Ancient Harmonic Law (version 2)

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Abstract

The matrix arithmetic for ancient harmonic theory is presented here for two tuning systems with opposite defects: "Spiral fifths" as presented by Nicomachus, a Syrian Neo-Pythagorean of the second century A.D., and Plato's "Just tuning" as reconstructed by the ethnomusicologist, Ernest McClain, from clues preserved by Nicomachus and Boethius (6^{th} c. AD). These tables lie behind the system of architectural proportions used during the Renaissance, and their basic ratios now pervade modern science as the foundation of a "string theory" formally presented first in Euclid. Calculation employs an early form of log table governed by vectors of 2-3-4 in the first, and by 3-4-5 in the second. The square root of 2 plays a central role in integrating these systems governing 12-tone theory from the perspective of four primes--2, 3, and 5 generate all ratios under the overview of 7—as disciplined "self-limitation" within a "balance of perfect opposites. "

1. Introduction

Nicomachus was a Syrian mathematician writing about 150 A.D. His work forms one of the best links to what survived from his day about Greek theory of numbers and music [1,2]. I shall describe how the sequence of integers shown in Table 1, and attributed to Nicomachus, defines musical octaves, fifths, and fourths the only consonances recognized by the Greeks, and lies at the basis of ancient musical scales sometimes attributed to Pythagoras. A second Table inferred by Plato but brought to light by the ethnomusicologist, Ernest McClain [3,4,5], will be shown to be the basis of the Just scale, another ancient musical scale. This table, which I shall refer to as the McClain Table, will also provide a link to the modern theory of music. In his books and papers, McClain has made a strong case for music serving as the lingua franca of classical and sacred texts, providing plausible explanations to otherwise difficult to understand passages and providing metaphors to convey ideas and meaning. In this paper I will focus primarily on the mathematics and music at the basis of ancient musical scales.

Table 1. Nicomachus' table for expansions of the ratio 3:2 (as string lengths)

1	2	4	8	16	32	64 …	
	3	6	12	24	48	96 …	
		9	18	36	72	144 …	
			27	54	108	216 …	
				81	162	324 …	
					243	486 …	
						729 …	

The construction of ancient musical scales will be shown to be based strongly on the integers, 2,3,7, and 12. There were no easy notations for fractions in ancient civilizations so the problem was to represent ratio in terms of whole numbers. To this Plato states:

> For surely you know the ways of men who are clever in these things. If in the argument someone attempts to cut the one itself [i.e., use a fraction], they laugh and won't permit it. If you try to break it up into small coin, they multiply… Republic 525

The ancient musical scale poses an elegant solution to this problem.

2. Musical fundamentals

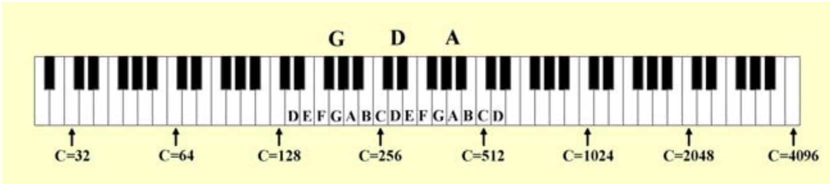


Figure 1. The piano keyboard

Our story begins with a look at the 88 keys of the piano shown in Fig. 1. Three tones, A, D, and G, have been highlighted. D is located at the center of symmetry of the black and white pattern of keys with G and A seven tones below and above D. The interval from D to G spans five tones DCBAG called a *falling musical fifth* while from D to A is *a rising fifth* (DEFGA). The pattern of black and white keys repeats every 12 tones, called the *chromatic scale*, with the white keys repeating every 7 keys and assigned the tone names DEFGABCD, the interval from D to D being called an *octave*, and with the black keys augmented or diminished by sharps (♯) and flats (♭). The white keys produce *heptatonic* scales in different modes depending on the starting key. Because of the periodicity of the musical scale any tone can be chosen as the first or fundamental, and since any tone is perceived by the ear to be identical when played in different octaves, each tone of the scale refers to a *pitch class* of tones differing by some number of octaves. The D mode, or ancient *Phrygian mode*, was the preferred scale of ancient times because of its symmetry, and I will represent all scales in this paper in the D mode. The heptatonic scale beginning on C: CDEFGABC is the famous do re mi .. scale revered by Western civilization. The tones of the piano are *equal-tempered* so that beginning on any tone, after twelve musical fifths one arrives again at the same tone seven octaves higher in pitch. This will not be the case when the scale is derived from *string length* as it was in ancient times. Because of the periodicity of the chromatic scale, it can be comfortably represented on a tone circle as shown in Fig. 2 with *semitone* intervals between adjacent tones, and with pairs of semitones equaling a *wholetone*. Proceeding clockwise around the circle leads to a rising scale while a counterclockwise rotation leads to a falling scale. The numbers 0-12 on the clock positions number the tones of the chromatic scale, and they can be thought of as numbers in a mod 12 number system where any multiple of 12 can be added to one of the integers to get another tone in the same pitch class. I will refer to the numbers on the face of the clock as the *digital roots* of the pitch class more commonly called *principal values* in mathematics. Notice that the digital roots of two tones symmetrically placed around the tone circle sum to 12. Such tonal pairs are called *complementary*.

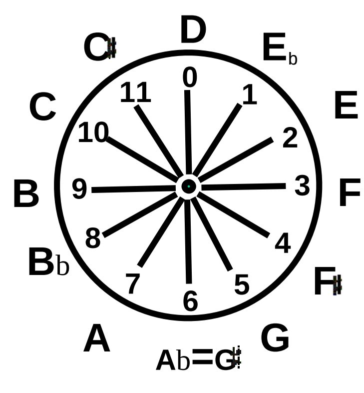


Figure 2. The tone circle

Creation of the ancient musical scale begins with the numbers 1,2, and 3. The length of a vibrating string is arbitrarily assigned the value of 1 unit which is taken to be the *fundamental* tone when bowed or plucked. If the length of the fundamental is multiplied by any power of 2, the resulting tone sounds to the ear identical to the fundamental with the resulting pitch higher or lower by some number of octaves. In other words shortening the string to 1/2 gives a tone one octave higher in pitch while increasing the length of the string to 2 results in a falling octave. Therefore each tone is a representative of a pitch class of tones all differing by a multiple of 2 in string length. The modern concept of *relative frequency* is reckoned as the inverse of the *relative string length*. A new tone emerges by increasing or decreasing string length by a factor of 3. It is therefore easy to comprehend why the musical scale served well as a metaphor for the act of creation with "Deity" commensurate with "1", the "Great Mother" as "2" the female number, while the act of "procreation" was associated with the male number "3".

2. The Tetrapolarity of the Ancient Musical Scale

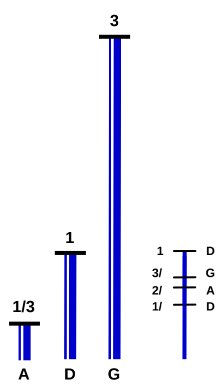
How does a new pitch class emerge from the number 3? In the ancient world, lacking the concept of a fraction, the integer 3 had four musical interpretations. First of all 3 can be associated with the ratio 3:1. This ratio is equivalent, after suppressing octave powers of 2, to relative string lengths 3/2 and 3/4 of a falling musical fifth or a rising fourth.. In this case, taking D to be the fundamental, 3:1 is represented by the pitch class G. Alternatively, 3 can be associated with the ratio 1:3 which in turn corresponds to relative string lengths 2/3 and 4/3, a rising musical fifth or a falling fourth, i.e., inverse ratios. Now with D as the fundamental, 3 is represented by pitch class A. So the integer 3 represents two pitch classes and takes on the four inverse interpretations : 3/2, 3/4 , and 2/3, 4/3.

Since the modern concept of relative frequency is the inverse of relative string length, the second interpretation could be related to relative frequency. In what follows, I will follow the first interpretation and consider musical integers directly as relative string length while at the same time referring to the inverse interpretation in terms of relative frequency. This can be very confusing to someone trying to orient themselves with regards to interpreting the musical scale in terms of integers, and it has led to a good deal of confusion.

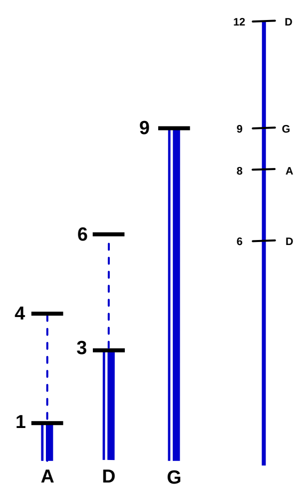
The situation is completely analogous to the paradox of the relative motion of a pair of railway trains. The question one asks is: Am I moving forward and the other train stationary, or am I stationary and the other train moving backwards? This paradox is then made more confusing by witnessing it in a mirror in which forward and backwards reverse leading to the tetrapolarity: moving-stationary, and forward-backward.

3. The Greek scale

Three string lengths are shown in Fig. 3, the fundamental at 1 unit between the lengths of 1/3 and 3 units respectively representing three pitch classes. In what follows we always place the fundamental in the middle and define the other



Figures 3a and b. String lengths of the Greek Scale with unity as the whole



Figures 4a and b. A rising and falling musical in terms of unity as part or as the "cornerstone."

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tones as rising and falling from that middle. Since factors of 2 are irrelevant to pitch, the 1/3 length is expanded to 2/3 while the 3 length is contracted to 3/4 and when placed within the fundamental unit along with the octave 1/2 gives rise to quartet of tones of increasing pitch:

D	G	A	D
1	3/4	2/3	1/2

Alternatively, 1/3 could be expanded to 4/3 while 3 is contracted to 3/2 resulting in the falling scale,

D	A	G	D
1	4/3	3/2	2

A and G reverse when relative frequency is used instead of relative string length. When the string length has a ratio to the fundamental of 2:3 it corresponds to a rising fifth, while 3:4 is a rising fourth, and 1:2 an octave. Notice that these ratios are represented in Table 1 with the integers along each row expressing the ratio 1:2, the columns 2:3, and the right leaning diagonals 3:4. The rising and falling fifths arrived at by string length differ inaudibly in pitch from the piano fifths described above. In fact, when the relative frequency of the octave is given a value of 1200 cents on a *logarithmic scale* (see the Appendix) with the equal-tempered relative frequencies of A and G, 700 cents and 500 cents, respectively, the string length values are 702 cents and 598 cents. The small discrepancy between tonal values is referred to in musical theory as a *comma*. We have once again arbitrarily chosen the fundamental as D with the *"twin tones"* A and G derived from the *"middle."* For this scale, the fundamental at D, taken as unity, can be thought of as *"containing the whole."* Notice in Fig. 2 that the twin tones G and A occupy symmetric positions on the tone circle as would any pair of tones with reciprocal ratios. In this case the symmetry is the result of the geometric sequence : 1/3, 1, 3 in which D as 1 is the geometric mean of AG as 1/3 and 3 respectively following harmonic law which states that *as relative string length multiply, intervals add*.

An alternative representation in Fig. 4a maintains the same ratios but considers the fundamental to be 3 units at D surrounded by string lengths of 1 at A and 9 at G. In Fig. 4b and the third column of Table 1, when multiplied by the proper power of 2, the strings are represented by falling musical fifths from left to right or rising fifths from right to left:

A	D	G	
4	6	9	(1)

When the fundamental is doubled to 12 in order to enclose or "seal" the larger integer 9, the twin tones multiply to 8 and 9, with ratio 9:8, without changing pitch class so that the quartet of tones of falling pitch contained in the 12/6 octave, now represented by integers, is:

D	A	G	D'	
6	8	9	12	(2)

This is read as a falling scale from left to right or a rising scale from right to left. Again, rising and falling and A and G reverse when considering the numbers to be relative frequency. Unity at A is now the building block or "cornerstone" of the system. In other words unity is now "part." In this way we have dual interpretations of scale building in which unity can be considered either whole or part in the spirit of ancient music theory and philosophy.

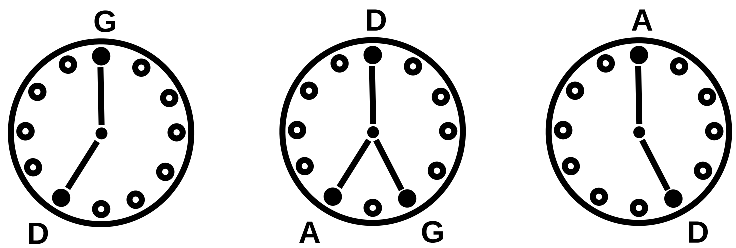
Greek musical theory was founded on this scale, and I shall refer to it as the "*Greek scale*." Observe from Sequence 2 that for the twin tones, G functions as the *arithmetic mean* of the octave DD' while A is the *harmonic mean*, reversing when the integers represent relative frequency. The integer 9 is the arithmetic mean of 6 and 12 because, 9-6 = 12-9 whereas, 8 is the harmonic mean because, $\frac{8-6}{6}=\frac{12-8}{12}$. Plato gave great significance to the Greek scale when he stated:

…In the potency of the mean between these terms (6,12), with its double sense, we have a gift from the blessed choir of the Muses to which mankind owes the boon of the play of consonance and measure, with all they contribute to rhythm and melody. [Epinomis 991]

Also it is significant that the fundamental is chosen as 6, a perfect number, i.e. the sum of its factors 1,2,3 to which Plato states: *For a divine birth there is a period comprehended by a perfect number. [Rep.* 546] It will also be shown later that all consonant tones of the chromatic scale can be expressed as ratios of the first six integers.

4. The Nicomachus Table

Figure 5. D as a) geometric mean; b) arithmetic mean; c) harmonic mean



c) a) b)

The centrality of the Greek scale to musical theory can be appreciated by looking at Fig. 5. In Fig. 5a, D is the geometric mean between A and G as harmonic and arithmetic means, respectively. In Fig. 5b, A is elevated to the fundamental position on the tone circle, and now D is in the position of the arithmetic mean of the octave AA'. In Fig. 5c, G is elevated to the fundamental, and D is now the harmonic mean of the octave GG'. So we see that D functions in the tri-partite role of geometric, arithmetic, and harmonic mean. Once again, arithmetic and harmonic means change meanings when the scale numbers are interpreted as relative frequency. This becomes evident when a segment of the Nicomachus Table (Table 1) is rewritten as follows:

Table 2

1 2 4 8 16 … (3)
3 6 12 24 …
9 18 …
27 …

We see that any integer is the arithmetic mean of the two integers that brace it from above and the harmonic mean of the two integers that brace it from below. For example, the Greek scale is gotten by placing 8 and 9 into the octave interval [6,12]. Since each row consists of octave doubles, each row represents a different pitch class. We take the row headed by 3 as D making the row headed by 1 as A, and the 9 row G. Consider the hexagonal pattern in Table 2 surrounding 12. D at 12 is the arithmetic mean of AA' at [8,16], and the harmonic mean of GG' at [9,18] while at the same time being the geometric mean of AG at [8,18] and also DD at [6,24] as shown in Fig. 5.

Notice the pattern of integers at the edge of Table 2:

1
2 3
4 9
8 27

This pattern appears in Plato's Timaeus where it is referred to as the "World Soul." The architect Leon Battista Alberti reproduced this pattern which now appears in the Alberti Museum in Florence and used it as a template for the proportions of his buildings. He clearly states that he will look to the great architecture of antiquity to create his buildings, and stated that "what pleases the ear should please the eye." Notice the hexagon of integers surrounding 12 in Table 2. Alberti took adjacent integers from this pattern to serve as relative length, width and heights of key dimensions within his buildings. From this Table the musical proportions of Renaissance architecture had its origin [2,6].

I have also noticed that the integers in a sequence of matrices discovered by the Russian bio-physicist Sergey Petoukhov, and related to DNA coding are integers from successive columns of the Nicomachus matrix [7].

5. The Pentatonic scale

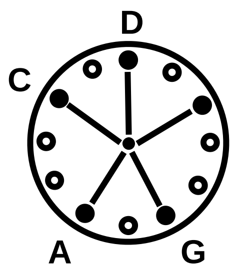


Figure 6. A pentatonic scale

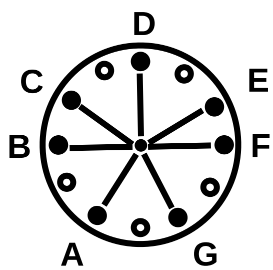


Figure 7. A heptatonic scale when inverted forms a pentatonic

The Greek scale was obtained by moving along the tone circle 7 tones clockwise and counterclockwise from D at its zenith. We can repeat our analysis but now moving twice 7 clockwise and counterclockwise from D. The two clockwise and counterclockwise moves of seven tones produce the sequence: 14, 7, 0, -7, -14. But on the mod 12 clock this sequence can be rewritten as the sequence of digital roots: 2, 7, 0, 5, 10 which according to Fig. 2 produces the sequence: E, A, D, G, C as it appears on the tone circle of Fig. 6. These tones can also be derived from the five pitch classes: 1/9, 1/3, 1, 3, 9 or 1, 3, 9, 27, 81 as we did for the Greek scale. Multiplying by the appropriate power of 2 results in the succession of falling fifths from left to right or rising fifths from right to left, also shown in column 5 of Table 1,

16	24	36	54	81	(4)
E	A	D	G	C	

Since the largest integer of this sequence is 81, the other integers can be "sealed" in the 144/72 octave by multiplying them by appropriate powers of 2 without changing pitch class to reproduce Sequence 5, a pentatonic scale in a falling octave,

72	81	96	108	128	144	(5)
D	C	A	G	E	D	

Recent research of Anne Bulckens has shown that principal dimensions of the Parthenon are reckoned by the integers of this pentatonic sequence [8,9].

6. The heptatonic scale

A heptatonic scale is derived from three clockwise and counterclockwise movements of 7 around the tone circle represented by the sequence: 21, 14, 7, 0, -7, -14, -21 or in terms of the digital roots as: 9, 2, 7, 0, 5, 10, 3 which, according to Fig. 2, is the sequence of falling fifths: B, E, A, D, G, C, F. These seven musical fifths are represented in column 7 of Table 1 by their relative string lengths, reading from top to bottom. They can also be represented by the placement of the seven pitch classes: 1/27, 1/9. 1/3, 1, 3, 9, 27 into the octave limit 864/432 by the same process that we used to create the tones of the Greek scale. The symmetric heptatonic scale in the D mode is shown on Fig. 7 by the seven darkened circles. If the tone circle is inverted, notice that the five unmarked circles form the pattern of the pentatonic scale (see Fig. 6). Since these circles lie at the positions of the black keys of the piano, we see that the black keys represent all of the pentatonic modes. Because pentatonic scales have no semitone intervals, they give rise to pleasant sounding melodies however played, and they have been used to create much of the world's folk music. On the other hand the two semitone intervals BC and EF of the heptatonic scale must be cleverly manipulated by composers in order to create interesting musical patterns.

Toussaint [10] has shown that if one goes around the tone circle of a heptatonic scale clapping on the black tones and pausing at the unmarked tones one gets one of the clapping patterns found in African rhythms. Different African clapping patterns are gotten by rotating the heptatonic scale pattern so that different of the tones are in the position of the fundamental.

7. The Chromatic scale and the Pythagorean comma

The twelve tones of the chromatic scale are derived, beginning with D, from six clockwise and counterclockwise musical fifths or movements of 7 spaces around the tone circle to produce the sequence of falling musical fifths and their digital roots:

G♯	C♯	F♯	B	E	A	D	G	C	F	B_b	E_b	A_b
6	11	4	9	2	7	0	5	10	3	8	1	6

Notice that the digital roots are multiples of 5 in the mod 12 system. The sixth clockwise and sixth counterclockwise fifth at G♯ and A_b overshoot the *tritone* at 6 o'clock on the tone circle in both directions. Although on the piano these tones would be identical, when the tones are reckoned by string lengths they differ by a barely audible pitch amounting to 24 cents (2 cents per fifth) or approximately a quarter of a semitone known as the *Pythagorean comma*. Six multiples of 3 and their inverses are defined with unity at the center: $\frac{1}{729},...,\frac{1}{3},1,3,...729$. The sequence of relative string lengths for the pitch classes of these twelve tones: 1,3,9, 27, 81, 243, 729, …, 177147, 531441 appear along the bottom edge of the Nicomachus Table 1, and when placed in scale order in a single octave I shall refer to them as the *scale of spiral fifths*. The thirteen tones (including a repeat of the fundamental) of the chromatic scale, in terms of relative frequency, are pictured in Fig. 8 as the meandering form of a serpent eating its tail since the first and thirteenth tones nearly close up missing by the Pythagorean comma.

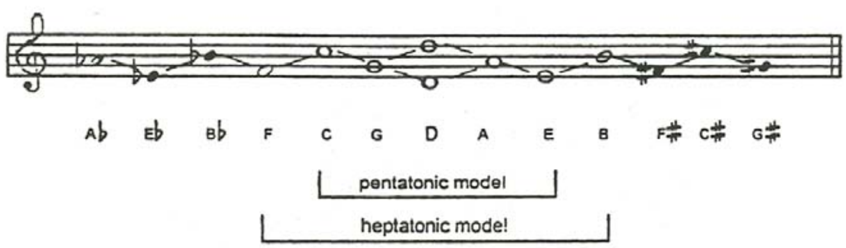


Figure 8. The serpent tuning of thirteen musical fifths arranged according to increasing frequency

8. The McClain Table

The ancients were aware that these numbers proliferated in their magnitudes. The string would have to be sealed in an octave great enough to contain the twelfth tone 177,147 of the chromatic scale. McClain, suggests that, in the Hebrew Scriptures [11], these huge integers were associated with Anakim or giants that walked the Earth and must be slain. So the serpent was cut in two places, C and E. The tones between E and G♯ were multiplied by the ratio 80:81, known as the *syntonic comma*, and the tones between A_b and C were multiplied by 81:80 with the three strands of the severed rope juxtaposed to form three rows of the *McClain Table* shown in Fig. 9a with the integers representing the relative string lengths of the twelve tones referred to as the *Just scale*, and with tones of the *scale of spiral fifths* along the center row of the three. The research of McClain suggests that this scale may have been known as far back as the 3rd millennium B.C. in ancient Sumeria. The tonal names of these pitch classes are presented in Fig. 9b when the fundamental at length 45 is chosen as D with uppercase letters for spiral fifths and lowercase for Just scale equivalents.

a)			25	75	135	675	2025 …	b)	c	f	b_b	e_b	a_b
		5	15	45	135	405…			E	A	D	G	C
	1	3	9	27	81…			g♯	c♯	f♯	b	e	

c)		400	600	450	675	d)		0(18)	3(16)	8(14)	1(12)	6(10)
	640	480	720/360	540	405		2(4)	7(2)	0(0)	5(-2)	10(-4)	
512	384	576	432	648		6(-10)	11(-12)	4(-14)	9(-16)	2(-18)		

Figure 9. The McClain Table. a) Multiplication table of powers of 3 and 5; b) tones of the Just scale; c) integers are sealed in the 720/360 octave; d) digital roots of tones from the Just scale.

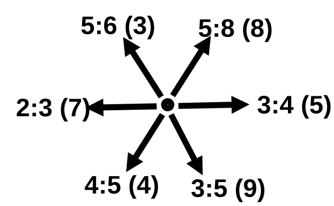
Since a_b and g♯, both representing the tritone at 6 o'clock on the tone circle, do not appear together within the same chromatic octave, a_b is eliminated and the largest integer is then 675. Therefore, the fundamental at D = 45 must be multiplied by 8 to the 720/360 octave in order to seal 675. The string lengths of Fig. 9a are now multiplied by the appropriate power of 2 to project them into the 720/360 octave of Fig 9c. All twelve tones of the chromatic scale are now represented within a relative string length limit of between 360 and 720, a great reduction in size. The octave limits prevent the proliferation of tones which results from non-cyclic nature of a scale based on string length. Note also that 360 represents the length of the ancient canonical year in all cultures, a compromise between the solar year (365.25 days) and lunar year (354 days).

The fragment of Fig. 9a,

	5		or		5	
1		3		4		3

reveals the 3,4,5-relationship underlying the musical scale. Plato was aware of this when he stated in The Republic: *"four-three mated with five … produces two harmonies."* All tones of the Just scale can be represented in terms the three ratios: 4:5, 3:4, and 5:3, and their inverses: 5:4, 4:3, and 3:5, with product of unity. This is analogous to the color wheel with primary colors and their complements. The primary colors mix to create the neutral colors white, gray, and black. The frequencies of the primary colors are also in the ratio of 3:4:5 [3]. The integers of Fig 9c can be organized according to a system of three *vectors* and their opposites governed by these ratios as shown in Fig. 10. Any integer can be chosen as the origin of this vector system. The 3:4 direction moves to a tone higher by a musical *fourth* (5) where the number in parentheses is the digital root or interval length corresponding to the fourth; the opposite direction, 4:3 ≡ 4:6, where 4:3 and 4:6 are in the same pitch class, is a move to a tone higher by a *fifth* (7). The 4:5 direction moves to a tone higher by a *major 3rd* (4) and its opposite, 5:4 ≡ 5:8 results in a tone higher by a *minor 6th* (8). The direction 5:3 ≡ 5:6 results in a tone higher by a *minor 3rd* (3) while its opposite 3:5 is a *major 6th* (9). The ratios invert for the system of relative frequencies.

Figure 10. The system of musical vectors



Beginning at D one can steer by this vector system to any tone in the system and derive the ratio corresponding to this tone by multiplying the ratios along this path, or one can determine its digital root by adding the digital roots corresponding to the vector directions. Since multiplication and addition are equivalent operations, this table may have served as a kind of *proto-log table* enabling rapid computations to be carried out as far back as the 3rd millennium B.C. For example, in

Fig 9b a move from D to e_b by the route: D→ G→e_b results in the ratio: 3/4 x 5/8 = 15/32 or, when placed into the octave, 15:16, the Just semitone ratio. The sum of the digital roots along this path are: 0 + 5 + 8 = 13 which has digital root 1 and according to Fig. 2, corresponds to e_b. The same digital root results from any other path between the same two tones.

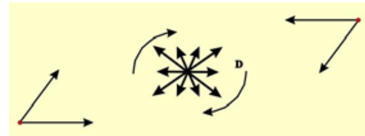


Fig. 11. Five pairs of symmetric tones as "sparks of light

The digital roots of all tones with respect to D are shown in Fig. 9d. Notice that five reciprocal tone pairs are symmetrically placed with respect to D according to the pattern of Fig. 11 in which ten "sparks of light" appear to be emanating from the fundamental tone. Fig . 9d shows that these tone pairs are complements or inverse tones in the octave since their digital roots sum to 12. Only the tritone does not have a mate in this picture. The inverse tone ratios of these complementary tones follow from Fig. 9c. For example, 450/360 = 5/4 while 576/720 = 4/5; 675/360 = 15/8 while 384/720 = 8/15; etc. show that the ancient musical scale is based on a balance of opposites which along with self-limitation were Plato's primary concerns. These ratios invert when the tones are expressed in terms of relative frequency. When considering the integers in Fig. 9c as relative frequencies, complementary tones switch places. The ratios of the twelve tones of the Just scale are listed in Table 3 along with the common names of the intervals and their digital roots.

Table 3. Tones of the Just scale

0	D	unison	1	0	0	0	12	D'	octave	1/2	24	1224	0
1	e_b	semitone	15/16	12	122	22	11	c♯	major 7th	8/15	-12	1088	-22
2	E	2^{nd}	8/9	4	204	0	10	C	minor 7^{th}	9/16	-4	996	0
3	f	minor 3rd	5/6	16	316	22	9	b	major 6th	3/5	-16	884	-22
4	f♯	major 3rd	4/5	-14	386	-22	8	b_b	minor 6th	5/8	14	814	22
5	G	fourth	3/4	-2	498	0	7	A	fifth	2/3	-2	702	0
6	g♯	tritone	32/45	-10	590	-22							

The tones in Table 3 are juxtaposed so that complementary or symmetric tones on the tone circle are adjacent. The major and minor thirds, fourth and fifth are the consonant intervals, and these can be represented by ratios of the first six integers: 1,2,3,4,5,6. Deviations in cents of the Just scale from the equal temperament appear in columns 5 and 12 (also the numbers in parentheses in Fig. 9d) while deviations in cents of the Just scale from the scale of spiral fifths are listed in columns 7 and 14. Frequency values of tones of the Just scale in cents are found in columns 6 and 13.

As a second example of how this vector system operates, C→G→D→A→E→c is a route from C to c in Fig. 9b with ratios: 2/3 x 2/3 x 2/3 x 5/8 = $\frac{80}{81\times 8}$. Multiplying by the octave, 8, yields 80:81, corresponding to 22 cents on the logarithmic scale. Assigning C the value 0 and summing the digital roots along this path: 0 + 7 + 7 + 7 + 3 = 24 with digital root 0. Of course, C and c represent the same tone with C from the scale of spiral fifths and c from the Just scale as shown in Fig. 9b and c where the ratio of C to c: 405:400 = 81:80. Similarly, the ratio of E:e = 640:648 = 80:81. Just as E for the scale of spiral fifths overshoots its equal-tempered value by 6 cents, the Just scale value e has opposite defect and undershoots by 14 cents (see the Appendix).

Notice that a similar relationship exists, as it did for the Nicomachus Table (see Table 2), for the hexagon of integers surrounding 720/360 in Fig. 9c where tones in row 3 are arithmetic means of the pair of tones that brace them from below in row 2, while tones in row 1 are the harmonic mean of the pair of tones that brace them from above in row 2. This is a consequence of major an minor thirds being symmetrically placed within the musical fifth.

8. The Tetrapolarity of the Musical Scale Revisited

Ancient obsession with inverse symmetry and numerical exactitude included a mastery of Spiral Fifths and Just tuning systems through all thirteen pitch classes in each, with only three tones (G-D-A) in common. A *tetrapolarity* results--with both vertical and horizontal planes of reflection—.indifferent to right/left and up/down. McClain suggests that the three tones belonging to both systems were encoded in the name of David from the bible using Hebrew gematria, where letters are given integer values, as 4.6.4 which can be read 2:3 vs 3:2, rising and falling fifths.

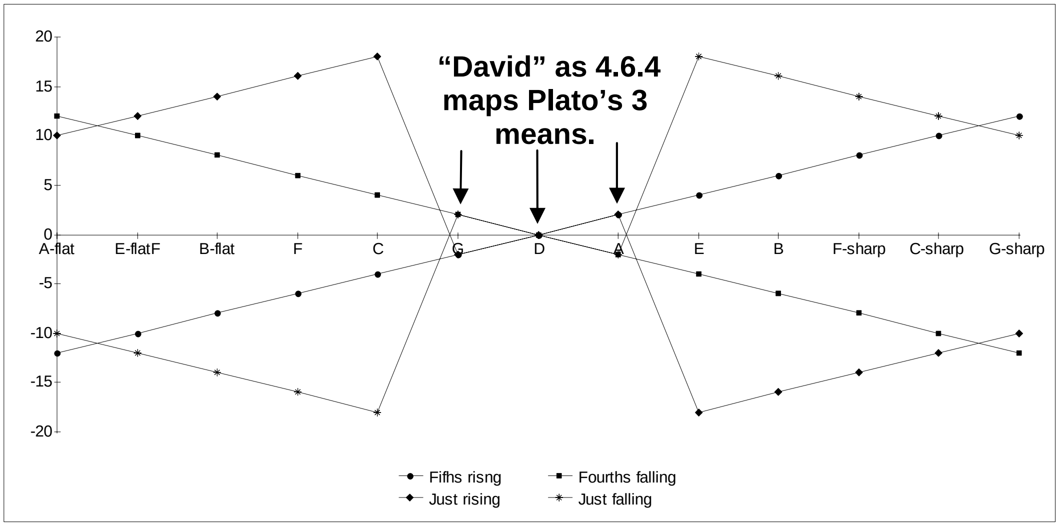


Figure 12. The tetrapolarity of the musical scale. Deviations in cents between the Just scale and spiral fifths and the equal-tempered scale

In Fig. 12 the tones of the thirteen pitch classes are plotted for the Just scale and the scale of spiral fifths against their deviations from the equal-tempered scale (see the numbers in parenthesis in Fig. 9d). A "straight line" geometric progression of rising perfect fifths 2:3 and/or falling perfect fourths 4:3 (increasing or decreasing 2 cents per fifth) allows their "Spiral" results to be imagined as a near-approximation to equal temperament, graphed as a reference on the central horizontal axis. Five other pitch classes in each direction beyond the three, G,D,A, of the Greek scale, held in common, display Just alternate thirds of 5:4 at contrary angles to their spiral fifth relatives sharing the same names, but parallel to opposites above and below. In the four outer corners, the two systems converge at the tritone--G-sharp with a-flat, and A-flat with g-sharp-- more closely than pure and tempered musical fifths at G and A. Vertical dimensions are grossly exaggerated to make them visible in modern "cents." Notice that the values of the Just scale and scale of spiral fifths have opposite deficiencies and differ from each other by 22 cents, the syntonic comma. This "dragonfly" projection offers a mainly "subliminal" window on processing acoustical events in the soul.

9. The square root of 2

Computation of an approximation to the square root of 2 was important for the astronomy, architecture, music, and mathematics of ancient civilizations. Although there was no concept of an irrational number, it was understood that the tritone of the musical scale provided an approximation to $\sqrt{2}$ where relative frequency of the equal-tempered tritone is $\sqrt{2}$ in an ideal sense. McClain has traced how this quest for an approximation to $\sqrt{2}$ provided a rich source of metaphors for the great sacred and philosophical works of antiquity including the Rg Veda, Hebrew Scriptures, and dialogues of Plato.

All tones of the ancient musical scales with digital roots of 6 provide approximations to $\sqrt{2}$. For example, A$_b$ and G# from the scale of spiral fifths provides two values: 729/512 = 1.423… and 1024/729 = 1.404…, where 512 and 1024 are powers of 2 representing unity. The cornerstone of the McClain Table at g# = 512 provides the ratio to the fundamental at 720 as 720/512 = 45/32 = 1.406… while its inverse at a$_b$ is 64/45 = 1.422.

10. The McClain Table and the modern theory of music

Western music is built around a chord structure of the fundamental, major third, and fifth known as a *major triad* and the fundamental, minor third and fifth known as the *parallel minor triad*. Table 3b when extended, exhibits major and minor triads in every key with the major triad as the downward triangle of tones consisting of a pair of tones from row 2 bracing the tone from below in row 1 while the minor triad is the pair of tones from row 2 bracing a tone from row 3. For example, in the key of C the major triad is: C,e,G while the parallel minor is: C,e$_b$,G. Notice that a similar relationship exists, as it did for the Nicomachus Table (see Table 2), for the hexagon of integers surrounding 720/360 in Fig. 9c where tones in row 3 are arithmetic means of the pair of tones that brace them from below in row 2, while tones in row 1 are the harmonic mean of the pair of tones that brace them from above in row 2. This is a consequence of major an minor thirds being symmetrically placed within the musical fifth.

11. Discussion and Conclusions

We have shown that embodied within the ancient musical scale are primitive notions of modular arithmetic base 12, vector analysis, logarithms, symmetry, and the interplay between the geometric, arithmetic and harmonic means all accomplished within the realm of integers. McClain suggests that his table goes back to ancient Sumeria in the 3rd millennium B.C. In fact at one point in the Gilgamesh legend Gilgamesh says that “a three strand rope cannot be broken” possibly referring to this matrix. Although there is no direct evidence of the McClain matrix appearing in ancient cultures, McClain has found many highly suggestive archaeological clues pointing strongly in this direction. Thus we see that ancient musical scales through the ages have taken on numerous meanings in both scientific and cultural contexts, the result of its strong underlying mathematical structure.

Appendix

The logarithmic measure of the length of an interval in terms of cents allots 1200 cents to the octave with each semitone of the equal-tempered scale given the value 100 cents. To express the length of an interval corresponding to relative frequency r in units of cents:

$$r_{cents} = 1200 \log_2 r = 1200 \times 3.322 \log_{10} r$$

Example: For a major 3rd , r = 5:4 so $r_{cents} = 1200 \times 3.322 \log_{10} 5/4 = 386.3$ cents which deviates from the equal-tempered value of 400 by approximately -14 cents.

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