Electric Discharge Sintering: A Mathematical Model

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Abstract

In this paper we mathematically model the densification of metallic powders and the sintering of ceramic powders by electric discharge. The ordinary and partial differential equations governing these processes are the same with the exception of the effective electrical conductivity. This function is a monotonically decreasing (increasing) function of temperature for the metallic (ceramic) powders. We employ asymptotic methods to approximate the solution to these equations in the limit as $\epsilon \to 0$, where $\epsilon$ is the ratio of the discharge to diffusion time scales. We find on the shortest time scale that the temperature, voltage, and density satisfy a system of nonlinear, coupled ordinary equation. We solve these and find the relationship between the temperature and density, as functions of the input energy.

The results on the short or discharge time scale do not take into account diffusion and heat loss into the surrounding medium. These occur on a much longer time scale which we identify and exploit to deduce a new approximation. On this time scale the capacitor has no more energy to deposit into the powder. The temperature relaxes to that of its surroundings and the density increases to its final value. Our results show the functional relationship between the final density and the initial energy stored in the capacitor, as well as the initial density of the powder.
1. Introduction

Over the last several years researchers have used electric discharges to sinter ceramic materials and to compact metal powders [1-3]. The experimental configurations have cylindrical dies, which contain a powdered material. The green powders are compacted with punches which also act as electrodes through which a short, powerful current pulse is passed. The source of the current is a capacitor bank that is charged to a few hundred volts and in some applications to metal powders, 20 kilovolts [2]. In all cases the dies are surrounded by a thermally insulating layer such as alumina, $\text{Al}_2\text{O}_3$. The short current pulse passing through the powder rapidly heats it to a temperature where the ceramic powder sinters or the metallic powders melt together. A schematic of the experimental set-up is shown in Figure 1.

In this paper we introduce a mathematical model to describe this sintering or densification process. Specifically, our model takes the form of a heat equation, a Laplace equation describing the electric potential, and an evolution equation describing the densification of the powder. All of these equations are coupled and non-linear. In addition, there is a differential equation relating the change in the capacitor voltage across the punches to the current flowing through them. Implicit in this mathematical description is the assumption that the powdered material can be treated as a continuum.

Our mathematical model is very similar to the one developed and analyzed to describe temperature surges in thermistors [4]. There are two obvious differences; the first is the addition of the evolution equation for the density and the second, the equation for the capacitor voltage. In Reference 4 the external circuit was a battery and resistor in series with the thermistor. More subtle and important differences are the time scales. These produce different physical balances and hence different physics. For example, the time scale describing the source in our heat equation is commensurate with the capacitive discharge time scale; both are very small compared with the diffusion time scale along the axis of the experimental device. In the thermistor model the source and diffusion time scales are of the same order; the result is a nonlinear, non-local heat equation.

Denoting the ratio of the source to diffusion time scales by $\epsilon$ we perform an asymptotic analysis of the solutions to our equations as $\epsilon \to 0$. Assuming that the sintering dynamics occur on the same time scale as the source, our analysis shows that the temperature, voltage, and density satisfy a system of three, non-linear ordinary differential equations to leading order. These are analyzed and the qualitative features of the solutions are described. In particular the capacitor voltage decays to zero, the density reaches a maximum, and the temperature achieves a maximum, too; all depend in a critical fashion on the initial energy stored in the capacitor. These results are only valid on the source time scale which we call the heating
regime. On the much longer convective time scale, the thermal energy decays as the sample loses heat to the surrounding medium through the metallic punches. In this span of time the temperature of the sample decays to that of the surrounding environment. On this time scale, diffusion plays a minor role yielding an $O(\epsilon^2)$ spatial variation along the axis of the experimental device.

If the sintering or the densification dynamics occur on the convective time scale, the results are more complicated. In the heating regime the voltage again decays to zero, the density remains at its initial value, and the temperature achieves a maximum. This maximum depends upon the initial density and the initial energy stored in the capacitor. These results are again only valid in the heating regime. On the convective time scale the density increases to its final value and the temperature decays to that of the surrounding medium. The final density depends critically upon its initial value and the energy stored in the capacitor. Our results explicitly show this dependence.

We close this introduction by giving a brief outline of the paper. Section 2 contains the mathematical formulation of our model. In Section 3 a careful dimensional analysis is given identifying the relevant time scales for our problem. Section 4 contains our asymptotic analyses in both the heating and convective regimes. Here the time scale associated with the sintering dynamics is taken to be commensurate with the source time scale. In Section 5 we present our analysis for the slow densification case. Here the sintering time scale is commensurate with the convective time scale. Finally, Section 6 contains our conclusions.

2. Problem Formulation

In Figure 1 we show a schematic of an electro-discharge consolidation experiment. The powder specimen to be densified is contained in an electrically non-conducting cylinder, of radius $R_0$ and height $L$, surrounded by a layer of alumina. Two punches compress the powder through an external force $F$. The punches also serve as electrodes connected to a capacitor charged to an initial voltage $V_0$ which can be on the order of a few hundred to several thousand volts. When the switch is closed the capacitor discharges and creates a current, $I$, which flows through the powder specimen. This rapidly heats the powder and causes it to sinter into a dense material.

We model the compacted powder as a continuum with a temperature $T$, defined inside the cylindrical region $0 < Z < L$, $0 < R < R_0$, satisfying

$$\frac{\partial}{\partial t}[\rho C_p T] = \nabla \cdot (K \nabla T) + \sigma(T)|\nabla \Phi|^2$$  \hspace{1cm} (1a)

where $\rho$ is an averaged density, $C_p$ the thermal capacity, $K$ the thermal conductivity, and $\sigma$ the electrical conductivity of the powder, which all can depend upon the
temperature and the external force $F$, with the exception of $C_p$. In this equation $\Phi$ is the electric potential, caused by the discharging capacitor, which satisfies in the same region

$$\nabla \cdot (\sigma \nabla \Phi) = 0.$$  \hspace{1cm} (1b)

Denoting the voltage across the capacitor as $V(t)$ and the current flowing from it as $I(t)$ we have from Kirchoff's current law

$$C \frac{dV}{dt} = -I$$ \hspace{1cm} (1c)

where $C$ is the capacitance.

We shall now determine a relationship between the current, the temperature, and the potential. First, we recall that the current density is defined by $J = \sigma \nabla \Phi$, where $\nabla \Phi$ is the electric field. Next, we assume that the $R$ component of this current density vanishes at $R = R_0$, that is, $\sigma \frac{\partial \Phi}{\partial R} = 0$ there. This assumes no current flows out of the powder into the insulation. Next, we integrate (1b) over the cylindrical domain $Z_0 < Z < Z_1$, $0 < R < R_0$, where $Z_0$ and $Z_1$ are arbitrary,
apply the divergence theorem, and use $\Phi_R = 0$ at $R = R_0$ to obtain
\[
\iint_{Z=Z_1} \sigma \frac{\partial \Phi}{\partial Z} dX dY = \iint_{Z=Z_0} \sigma \frac{\partial \Phi}{\partial Z} dX dY.
\]
This shows that the integral of $\sigma \Phi_Z$, the $Z$ component of the current density, is independent of $Z$, and hence a constant. This constant is the current in (1c) and is defined by
\[
I = \iint_{\Omega} \sigma \frac{\partial \Phi}{\partial Z} dX dY \quad (1d)
\]
where $\Omega$ is an arbitrary circular cross-section of the powder sample.

Finally, to close these equations we must model the evolution of the powder density as it sinters. We assume the phenomenological equation
\[
\frac{\partial \rho}{\partial t} = G(T)[\rho_1 - \rho] \quad (1e)
\]
where the reaction rate $G$ depends upon temperature and $\rho_1$ is the bulk density for a single species powder, or an average bulk density of an multi-species powder, qualitatively describes this evolution [5].

The boundary conditions we apply are
\[
\frac{\partial T}{\partial R} = 0, \quad \frac{\partial \Phi}{\partial R} = 0, \quad R = R_0, \quad 0 < Z < L, \quad (2a)
\]
where the first assumes a perfect thermal insulation and the second is a restatement that no current flows radially out of the sample,
\[
\Phi(X, Y, 0, t) = 0, \quad \Phi(X, Y, L, t) = V(t), \quad (2b)
\]
which relates the potential to the capacitor voltage, and
\[
K \frac{\partial T}{\partial Z} - h(T - T_A) = 0, \quad Z = 0, \quad K \frac{\partial T}{\partial Z} + h(T - T_A) = 0, \quad Z = L, \quad 0 < R < R_0. \quad (2c)
\]
Here, we thermally model the punches by a linear Newton’s law of cooling where $T_A$ is the ambient temperature of the surrounding air and $h$ is the effective heat transfer coefficient. This boundary condition removes the need to study heat diffusion in the punch and ultimately heat convection from the punch into the surrounding medium. It is valid when the mass of the punch is small and its thermal conductivity is large. Such are the cases in References 1-3.

Finally, the initial conditions required are
\[
T = T_A, \quad 0 < R < R_0, \quad 0 < Z < L, \quad V(0) = V_0, \quad \rho(0) = \rho_0(F) \quad (2d)
\]
where $V_0$ is the initial voltage of the charged capacitor and $\rho_0(F)$ is the initial density of the powder which depends upon the forces on the punches.
3. Nondimensional Analysis

We begin this section by introducing the dimensionless dependent and independent variables

\[ u = \frac{T}{T_A} - 1, \quad \phi = \frac{\Phi}{V_0}, \quad v = \frac{V}{V_0}, \quad i = \frac{I}{\sigma_A V_0 L}, \quad w = \frac{\rho}{\rho_1} \quad (3a) \]

and

\[ \tau = \frac{t}{\theta_S}, \quad x = \frac{X}{L}, \quad r = \frac{R}{L} \quad (3b) \]

respectively. Here and henceforth, the subscript \(A\) denotes that a function is evaluated at the ambient temperature and \(\theta_S = C_p \rho / \sigma_A V_0^2\) is the time scale associated with the capacitive source. We also define the dimensionless functions

\[ k(u) = \frac{K(T)}{K_A}, \quad f(u) = \frac{\sigma(T)}{\sigma_A}, \quad g(u) = \frac{G(T)}{G_A}. \quad (3c) \]

Introducing (3a-3c) into (1a-1e) and combining (1c) and (1d), we find that our dimensionless equations are

\[ \frac{\partial [wu]}{\partial \tau} = \epsilon \nabla \cdot (k \nabla u) + f(u) |\nabla \phi|^2, \quad 0 < z < 1, \quad 0 < r < r_0 = \frac{R_0}{L}, \quad \tau > 0, \quad (4a) \]

\[ \nabla \cdot (f \nabla \phi) = 0, \quad 0 < z < 1, \quad 0 < r < r_0, \quad \tau > 0 \quad (4b) \]

\[ \frac{dv}{d\tau} = -\lambda \int \int_{r<r_0} f(u) \frac{\partial \phi}{\partial z} \, dx \, dy, \quad (4c) \]

\[ \frac{\partial w}{\partial \tau} = \gamma g(u)(1 - w). \quad (4d) \]

Here we have defined three new parameters, each a ratio of time scales. The first is \(\lambda = \theta_S / \theta_C\), where \(\theta_C = C / \sigma_A L\) is the capacitive time scale. The second is \(\gamma = \theta_S G_A\). We assume here that both these parameters are order one quantities. On the other hand, the third parameter \(\epsilon = \theta_S / \theta_Z \ll 1\) where \(\theta_Z = \rho_0 C_p / K_A L^2\) is the diffusive time scale in the \(Z\) direction. That is, we are assuming that heat diffuses much more slowly than the source time scale. The corresponding boundary conditions become

\[ \frac{\partial u}{\partial r} = 0, \quad \frac{\partial \phi}{\partial r} = 0, \quad r = r_0, \quad 0 < z < 1, \quad (5a) \]

\[ \phi(x, y, 0, \tau) = 0, \quad \phi(x, y, 1, \tau) = v(t), \quad 0 < r < r_0, \quad (5b) \]
\[ k(u) \frac{\partial u}{\partial z} - Bu = 0, \quad z = 0, \quad k(u) \frac{\partial u}{\partial z} + Bu = 0, \quad z = 1. \quad (5c) \]

In equation (5c) the Biot number is defined by \( B = \theta Z/\theta_{\text{con}} \) where \( \theta_{\text{con}} = \rho_1 C_p L/h \) is the convective time scale at which the sample cools. The Biot number is also a small parameter for \( L \sim \) several centimeters. We relate it to \( \epsilon \) by \( \beta = (\theta Z/\theta_{\text{con}})(\theta Z/\theta_S) = O(1) \). Finally, the dimensionless initial conditions are

\[ u = 0, \quad 0 < r < r_0, \quad 0 < z < 1, \quad v(0) = 1, \quad w(0) = \frac{\rho_0(F)}{\rho_1}. \quad (5d) \]

Equations (4-5) constitute our dimensionless initial, boundary value problem.

## 4. Analysis

This section contains two parts. In the first we analyze the evolution of \( u, v, \) and \( w \) on a time scale where \( \tau = O(1) \). In this time frame diffusion and convective heat losses play a minor role as the powder sample rapidly heats. We shall provide a leading order asymptotic analysis of our problem. This result becomes nonuniform for large \( \tau \) when diffusion and convection become important. The analysis on a much larger time scale is the subject of the second part of this section.

### 4.1 The Heating Regime.

We begin our analysis with the observation that the boundary data and initial conditions are all independent of \( r \) and the angle \( \theta \). Thus, we shall seek a solution of Equations (4)-(5) that depends only upon \( z \) and \( \tau \). Accordingly, Equations (4a-c) become

\[ \frac{\partial [uw]}{\partial \tau} = \epsilon \frac{\partial}{\partial z} (k \frac{\partial u}{\partial z}) + f(u) \frac{\partial \phi}{\partial z}^2, \quad 0 < z < 1, \quad \tau > 0, \quad (6a) \]
\[ \frac{\partial}{\partial z} (f(u) \frac{\partial \phi}{\partial z}) = 0, \quad 0 < z < 1, \quad \tau > 0, \quad (6b) \]
\[ \frac{dv}{d\tau} = -\Lambda f(u) \frac{\partial \phi}{\partial z}, \quad (6c) \]

where \( \Lambda = \pi r_0^2 \lambda \). Equation (4d) remains the same as do the boundary and initial equations, except now (5c) is replaced by

\[ k(u) \frac{\partial u}{\partial z} - \beta \epsilon u = 0, \quad z = 0, \quad k(u) \frac{\partial u}{\partial z} + \beta \epsilon u = 0, \quad z = 1. \quad (6c) \]

We now exploit the smallness of the parameter \( \epsilon \) and seek an asymptotic solution of the form

\[ u \sim u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots \quad (7a) \]
\[ \phi \sim \phi_0 + \epsilon \phi_1 + \epsilon^2 \phi_2 + \cdots \] (7b)
\[ v \sim v_0 + \epsilon v_1 + \epsilon^2 v_2 + \cdots \] (7c)
\[ u \sim w_0 + \epsilon w_1 + \epsilon^2 w_2 + \cdots . \] (7d)

Inserting these expansions into our equations, boundary and initial equations we find to leading order that \( u_0, v_0, \phi_0, \) and \( w_0 \) satisfy

\[ \frac{\partial[u_0 w_0]}{\partial \tau} = f(u_0)\left| \frac{\partial \phi_0}{\partial z} \right|^2, \quad \frac{\partial u_0}{\partial z} = 0, \quad z = 0, 1 \quad u_0(z, 0) = 0, \] (8a)

\[ \frac{\partial}{\partial z} (f(u_0) \frac{\partial \phi_0}{\partial z}) = 0, \quad \phi_0(0, \tau) = 0, \quad \phi(1, \tau) = v(\tau), \quad \tau > 0, \] (8b)

\[ \frac{dv_0}{d\tau} = -\Lambda f(u_0) \frac{\partial \phi_0}{\partial z}, \quad v_0(0) = 1, \] (8c)

\[ \frac{\partial w_0}{\partial \tau} = \gamma g(u_0)(1 - w_0), \quad w_0(0) = \frac{\rho_0(F)}{\rho_1}. \] (8d)

We now integrate (8b) twice and employ the boundary conditions at \( z = 0 \) and \( z = 1 \) to find

\[ \phi_0 = v_0(\tau) \frac{Q(z)}{Q(1)}, \quad Q(z) = \int_0^z \frac{1}{f(u_0)} dz'. \]

Now from this result we observe that \( \frac{\partial \phi_0}{\partial z} = v_0/f(u_0)Q(1) \) which is independent of \( z \). Thus, the right hand side of (8a) is independent of \( z \), too, and this implies that \( u_0 \) and \( w_0 \) are functions of \( \tau \) alone. This observation yields

\[ \phi_0 = z v_0(\tau). \] (9)

and upon inserting this result into equations (8a), (8c),and (8d) we arrive at the third order system of ordinary differential equations

\[ \frac{d[u_0 w_0]}{d\tau} = f(u_0)v_0^2, \quad u_0(0) = 0 \] (10a)

\[ \frac{dv_0}{d\tau} = -\Lambda f(u_0)v_0, \quad v_0(0) = 1, \] (10b)

\[ \frac{dw_0}{d\tau} = \gamma g(u_0)[1 - w_0], \quad w_0(0) = \frac{\rho_0(F)}{\rho_1}. \] (10c)
We next divide equation (10a) by (10b), integrate this result, and use the initial conditions to obtain the first integral

\[ v_0^2 = 1 - 2\Lambda u_0 w_0. \]  

Combining this with equation (10a), differentiating the left hand side, and using (10c) to eliminate the derivative of \( w_0 \), we find

\[
\frac{du_0}{d\tau} = \frac{1}{w_0} \left\{ (1 - 2\Lambda u_0 w_0) f(u_0) - \gamma u_0 g(u_0)(1 - w_0) \right\} \quad w_0(0) = \frac{\rho_0(F)}{\rho_1},
\]

and this combined with (10c) yield a second order system, which describes the leading order heating process.

We begin by noting that \( f(u) \) is a positive function; it may increase or decrease with \( u \) depending upon the powdered material. If we assume for the moment that \( g(u) \) is also a positive function, then a simple phase plane analysis of our system reveals that \( u_0 \) and \( w_0 \) both evolve from their initial conditions \((0, \rho_0/\rho_1)\) to their steady state values \((u_0^*, w_0^*)\), where \( u_0^* = 1/2\Lambda \) and \( w_0^* = 1 \), as \( \tau \to \infty \). Coupling this information with equation (10b) it is easy to deduce that \( v_0 \) decreases monotonically from its initial value of 1 to its steady state \( v_0^* = 0 \) as \( \tau \to \infty \). It is interesting to note that the final state is independent of the dimensionless electrical conductivity \( f(u) \) and the dimensionless reaction rate \( g(u) \). These functions will of course affect the rate at which \( u_0, w_0, \) and \( v_0 \) approach their steady state values.

A more physically realistic assumption about the function \( g(u) \) is to require it to vanish for \( u < u_T \) and increase monotonically and smoothly for \( u > u_T \). Here \( u_T \) denotes a dimensionless threshold temperature. This characterization is equivalent to assuming that no sintering or densification occurs below this temperature. For simplicity and concreteness of presentation we take

\[
g(u) = \begin{cases} 
0, & 0 \leq u < u_T \\
\tanh^2(\nu(u - u_T)), & u_T \geq u,
\end{cases}
\]

where \( \nu \) controls how rapidly \( g \) switches. For this choice of \( g \), or any other smooth switching function, there will be two cases to consider: Low power where \( \frac{1}{2\Lambda} < u_T \) and high power where \( \frac{1}{2\Lambda} > u_T \).

We begin by considering the high power case. Figure 2a shows two trajectories for different values of \( w_0(0) = \rho_0/\rho_1 \). Both trajectories remain horizontal, where \( w_0 \) is essentially fixed at its initial value, until they cross the threshold \( u_T \) where \( w_0 \) increases with \( u_0 \). This continues until the right hand side of (12) changes sign and \( u_0 \) decreases with increasing \( w_0 \). The functions \( u_0 \) and \( w_0 \) then approach the
steady state values, \( u_0^* = 1/2\Lambda \) and \( w_0^* = 1 \), respectively. The function \( v_0 \) converges to \( v_0^* = 0 \) as can deduced from (11).

We next consider the case of low power where the dynamics are more interesting. Figure 2b also shows two trajectories for different values of \( w_0(0) \). For the larger initial condition the trajectory never crosses the threshold temperature \( u_T \), remains horizontal, and approaches the hyperbola \( w_0 = 1/2\Lambda u_0 \) as \( \tau \to \infty \). There \( w_0^* = w_0(0) \) and \( u_0^* = 1/2\Lambda w_0(0) \). The trajectory for the lower initial condition remains horizontal until it crosses \( u_T \) where \( w_0 \) increases with \( u_0 \). This continues until the right hand side of (12) changes sign and \( u_0 \) decreases with increasing \( w_0 \). The trajectory approaches its steady state on the hyperbola where \( u_0^* = u_T \) and \( w_0^* = 1/2\Lambda u_T \). Again, we deduce from (11) that \( v_0^* = 0 \).

In closing this subsection we express the final state of the system in terms of its dimensional quantities. We first consider the high power case \( 1/2\Lambda > u_T \), or in dimensional terms

\[
\frac{1}{2} \frac{CV_0^2}{\rho_1 C_p V_S} > T_T - T_A.
\]

Here, \( T_T \) is the threshold temperature, \( V_S \) is the volume of the sample, and \( \frac{1}{2}CV_0^2 \) is the total energy initially stored in the capacitor. The steady state values of the density and temperature for this case are

\[
\rho^* = \rho_1 w_0^* = \rho_1
\] (13a)
and

\[ T^* = T_A (1 + u_0^*) = T_A + \frac{1}{2} \frac{C V_0^2}{\rho_1 C_p V_S}, \]  

(13b)

respectively. The result (13a) is intuitively obvious, but (13b) is not. It states that the increase in the system's temperature above its ambient value depends very little upon the physical properties of the powder sample. In fact, it only depends upon the volume of the sample, the final density \( \rho_1 \), and the total energy initially stored in the capacitor.

In dimensional terms the low power case occurs when

\[ \frac{1}{2} \frac{C V_0^2}{\rho_1 C_p V_S} < T_T - T_A. \]

If the initial density \( \rho_0 \) is sufficiently large, then, according to Figure 2b, the density remains constant and the temperature increases to \( \frac{1}{2} \Lambda w_0(0) \). In terms of dimensional quantities the steady state density and temperature are

\[ \rho^* = \rho_1 w_0(0) = \rho_0 \]  

(13c)

and

\[ T^* = T_A + \frac{1}{2} \frac{C V_0^2}{\rho_0 C_p V_S}, \]  

(13d)

**Figure 2b. Phase Plane: Low Power**
respectively. The only difference between this final temperature and the one given for higher power (13b) is the presence of \( \rho_0 \) in the denominator. Finally, if the initial density is low enough, then the system evolves along the lower trajectory shown in Figure 2b. The steady state density and temperatures are now given by

\[
\rho^* = \rho_1 \frac{1}{2\Lambda u_T} = \rho_1 \left\{ \frac{CV_0^2}{\rho_1 C_P V_S} \right\} \frac{1}{T_T - T_A} \tag{13e}
\]

and

\[
T^* = T_A (1 + u_T) = T_T, \tag{13f}
\]

respectively, where it must be recalled that the low power constraint implies \( \rho^* < \rho_1 \).

4.2 The Cooling Period.

The analysis in the preceding section was concerned with times on the order of \( \theta_S \), the source time scale. Convective heat losses, modeled by Newton’s Law of Cooling, are unimportant here. However, they do become important on a much longer time period where the powder sample cools back to the ambient temperature, \( T_A \). We can analyze this behavior by introducing the long time scale \( \bar{t} = \epsilon^2 \tau \). In dimensional terms this \( \bar{t} = t/\beta \theta_{con} \), that is we are now considering the temperature evolution on the convective time scale. Inserting this change of variable into (6a) gives

\[
\epsilon \frac{\partial}{\partial \bar{t}}(uw) = \frac{\partial}{\partial z} \left( k \frac{\partial u}{\partial z} \right) + \frac{1}{\epsilon} S(z, \bar{t}), \quad 0 < z < 1, \quad \bar{t} > 0, \tag{14a}
\]

where the source \( S \) is defined by \( S = f(u)|\frac{\partial \phi}{\partial z}|^2 \). The source term can be simplified by integrating (6b) twice and using the boundary conditions (5b) to obtain \( \phi = v(\tau)Q(z)/Q(1) \) where now \( Q(z) = \int_0^z \frac{1}{f(u)} \, dz' \). Differentiating this result with respect to \( z \) and replacing \( \tau \) by \( \bar{t}/\epsilon^2 \) we find that the source term becomes

\[
S(z, \bar{t}) = \frac{v^2(\bar{t}/\epsilon^2)}{f(u)Q^2(1)}. \tag{14b}
\]

Now, on the cooling time scale \( \bar{t} = O(1) \) which implies that the argument of \( v \) in (14b) is very large; alternatively \( \tau >> 1 \). But, for large values of \( \tau \) we have shown that \( v \sim v_0 \to 0 \), exponentially. Thus, the \( S/\epsilon \) is negligible and (14a) becomes a homogeneous diffusion equation on this time scale. Also, by inserting \( \tau = \bar{t}/\epsilon^2 \) into equation (14) and letting \( \epsilon \to 0 \) we obtain \( \gamma g(u) \{1 - w\} = 0 \). This is satisfied in both the high and low power cases. In the former, \( w \sim w_0^* = 1 \). In the latter, \( u \sim u_0^* \leq u_T \) so that \( g = 0 \).
Taking the facts that $S = 0$ and $w = w_0^*$ into consideration we find that $u$ satisfies

$$
\epsilon w_0^* \frac{\partial}{\partial \bar{t}} (u) = \frac{\partial}{\partial z} (k \frac{\partial u}{\partial z}), \quad 0 < z < 1, \quad \bar{t} > 0.
$$

(15a)

In addition to this equation $u$ must still satisfy the boundary conditions (6c) at $z = 0$ and $z = 1$. Finally, an initial condition must be prescribed to close the initial boundary value problem for $u$. Intuitively, this is given by the large $\tau$ value of $u_0$, namely,

$$
u|_{\bar{t}=0} = u_0^*.
$$

(15b)

We note here that the arguments of this paragraph can be put on more formal grounds by asymptotically matching the long time behavior of the solution on the heating time scale with the short time behavior of the solution on the cooling time scale.

We now proceed to find the behavior of $u(z, \bar{t})$ as $\epsilon \to 0$. As usual we take

$$
u \sim U_0(z, \bar{t}) + \epsilon U_1(z, \bar{t}) + \epsilon^2 U_2(z, \bar{t}) + \cdots
$$

and insert this expression into (15) and (6c). Equating to zero the coefficients of the powers of $\epsilon$ yields a sequence of initial boundary value problems. The leading order problem is

$$
\frac{\partial}{\partial z} (k \frac{\partial U_0}{\partial z}) = 0, \quad 0 < z < 1, \quad k \frac{\partial U_0}{\partial z} = 0, \quad z = 0, 1, \quad U_0(z, 0) = u_0^*.
$$

(16)

Integrating this equation once and applying the boundary conditions we find that $U_0(z, \bar{t}) = U_0(\bar{t})$, a function of $\bar{t}$ alone.

The first order correction $U_1$ satisfies

$$
\frac{\partial}{\partial z} (k \frac{\partial U_1}{\partial z}) = w_0^* \frac{d U_0}{d \bar{t}}, \quad 0 < z < 1, \quad U_1(z, 0) = 0
$$

(17a)

and the inhomogeneous boundary conditions

$$
\frac{\partial U_1}{\partial z} = \beta U_0, \quad z = 0, \quad \frac{\partial U_1}{\partial z} = -\beta U_0, \quad z = 1.
$$

(17b)

Integrating the equation in (17a) between $z = 0$ and $z = 1$, and applying the boundary conditions in (17b) we find that $U_0$ satisfies the ordinary differential equation

$$
\frac{d U_0}{d \bar{t}} = -2 \frac{\beta^2}{w_0^*} U_0.
$$

The solution of this equation satisfying the initial condition in (16) is

$$
U_0 = u_0^* e^{-2 \frac{\beta^2}{w_0^*} \bar{t}}.
$$

(18)
The result given in (18) is the leading order approximation to $u$ on the cooling time scale. For the high power case the result in terms of dimensional quantities becomes

$$T \sim T_A + \frac{1}{2} \frac{C V_A^2}{\rho_1 C_p V_S} e^{-2t/\theta_{con}}$$

which explicitly shows that the temperature decays back to its ambient value on the cooling time scale $\theta_{con}$. This scale depends upon the punch's ability to lose heat to its surroundings. For the low power, high initial density case we have

$$T \sim T_A + \frac{1}{2} \frac{C V_A^2}{\rho_1 C_p V_S} e^{-2\frac{w_0 t}{\rho_0}}$$

and for the low power, low initial density,

$$T \sim T_A + \frac{1}{2} \frac{C V_A^2}{\rho_1 C_p V_S} e^{-2\frac{w_0^* t}{\rho_0}}.$$  

In Equation (19c) the final density $\rho^*$ is given in (13e). It is interesting to note that the low power cases decay more quickly to the ambient temperature than the high power scenario.

5. Slow Sintering

Up until now, we have taken the dimensionless parameter $\gamma$, appearing in equation (4d), as an $O(1)$ quantity. That is, we have assumed that sintering occurs on the source time scale, $\tau$. In this section we investigate the case where this process occurs on the much longer cooling time scale, $\bar{\tau}$. This assumption implies that $\gamma = \gamma_0 \epsilon^2$, where $\gamma_0$ is now an order one quantity.

The heating regime analysis precedes as before, the only difference being that Equation (10c) is replaced by

$$\frac{dw_0}{d\tau} = 0, \quad w_0(0) = \frac{\rho_0(F)}{\rho_1},$$

from which it follows that $w_0 = \frac{\rho_0(F)}{\rho_1}$, that is, the density remains a constant on the source time scale. Equations (11) and (12) still remain the same. However, $u_0 \rightarrow u_0^* = \frac{1}{2A w_0}$ as $\tau \rightarrow \infty$. Thus, the final temperature in the heating regime depends upon $w_0$. In terms of dimensional quantities (13b) is now replaced by

$$T^* = T_A(1 + u_0^*) = T_A + \frac{1}{2} \frac{C V_A^2}{\rho_0(F)C_p V_S}.$$
Since $\rho_1 / \rho_0 > 1$, the final temperature is higher in this case, although increasing the external force $F$ on the punches diminishes the difference. Finally, we observe that $v_0$ still satisfies (11) and hence approaches zero as $\tau \to \infty$.

The cooling period analysis again proceeds as before. Equation (15) is now replaced by

$$
\epsilon \frac{\partial (uw)}{\partial \bar{t}} = \frac{\partial}{\partial z} \left( k \frac{\partial u}{\partial z} \right), \quad u|_{\bar{t}=0} = \frac{1}{2\Lambda w_0}
$$

and equation (4d) by

$$
\frac{dw}{d\bar{t}} = \gamma_0 g(u) (1 - w), \quad w|_{\bar{t}=0} = w_0 = \frac{\rho_0(F)}{\rho_1},
$$

where we have replaced $\gamma$ by $\gamma_0 \epsilon^2$ and changed the time variable to $\bar{t}$. The temperature $u$ still satisfies the boundary conditions contained in (6c).

We now expand both $u$ and $w$ in the asymptotic series $u \sim U_0 + \epsilon U_1 + ..$ and $w \sim W_0 + \epsilon W_1 + ..$, respectively, and perform the same analysis as in Section 4.2. We find to leading order that $U_0$ and $W_0$ satisfy the ordinary differential equations and initial conditions

$$
\frac{d}{dt} W_0 = \gamma_0 g(U_0) (1 - W_0), \quad W_0(0) = \frac{\rho_0(F)}{\rho_1}.
$$

$$
\frac{d}{dt} U_0 = -\frac{1}{W_0} \{2\beta U_0 + \gamma U_0 g(U_0) (1 - W_0)\}, \quad U_0(0) = \frac{1}{2\Lambda W_0(0)}
$$

Equation (23a) is the same as (10c) and (23b) is similar to (12). The difference in the latter is the new initial condition and the change in its right hand side.

We again have the high and low power cases to consider, $\frac{1}{2\Lambda} \geq u_T$ and $\frac{1}{2\Lambda} \leq u_T$, respectively. The phase plane with typical trajectories is shown in Figure 3a for the former case. These trajectories begin on the hyperbola $W = 1/2\Lambda U$ and flow to the left with $U_0$ decreasing and $W_0$ increasing. When this curve crosses $U_0 = u_T$ it becomes horizontal, $W_0$ remains constant, and $U_0$ approaches zero. The final state is then $U_0^* = 0$ and $W_0^* = \psi(W_0(0))$, where $\psi$ denotes the function mapping the initial data to its final state. This function is shown in Figure 3b for different values of $\Lambda$. Since $1/\Lambda$ is proportional to the power initially stored in the capacitor, i.e., $CV_0^2$, our results show that higher power levels produce higher densification.
Figure 3a. Phase Plane: High Power

Figure 3b. Final Density: High Power
The trajectories for the low power case are shown in Figure 4a, each again beginning on the hyperbola. If the initial density is large enough, the trajectory remains horizontal, $U_0$ monotonically decreases, and $W_0 = \rho_0/\rho_1$. If the initial density is sufficiently low, then $W_0$ initially increases and $U_0$ decreases until the threshold $U_0 = u_T$ is crossed. After this time $W_0$ remains fixed and $U_0$ approaches zero as $\bar{t} \to \infty$.

![Figure 4a. Phase Plane: Low Power](image)

The final density $W_0^*$ for the lower power case is also a function of the initial density $W_0(0)$, i.e., $W_0^* = \psi(W_0(0))$. This function is shown in Figure 4b for several values of $\Lambda$. Again higher power levels for a given initial density produce more densification. However for densities sufficiently large, the temperature is lower than the threshold and no sintering occurs. This is born out in the linear behavior of $\psi$ as a function of $\rho_0/\rho_1$. 
We have developed a model to describe a class of experiments that use an electric discharge to sinter or compact ceramic and metallic powders, respectively. Our mathematical description begins with the heat equation, Laplace’s equation, and an evolution equation describing the densification of the powder. All these equations are coupled and nonlinear, and have appended to them the appropriate initial and boundary conditions.

We have performed an asymptotic analysis to approximate the solutions of these equations. The appropriate small parameter is the ratio of the source to diffusion time scales. Our analysis shows that the leading order temperature, voltage across the sample, and relative density satisfy a third order system of nonlinear, ordinary differential equations. These are analyzed for a particular densification rate which possesses a temperature threshold, below which sintering or densification does not occur. In particular, we have identified a critical power below which no densification occurs. We have presented a physical interpretation of our results.

The results described above are only valid on the source time scale where diffusion and heat loss to the surrounding environment are unimportant. We have identified a much longer convective time scale on which the heated powder sample looses energy through the punches to the surrounding medium. We have analyzed our equations on this time scale and found the temperature decays back to its ambient value. The rate depends upon the energy initially stored in the capacitor as does the final density.
All of the preceding results are based on the assumption that the time scale associated with the densification rate is commensurate with the source time scale. We have also modified our analysis to take into account a much slower densification process that occurs on the convective time scale. During the initial heating of the powder, the temperature evolves to a steady state that depends upon the initial density and the energy stored in the capacitor. The density remains at its initial value. Then on the long convective time scale the temperature decays back to its ambient value and the density increases to its final value. This value depends upon the initial density and the energy stored in the capacitor.

We note that our model assumes that the die, which holds the powder sample, is perfectly insulated. If this restriction is removed, then ultimately heat will escape through the cylindrical sides. If the time scale associated with this process is commensurate with the other convective time scales, then a refined theory can be derived. Finally, we observe that heat conduction in the punches will become important when the experimental apparatus becomes larger and the punches more massive. Indeed, if the punches are massive enough, the temperature in them is $T_A$, the ambient value, and Equation (6c) is replaced by $u = 0$ at $z = 0$ and $z = 1$. This and Equation (6a) strongly suggest that boundary layers occur at the ends of the sample, and through them heat is transferred into the punches. This and the incorporation of more complicated die structures are the topics of ongoing research.

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