K_{\delta c}^u \cdot n_S) |_{\Sigma} & (H_{\delta c}^u \cdot n_D) |_{\Sigma} \\
(K_{\delta c}^p - 2\mu K_{\delta c}^{\gamma}) |_{\Sigma} & -H_{\delta c}^p |_{\Sigma} \\
(K_{\delta c}^r - \frac{\gamma}{2\mu} K_{\delta c}^u \cdot \tau_S) |_{\Sigma} & 0 \\
0 & H_{\delta c}^u |_{\Gamma_D}
\end{pmatrix}, \quad F = \begin{pmatrix} f \\ 0 |_{\Sigma} \end{pmatrix}, \quad b = \begin{pmatrix} 0 |_{\Sigma} \\ g \end{pmatrix}, \quad p = \begin{pmatrix} 0 |_{\Sigma} \\ p_0 |_{\Gamma_D} \end{pmatrix}
\]

Observe that the non-symmetric matrix of this linear system is composed of dense blocks (from integrals with different kernels). Numerical tests performed in [27] show the accuracy improvement when using corrections. Another important observation is the reduction of the condition numbers. Indeed, without corrections, these numbers grow exponentially when increasing the regularization parameter $\delta$ ([27]). Adding corrections improves the matrix condition numbers and the convergence of the iterative technique. This will be illustrated in a Stokes flow example in a bounded domain, where we also show the significant increase of the convergence speed of GMRES using the preconditioner proposed in this paper.

### 4. Preconditioner

As mentioned, the correction process improves the efficiency of the iterative technique to solve the linear system. From these results and observations, we propose a preconditioner for the system (38) provided by the correction terms. We first describe the general case of potentials. We write (21) as

\[
w(x) = \mathcal{P}_{\delta c}[\phi; \psi] = \mathcal{P}_\delta[\phi; \psi] + \mathcal{P}^*[\phi; \psi] - \mathcal{P}_\delta^*[\phi; \psi],
\]

where $\mathcal{P}_\delta$, $\mathcal{P}^*$, $\mathcal{P}_\delta^*$ represent the integrals, the singular and the regularized parts of corrections in (21), respectively. With this notation, the preconditioner for $\mathcal{P}_{\delta c}$ is defined as

\[
\tilde{\mathcal{P}} = (\mathcal{P}^*)^{-1}.
\]

Applying this preconditioner to $w$, we get

\[
\tilde{\mathcal{P}} \mathcal{P}_{\delta c}[\phi; \psi] = (\tilde{\mathcal{P}} \mathcal{P}_\delta + \tilde{\mathcal{P}} \mathcal{P}^* - \tilde{\mathcal{P}} \mathcal{P}_\delta^*)[\phi; \psi] = (I - \Delta)[\phi; \psi],
\]

where $\Delta$ is defined by (17).
where $\Delta = \tilde{P}(P^*_S - \mathcal{P}_d)$ depends on the regularization parameter $\delta$ with a property that $\Delta[\phi^*, \psi^*] = 0$. This preconditioner corresponds to (21), which is a superposition of a single and double layers. The same approach will be used to construct the preconditioner, noted here by $((SD)^*)^{-1}$, for the Stokes-Darcy problem (38). The matrix $(SD)^*$, which has the same structure as $(SD)$, is obtained by computing the corresponding $\mathcal{P}^*$ for each block in $(SD)$. For the sake of simplicity, we present in the next subsection the preconditioner in the context of Stokes velocity. Then, we show the structure and the elements of the preconditioner for the coupled system.

4.1. Stokes velocity

Consider Stokes flow in a domain $\Omega_S$ where the velocity of the fluid is given on the boundary:

$$
-\nabla p_S + \mu \Delta \mathbf{u}_S = 0, \quad \mathbf{u}_S = \mathbf{u}_0, \quad \mathbf{x} \in \Omega_S,
$$

(42)

$$
\mathbf{u}_S = \mathbf{u}_0 \quad \mathbf{x} \in \partial \Omega_S.
$$

(43)

We use the representation (27) of the solution in terms of potentials. Evaluating it on the boundary $\mathbf{x} \in \partial \Omega_S$, we get

$$
\mathbf{u}_S(\mathbf{x}) = K^u_{\partial \Omega_S} \phi = K^u_\delta [f^{(\tau)}; f^{(n)}] + (K^u)^*[f^{(\tau)}; f^{(n)}] - (K^u)^*[f^{(\tau)}; f^{(n)}] - (K^u)^*[f^{(\tau)}; f^{(n)}],
$$

where the unknown density components for each operator are shown in square brackets $[ ; ]$, so that

$$
K^u_\delta [f^{(\tau)}; f^{(n)}] = \int_{\partial \Omega_S} \left\{ -G_\delta(\mathbf{x} - \mathbf{x}(s)) \frac{f(s)}{2\mu} + \nabla G_\delta(\mathbf{x} - \mathbf{x}(s)) \cdot \mathbf{f}(s) \frac{\mathbf{x} - \mathbf{x}(s)}{2\mu} \right\} \, ds,
$$

$$
(K^u)^*[f^{(\tau)}; f^{(n)}] = I^G(\mathbf{x}) \left\{ -\frac{\tau(s^*)}{\mu} f^{(\tau)}(s^*) - \frac{n(s^*)}{2\mu} f^{(n)}(s^*) \right\},
$$

$$
(K^u)^*[f^{(\tau)}; f^{(n)}] = I^G(\mathbf{x}) \left\{ -\frac{\tau(s^*)}{\mu} f^{(\tau)}(s^*) - \frac{n(s^*)}{2\mu} f^{(n)}(s^*) \right\}.
$$

Without using a preconditioner, enforcing condition (43) results in solving

$$
K^u_{\partial \Omega_S} \mathbf{f} = \mathbf{u}_0
$$

(44)

for the unknown forces $\mathbf{f} = (f^{(\tau)}, f^{(n)})^T$. On the other hand, to improve the convergence properties of the iterative technique, we solve equation (43) as

$$
\tilde{K}K^u_{\partial \Omega_S} \mathbf{f} = \tilde{K}\mathbf{u}_0,
$$

(45)

with the preconditioner defined as

$$
\tilde{K} = ((K^u)^*)^{-1} = \begin{pmatrix} D_u^{(\tau)} & D_u^{(n)} \\ D_v^{(\tau)} & D_v^{(n)} \end{pmatrix}^{-1}.
$$

(46)

Each of the four blocks of the matrix $K^*$ is a diagonal matrix with the elements equal to the coefficients of the forces in the corresponding component of velocity $\mathbf{u} = (u, v)$:

$$
D_u^{(\tau)}[i, i] = -\frac{\tau_i(s_i)}{\mu} I^G(x_i), \quad D_u^{(n)}[i, i] = -\frac{n_1(s_i)}{2\mu} I^G(x_i),
$$

$$
D_v^{(\tau)}[i, i] = -\frac{\tau_i(s_i)}{\mu} I^G(x_i), \quad D_v^{(n)}[i, i] = -\frac{n_2(s_i)}{2\mu} I^G(x_i),
$$

and $x_i = x(s_i)$. Due to the block-diagonal structure of $K^*$, the inverse can be computed easily. In the Numerical results section, we present experiments to show improvement of the GMRES iterations for this problem when solving the preconditioned system (45) instead of (44).
4.2. Coupled system

As mentioned, the same method is applied to derive the preconditioner for the coupled Stokes-Darcy system (38). The matrix \((SD)^*\) based on the singular part of corrections has the same block structure as (38):

\[
(SD)^* = \begin{pmatrix}
(K_u^u)^* & 0 \\
(K_u^u \cdot n_S)^* & (H_u^u \cdot n_D)^* \\
(K_p^p - 2\mu K''_\delta)^* & -(H_p^p)^* \\
(K'_\delta)^* - \frac{\gamma}{\sqrt{\kappa}}(K_u^u \cdot \tau_S)^* & 0 \\
0 & (H'_\delta)^*
\end{pmatrix}
\]

(47)

For each block of (38), we evaluate the corresponding fluid quantities (26)-(27), (32)-(33) at a boundary point \(x = x(s^*)\), and get the following expressions for singular part of corrections:

\[
(K_p^p)^*[f^{(\tau)}; f^{(n)}] = I^G(x)f_\tau^{(\tau)}(s^*) - f^{(n)}(s^*),
\]

\[
(K_u^u)^*[f^{(\tau)}; f^{(n)}] = I^G(x)\Phi(s^*) - \Psi(s^*) = I^G(x) \left\{ -\frac{f(s^*)}{2\mu} - \frac{x(s^*)}{2\mu}f^{(\tau)}(s^*) \right\},
\]

\[
(H_p^p)[g^{(n)}] = -g^{(n)}(s^*),
\]

\[
(H_u^u)[g^{(n)}] = -\frac{k}{\mu} \left\{ I^G(x) \left[ -g^{(n)}(s) \right] n(s) \right|_{s=s^*} - \left[ g^{(n)}(s) \tau(s) \right] n(s) \right|_{s=s^*} \right\},
\]

since \(g^{(\tau)} = 0\). Then the coefficients of \(f^{(\tau)}, f^{(n)},\) and \(g^{(n)}\) form the elements of \((SD)^*\). The normal velocities \(u \cdot n\) and the velocity gradients give similar expressions. Since \(x = x(s_i)\) is a point on the boundary, these give the following elements on the diagonals of the corresponding block matrices:

\[
(K_p^p)[f^{(\tau)}; f^{(n)}] = I^G_i f_\tau^{(\tau)} - f_i^{(n)},
\]

\[
(K_u^u)[f^{(\tau)}; f^{(n)}] = -\frac{\tau_i}{\mu} I^G_i f^{(\tau)}_i - \frac{n_i}{2\mu} I^G_i f^{(n)}_i,
\]

\[
(H_p^p)[g^{(n)}] = -g_i^{(n)},
\]

(50)

\[
(H_u^u)[g^{(n)}] = \frac{k}{\mu} \left\{ [n_i I^G_i + \tau_i] g_i^{(n)} + n_i g_{ss_i}^{(n)} \right\}.
\]

(51)

Notice that terms of the form \(c f_i\) will form the diagonal elements of the particular block matrix in (47). If the term has the form \(c f_{ss_i}\) or \(c f_{ss_{ss_i}}\), then a few off-diagonal elements become non-zero, depending on the particular approximation of \(f_s\) and \(f_{ss}\) used. Therefore, the matrix \((SD)^*\) in (47) will have a block-diagonal or near diagonal structure and can be solved efficiently.

5. Numerical Results

5.1. Integration

The numerical accuracy of solutions depends on two parameters, the regularization \(\delta\) and the discretization quadrature. Once the kernels are regularized and all integrands are smooth functions, we use the trapezoidal rule

\[
\int_0^L K_\delta(x - x(s))f(s)ds \approx \sum_{j=1}^N K_\delta(x - x(s_j))f(s_j)\Delta s_j
\]

10
to approximate the integrals in (26)-(28), (32)-(33), including the integrals for $I^{(n)}_b(x)$ and $I^G_b(x)$. For accuracy estimates and improvements, see [2]. Higher-order integration techniques such as Gaussian quadrature could be used. However, it does not change the dependence of solutions on regularization when $\delta$ is large, and better accuracy is achieved only for smaller values of $\delta$ (see [27]). To compute $I^G_b(x)$ with better accuracy than trapezoidal rule, we use a Gaussian quadrature.

5.2. Cubic Splines

To ensure smoothness, the boundary and the force distributions are parametrized using Cubic Splines:

$$x = (x, y) = a_k + b_k(\alpha - \alpha_k) + c_k(\alpha - \alpha_k)^2 + d_k(\alpha - \alpha_k)^3,$$

$$f^r(\alpha) = a_k^r + b_k^r(\alpha - \alpha_k) + c_k^r(\alpha - \alpha_k)^2 + d_k^r(\alpha - \alpha_k)^3,$$

$$f^n(\alpha) = a_k^n + b_k^n(\alpha - \alpha_k) + c_k^n(\alpha - \alpha_k)^2 + d_k^n(\alpha - \alpha_k)^3,$$

for $\alpha_k \leq \alpha \leq \alpha_{k+1}$, $k = 1, \ldots, N$, with periodic boundary conditions. The arclength $s = s(\alpha)$ is a smooth map with $ds/d\alpha = |dx/d\alpha|$, so that $d(\cdot)/d\alpha = (dx/d\alpha)/|dx/d\alpha|$. The unit tangent and normal vectors are $\tau(\alpha) = T(\alpha)/|T(\alpha)|$, $n(\alpha) = N(\alpha)/|N(\alpha)|$, where $T(\alpha) = (x'(\alpha), y'(\alpha))$ and $N(\alpha) = (y'(\alpha), -x'(\alpha))$. With this parametrization, we rewrite the integrals with respect to $\alpha$.

Using this representation of the unknown forces, the derivatives are approximated by the Spline coefficients $\alpha_i$. Recall that the number of unknowns reduces to $a_i$ by using $b = Ba$ and $c = Ca$ with matrices $B$ and $C$ defined by the periodic spline conditions. Therefore, when constructing the matrix $(SD)^*$ as in (47), equations (49) and (50), for example, will form diagonal blocks, whereas equations such as (48) and (51) include terms with Spline coefficients $b_i$ and $c_i$. These will form near diagonal block matrices with coefficients of $a_i$. Other approximations of the derivatives are possible, and we later compare results using Splines and finite differences for the coefficients of $f_s$ and $f_{ss}$ to compute the matrix $(SD)^*$.

5.3. Stokes problem

In Section 4.1, we have introduced the preconditioner for the Stokes problem. Here we compare the efficiency of GMRES for the original matrix system for forces (44) with the system using preconditioner (45). We solve the following problem: compute Stokes flow in $\Omega_S = (0, 1) \times (1, 2)$ using the boundary condition

$$u_S = (0, x(x - 1)),$$

$$x \in \partial \Omega_S.$$

We assume viscosity $\mu = 1$ and discretize the boundary using $N$ points. Both systems have dimensions $2N \times 2N$. Figure 3 compares how the number of GMRES iterations grows with the matrix size. The tolerance was kept fixed at $10^{-7}$. Table 1 shows the corresponding numbers.

<table>
<thead>
<tr>
<th>$N$</th>
<th>original, no corrections</th>
<th>corrected, no preconditioner</th>
<th>corrected, preconditioner</th>
</tr>
</thead>
<tbody>
<tr>
<td>168</td>
<td>242</td>
<td>78</td>
<td>24</td>
</tr>
<tr>
<td>328</td>
<td>456</td>
<td>110</td>
<td>31</td>
</tr>
<tr>
<td>648</td>
<td>950</td>
<td>152</td>
<td>36</td>
</tr>
<tr>
<td>1288</td>
<td>2008</td>
<td>212</td>
<td>42</td>
</tr>
</tbody>
</table>

Table 1
Stokes problem: number of GMRES iterations.

The number of iterations with the preconditioner is significantly reduced, and although still growing as the matrix size increases, the growth rate is almost negligible in comparison with the original matrix (Figure 3).
5.4. Coupled problem

The matrix in the previous example corresponds to the upper left block of the system (38). Although the Stokes problem alone is simpler to analyze, we expect the observations to be reflected in the coupled system. Consider Darcy flow in \(0 \leq x \leq 1, 0 \leq y \leq 1\) and Stokes flow in \(0 \leq x \leq 1, 1 \leq y \leq 2\), satisfying continuity of normal components of velocity and normal stress and the no-slip condition \((\gamma = \infty)\) along the interface \(y = 1\). The case with a non-zero slip and slight variations of the boundary conditions will give similar results. The exact solution is

\[
\begin{align*}
p_D &= \frac{\mu}{k} \left[ x(1-x)(y-1) + \frac{(y-1)^3}{3} \right] + 2\mu, \\
u_D &= \left( (2x-1)(y-1), x(x-1) - (y-1)^2 \right), \\
p_S &= 2\mu y, \\
u_S &= (0, x(x-1)).
\end{align*}
\]

We use this solution to assign the velocity along the outer Stokes boundary and pressure along the outer Darcy boundary. The numerical solution of this problem was presented in [27].

5.4.1. Dependence on the discretization and regularization parameters

In this section, we fix viscosity \(\mu = 1\), permeability \(k = 1\), and analyze the convergence properties of the iterative scheme for different discretization \(\Delta s\) and regularization \(\delta\) values. Each boundary is discretized using \(N\) points. Then the matrix that results from imposing the boundary conditions has dimensions \(3N \times 3N\), since there are \(2N\) unknowns for Stokes forces and \(N\) unknowns for Darcy. It should be noted here that while using regularization \(\delta/\Delta s\) for Stokes, we use a shifted regularization \(\delta/\Delta s + 2\) for Darcy, since the kernels in Darcy integrals are more singular and thus require stronger regularization in practice. The system is solved using GMRES with a dimension of Krylov subspace denoted here by \(N_{\text{Krylov}}\) and a given tolerance for the residual.

We first solve the system (38) without the preconditioner. Figures 4 and 5 show the eigenvalues of the matrix for \(N = 328\) points on each boundary, so that \(\Delta \alpha = \Delta s \approx 0.0122\). The optimal regularization \(\delta/\Delta s = 1\) for Stokes and \(\delta/\Delta s + 2\) for Darcy was used. The system is not well-conditioned and most of the eigenvalues being near zero is reflected in slow convergence of the iterative method. Figure 6 shows the matrix condition numbers and the number of GMRES iterations with tolerance \(10^{-9}\) for different values of discretization parameter \(\Delta s\) and regularization with \(\delta/\Delta s\) fixed. As we can observe, refining the boundary discretization impairs these numbers, so that condition numbers grow roughly quadratically, with the iteration numbers growing linearly.
Fig. 4. Coupled system: matrix without preconditioner. Right: close-up of eigenvalues 506 to 984.

Fig. 5. Coupled system: matrix without preconditioner. Right: close-up of eigenvalues 450 to 984.

Fig. 6. Coupled system: convergence without preconditioner.

Table 2 shows convergence rates of the original and the preconditioned matrices in more detail. Each boundary is discretized with \( N \) points, cond # is the matrix condition number, \( N_{Krylov} \) is the dimension of Krylov space used in GMRES with tolerance fixed at \( 10^{-9} \). Again, we roughly double the number of points on the boundary each time, also decreasing regularization while keeping the ratio \( \delta/\Delta s \) fixed. For each \( N \), the matrix size is \( 3N \times 3N \), and we notice a considerable reduction of the condition numbers when using preconditioners. We also compute the iteration number with respect to the dimension of the Krylov subspace.
N\_Krylov, keeping the tolerance fixed. Comparing the third and fourth columns in the corresponding tables, it is obvious that the preconditioned matrix not only converges much faster for the same Krylov space dimension, but it also allows using much smaller Krylov spaces and get the same convergence, which makes the algorithm even more robust. For instance, for the $504 \times 504$ matrix ($N=168$) the algorithm with the original matrix converges in 226 iterations, but a Krylov space of at least 250 is required for convergence. In the preconditioned case however, the algorithm converges in only 54 iterations, requiring a much smaller Krylov space. Doubling the matrix dimension results in 351 iterations with a roughly doubled Krylov space in the original calculations, whereas the preconditioned matrix gives convergence in only 66 iterations, which was possible to obtain without increasing the Krylov space.

Table 2
Coupled system: convergence for various grid sizes.

<table>
<thead>
<tr>
<th>N</th>
<th>cond #</th>
<th>N_Krylov</th>
<th>GMRES iter</th>
</tr>
</thead>
<tbody>
<tr>
<td>168</td>
<td>1.18 \cdot 10^5</td>
<td>200</td>
<td>5000+</td>
</tr>
<tr>
<td></td>
<td></td>
<td>250</td>
<td>226</td>
</tr>
<tr>
<td></td>
<td></td>
<td>500</td>
<td>226</td>
</tr>
<tr>
<td>328</td>
<td>4.61 \cdot 10^5</td>
<td>350</td>
<td>5000+</td>
</tr>
<tr>
<td></td>
<td></td>
<td>500</td>
<td>351</td>
</tr>
<tr>
<td></td>
<td></td>
<td>980</td>
<td>351</td>
</tr>
<tr>
<td>648</td>
<td>1.82 \cdot 10^5</td>
<td>500</td>
<td>5000+</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1000</td>
<td>550</td>
</tr>
<tr>
<td>1288</td>
<td>7.20 \cdot 10^6</td>
<td>3500</td>
<td>888</td>
</tr>
</tbody>
</table>

Table 3
Coupled system: eigenvalue range for various grid sizes.

<table>
<thead>
<tr>
<th>N</th>
<th>without preconditioner</th>
<th>with preconditioner</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>cond #</td>
<td>$</td>
</tr>
<tr>
<td>168</td>
<td>1.18 \cdot 10^5</td>
<td>0.0033</td>
</tr>
<tr>
<td>328</td>
<td>4.61 \cdot 10^5</td>
<td>0.0017</td>
</tr>
<tr>
<td>648</td>
<td>1.82 \cdot 10^5</td>
<td>0.0008</td>
</tr>
<tr>
<td>1288</td>
<td>7.20 \cdot 10^6</td>
<td>0.0004</td>
</tr>
</tbody>
</table>

Fig. 7. Coupled system: matrix with preconditioner. Right: close-up of eigenvalues 560 to 984.
The eigenvalues of the preconditioned matrix, shown for the case $N = 328$ in Figure 7, have smaller spectrum than the eigenvalues of the original matrix (Figure 5), since applying the preconditioner seems to keep the largest eigenvalue bounded, and the smaller eigenvalues are not as near zero. Table 3 shows the largest and the smallest eigenvalues of the matrices for different $N$. In the preconditioned case, the largest eigenvalue remains fixed. For the original matrix, the eigenvalue is growing with the matrix size. The smallest eigenvalue, on the other hand, is being halved each time we double the number of points in both cases. However, without using the preconditioner, they are closer to zero.

The next numerical experiments demonstrate the effect of regularization on convergence properties. We keep the discretization fixed at $N = 328$, and vary the regularization by changing the ratio $\delta/\Delta s$. Table 4 shows condition numbers and the eigenvalue range of the matrix system. If we do not use preconditioner, for all values of $\delta$, the condition numbers are of the order of $10^5$. Applying the preconditioner reduces the condition numbers significantly. An important observation is that the largest eigenvalue in magnitude is growing for larger $\delta$ without the preconditioner, but in the preconditioned case it remains bounded. The smallest eigenvalue $|\lambda_{\text{min}}|$ is closer to zero in the original matrix. The graphs of condition numbers and total GMRES iterations are shown in Figure 8. The number of iterations decreases when increasing the regularization parameter due to the eigenvalues increasing in magnitudes, see the $|\lambda_{\text{min}}|$ and $|\lambda_{\text{max}}|$ columns in Table 4. But $\delta$ cannot be increased further due to the deteriorating accuracy of the solution.

<table>
<thead>
<tr>
<th>$\delta/\Delta s$</th>
<th>without preconditioner</th>
<th>with preconditioner</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>cond #</td>
<td>$</td>
</tr>
<tr>
<td>1.0</td>
<td>4.61 · $10^5$</td>
<td>0.0017</td>
</tr>
<tr>
<td>1.5</td>
<td>3.73 · $10^3$</td>
<td>0.0023</td>
</tr>
<tr>
<td>2.0</td>
<td>3.21 · $10^3$</td>
<td>0.0031</td>
</tr>
<tr>
<td>2.5</td>
<td>2.90 · $10^3$</td>
<td>0.0038</td>
</tr>
<tr>
<td>3.0</td>
<td>2.69 · $10^3$</td>
<td>0.0046</td>
</tr>
<tr>
<td>3.5</td>
<td>2.53 · $10^3$</td>
<td>0.0053</td>
</tr>
<tr>
<td>4.0</td>
<td>2.42 · $10^3$</td>
<td>0.0061</td>
</tr>
</tbody>
</table>

Table 4
Coupled system: convergence for various regularization parameters.

Fig. 8. Coupled system: convergence without preconditioner (dashed) and with preconditioner (solid).

5.4.2. Dependence on the physical parameters
In this section, we investigate how the convergence of the iterative method behaves with different values of the fluid viscosity $\mu$ and the permeability of the porous medium $k$, which can differ by orders of magnitude in practice. This results in different blocks of system (38) having different scales, which may affect the
convergence. We investigate this by fixing one of the parameters, say $\mu$, and varying the other one. We used discretization with $N = 328$ and regularization as before, and GMRES with a tolerance $10^{-5}$, $N_{\text{Krylov}} = 500$ without preconditioner, and $N_{\text{Krylov}} = 100$ with preconditioner. Table 5 shows results with $\mu = 1$, $10^{-2}$, and $10^2$, and varying $k$. Although the number of iterations is still dependent on the parameter values, we again observe that the preconditioner improves the convergence of GMRES in each case.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$k$</th>
<th>without preconditioner</th>
<th>with preconditioner</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>cond #</td>
<td>iter</td>
</tr>
<tr>
<td>$1$</td>
<td>$1$</td>
<td>$4.61 \cdot 10^2$</td>
<td>$259$</td>
</tr>
<tr>
<td>$10^{-1}$</td>
<td>$4.61 \cdot 10^4$</td>
<td>$259$</td>
<td>$5.47 \cdot 10^2$</td>
</tr>
<tr>
<td>$10^{-2}$</td>
<td>$4.62 \cdot 10^3$</td>
<td>$264$</td>
<td>$3.08 \cdot 10^2$</td>
</tr>
<tr>
<td>$10^{-4}$</td>
<td>$1.26 \cdot 10^4$</td>
<td>$231$</td>
<td>$3.63 \cdot 10^3$</td>
</tr>
<tr>
<td>$10^{-2}$</td>
<td>$1$</td>
<td>$5.08 \cdot 10^2$</td>
<td>$192$</td>
</tr>
<tr>
<td>$10^{-1}$</td>
<td>$5.10 \cdot 10^4$</td>
<td>$196$</td>
<td>$5.47 \cdot 10^2$</td>
</tr>
<tr>
<td>$10^{-2}$</td>
<td>$5.40 \cdot 10^3$</td>
<td>$202$</td>
<td>$3.08 \cdot 10^2$</td>
</tr>
<tr>
<td>$10^{-4}$</td>
<td>$3.93 \cdot 10^3$</td>
<td>$198$</td>
<td>$3.63 \cdot 10^3$</td>
</tr>
<tr>
<td>$10^2$</td>
<td>$1$</td>
<td>$4.62 \cdot 10^2$</td>
<td>$316$</td>
</tr>
<tr>
<td>$10^{-1}$</td>
<td>$6.61 \cdot 10^4$</td>
<td>$312$</td>
<td>$5.47 \cdot 10^2$</td>
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<tr>
<td>$10^{-2}$</td>
<td>$6.61 \cdot 10^3$</td>
<td>$316$</td>
<td>$3.08 \cdot 10^2$</td>
</tr>
<tr>
<td>$10^{-4}$</td>
<td>$1.26 \cdot 10^6$</td>
<td>$281$</td>
<td>$3.63 \cdot 10^3$</td>
</tr>
</tbody>
</table>

Table 5
Coupled system: convergence rates for different physical parameters.

5.5. Finite difference approximation of derivatives

All the results above use coefficients of Cubic Splines to approximate the derivatives of forces. This results in a preconditioner with blocks that have a few non-zero off-diagonal elements. Alternatively, a different approximation for the derivatives can be used to reduce the block-bandwidth of the preconditioner. In particular, we use second-order finite differences for $f_s$, $f_{ss}$, $g_s$, and $g_{ss}$. This way, the blocks of the matrix $(SD)^*$ formed from the coefficients of $f$ and $g$ in (48)-(51) have at most a three-diagonal structure with periodic conditions, and computing the preconditioner becomes cheaper. As a test, we fixed the viscosity $\mu = 1$ and varied the permeability, comparing results of the two preconditioners. Again, tolerance $= 10^{-5}$ and $N_{\text{Krylov}} = 100$. As can be seen from Table 6, the GMRES iteration numbers with the preconditioned matrix using Cubic Splines (which were shown also in Table 5) and using finite differences are identical. This indicates that to construct the preconditioner, any approximation of the derivatives of forces could be used, and should be chosen so that the resulting matrix is easier to solve to obtain the preconditioner. As a comparison, we also modify the original preconditioner obtained with Cubic Splines to have a block-diagonal structure. This preconditioner has the simplest structure and therefore is cheapest to solve. We compute it by truncating the off-diagonal elements of the original preconditioner. The results are shown in the third column of Table 6. It can be seen that applying this simple preconditioner is already beneficial, it reduces the number of iterations by a factor of two.

6. Conclusions

In this paper, we have introduced a preconditioner to efficiently solve the coupled Stokes-Darcy integral formulation. This boundary integral formulation is obtained through a regularization-correction method. We derived the preconditioner based on the observation that the correction procedure improves condition
numbers of the system. Numerical results validate this approach and have shown significant improvement of the Krylov subspace method. Moreover, this preconditioner can be used in systems of integral equations involving the same kind of Green’s function, as we showed for a Stokes flow problem.

The preconditioner proposed in this work can be solved efficiently since it has a small bandwidth. Indeed, the blocks of this matrix are either diagonal or have only a few non-zero off-diagonal elements obtained because of the approximation of the derivatives on the densities. The approach used in this work to approximate these derivatives is Cubic Splines. However, to reduce the number of off-diagonal elements we also performed numerical differentiation using finite differences. In both cases, we observed that the preconditioner has the same effect on the iterative method.

Table 6
Coupled system: convergence rates with different preconditioners.

<table>
<thead>
<tr>
<th>k</th>
<th>Cubic Splines</th>
<th>Finite diff.</th>
<th>Block-diag. with Splines</th>
<th>Without preconditioner</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>40</td>
<td>40</td>
<td>123</td>
<td>259</td>
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<tr>
<td>10^{-1}</td>
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<td>124</td>
<td>259</td>
</tr>
<tr>
<td>10^{-2}</td>
<td>42</td>
<td>42</td>
<td>126</td>
<td>264</td>
</tr>
<tr>
<td>10^{-4}</td>
<td>51</td>
<td>51</td>
<td>83</td>
<td>231</td>
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</table>

References


