Proximity measure between samples with repetition factor greater than one

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Abstract. A new proximity measure between empirical samples, having values that may occur in a sample more than once, is constructed. This proximity measure is based on confidence intervals containing the bulk of population constructed by means of order statistics.

Key Words: proximity measure, atom, order statistics, confidence interval.

1. Introduction. Let \( x = (x_1, x_2, \ldots, x_n) \) be a sample drawn from general population \( G \) with distribution function \( F(u) \) by simple random sampling. The atom of the sample \( x \) is a sample value \( x_k \) that occurs in the sample \( x \) more than once:

\[
\begin{align*}
  x_k &= x_{k_1} = \cdots = x_{k_i}, \\
  k, k_1, \ldots, k_i &\in \{1, 2, \ldots, n\}.
\end{align*}
\]

The number of repetitions of the value \( x_k \) in the sample \( x \) shall be called a repetition factor \( t(x_k) \). Thus, atoms are sample values with repetition factor greater than 1. If \( F(u) \) is continuous and the values of \( x_k \) are exact, then the probability of atoms in \( x \) is zero and we shall refer to such a sample as hypothetical.

However, as a rule, sample values are the results of measuring some random variable. Since every measurement is subject to some error, the measured sample \( \tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n) \), \( \tilde{x}_i \in \tilde{x} \), may contain atoms (such sample we shall call empirical). Unfortunately, the well-known proximity measures between two samples (Kolmogorov-Smirnov statistics, Wilkoxon statistics, p-statistics [1, 2]) cannot be applied to atomic samples. The purpose of this paper is to modify the p-statistics in a such way that it may be used to calculate the similarity of empirical samples and to construct a corresponding test.

2. Proximity measure between empirical samples. Let us introduce the following notation: \( P_{\alpha} \) is the flooring operator up to the decimal number \( \alpha \),

\[
P_{\alpha} = \lfloor x \rfloor_{\alpha} \leq x - \alpha \delta, \quad \text{where } \delta \text{ is a rounding error, and}
\]

\[
\begin{align*}
  &x_{(1)} < x_{(2)} < \cdots < x_{(n)} , \\
  &\tilde{x}_{(1)} < \tilde{x}_{(2)} < \cdots < \tilde{x}_{(m)}
\end{align*}
\]

are variational series constructed on samples \( x \) and \( \tilde{x} \).

If \( x^* \) is a sample value drawn from the general population \( G \) and independent from \( x \), then it is well-known that

\[
p\left( x^* \in \left[ x_{(k)}, x_{(k+1)} \right] \right) = \frac{1}{n+1},
\]

\[
k = 0, 1, \ldots, n, \quad x_{(0)} = -\infty, \quad x_{(n+1)} = \infty.
\]

To extend this formula to empirical samples we need to prove the following lemma.

Lemma. If the hypothetical distribution function \( F_v \) of the general population \( G \) is differentiable and satisfies Lipschitz condition with module of continuity \( K \)

\[
|F(v) - F(u)| \leq K|v - u|.
\]

and the sample value \( x^* \) is independent from \( x \), then for every \( \delta > 0 \) the following inequality is true,

\[
p\left( x_{(k)} - \delta \leq x^* < x_{(k)} \right) \leq K\delta (n-k+1),
\]

Proof. Let \( \tilde{\xi} \) and \( \eta \) be random variables with continuous distribution functions \( F_{\tilde{\xi}}(u) \) and \( F_{\eta}(u) \) respectively. It was proved in [3] that
Therefore,\[ p(\xi < \eta) = \int_{-\infty}^\infty F_\xi(v) dF_\eta(v). \]

Therefore,\[ p\left(x^* < x_{(k)} - \delta\right) = p\left(x^* + \delta < x_{(k)}\right) = \int_{x^* + \delta}^{x_{(k)}} f_{x_{(k)}}(v) dv = \]

\[= nC_{n-1}^{k-1} \int_{-\infty}^{\infty} [F(v)]^{k-1} \left[1 - F(v)\right]^{n-k} \times \]

\[\times F(v - \delta) dF(v). \]

According to Lipschitz condition,\[ F(v - \delta) \geq F(v) - K \delta. \]

Substituting \( u = F(v) \), we have\[ p\left(x^* < x_{(k)} - \delta\right) \geq nC_{n-1}^{k-1} \int_{-\infty}^{\infty} [F(v)]^{k-1} \times \]

\[\times [1 - F(v)]^{n-k} \left[ F(v) - K \delta \right] dF(v) = \]

\[= nC_{n-1}^{k-1} \int_{-\infty}^{\infty} [F(v)]^k \left[1 - F(v)\right]^{n-k} \times \]

\[\times dF(v) - K \delta nC_{n-1}^{k-1} \int_{-\infty}^{\infty} [F(v)]^{k-1} \times \]

\[\times [1 - F(v)]^{n-k} dF(v) = \]

\[= \frac{k}{n+1} - K \delta nC_{n-1}^{k-1} \times \]

\[\times \int_0^1 u^{k-1}(1-u)^{n-k} du = J. \quad (4) \]

Integrating by parts, we obtain\[ \int_0^1 u^{k-1}(1-u)^{n-k} du = \]

\[= \frac{k-1}{n-k+1} \int_0^1 u^{k-2}(1-u)^{n-k+1} du, \]

\[= \frac{1}{n-k+2} \frac{(k-2)(k-1)}{(n-k+3)\ldots(n-1)n} \]

Therefore,\[ J = \frac{k}{n+1} - K \delta n \frac{(n-1)!}{(k-1)!(n-k)!} \times \]

\[\times \frac{(k-1)!}{n!} = \frac{k}{n+1} - K \delta (n-k+1) \quad (5) \]

Using relations (4) and (5), we have\[ p\left(x_{(k)} - \delta \leq x^* < x_{(k)}\right) = \]

\[= \frac{k}{n+1} - \frac{k}{n+1} + K (n-k+1) \delta = \]

\[= K (n-k+1) \delta. \]

The lemma is proved.

**Remark 1.** If the distribution function \( F(u) \) satisfies the Hölder condition \[|F(u) - F(v)| \leq K |u-v|^\theta \quad \forall u, v \in \mathbb{R}^k \] with index \( \theta \in (0,1] \), then in (3) the value \( \delta \) must be replaced with \( \delta^n \), \[ p\left(x_{(k)} - \delta \leq x^* < x_{(k)}\right) \leq \]

\[\leq K \delta^n (n-k+1). \quad (6) \]

**Remark 2.** Under the conditions of the above lemma it is not difficult to show that the following inequality holds,\[ p\left(x_{(k)} - \delta \leq x^* < x_{(k)}\right) \leq \]

\[\leq 2K (n-k+1) \delta. \quad (7) \]

**Theorem.** If conditions of the lemma are satisfied and the order statistics \( \tilde{x}_k = P_\alpha(x_{(i)}), \quad k \leq i \), of the empirical sample \( \tilde{x} = P_\alpha(x) = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_m) \) is an atom with repetition factor \( t(\tilde{x}_k) \), then the following inequality holds,\[ \gamma(\tilde{x}_{(k)}) + \frac{1}{n+1} - 2K (n-1 + \lambda) \delta \leq \]

\[\leq p\left(\tilde{x}^* \in [\tilde{x}_{(k)}, \tilde{x}_{(k+1)}]\right) \leq \]

\[\leq \gamma(\tilde{x}_{(k)}) + \frac{1}{n+1} + K \delta (n-i+1), \]

where \( \tilde{x}^* = P_\alpha(x^*), \quad \tilde{x}_{(k)} = P_\alpha(x_{(i)}), \]

\[\gamma(\tilde{x}_{(k)}) = t(\tilde{x}_{(k)})^{-1}, \quad 1 \leq k \leq m, \]

\[\lambda = t(\tilde{x}_k) - 1, \quad \text{and} \quad \delta = 10^{-\alpha} \text{is the rounding error.} \]
Proof. Let \( t(\tilde{x}_i) = \lambda + 1, \lambda = 0, 1, \ldots, \) and
\[ P_a(x_{(i)}) = \tilde{x}_i, \quad P_a(x_{(i+1)}) = \tilde{x}_{i+1}, \ldots, \]
\[ P_a(x_{(i+k)}) = \tilde{x}_{i+k}, \quad P_a(x_{(i+k+1)}) = \tilde{x}_{i+k+1}. \]
Consider the following random events:
\[ A = \{ x^* \in [x_{(i)}, x_{(i+1)}] \}, \]
\[ \bar{A} = \{ x^* \in [x_{(i)}, x_{(i+1)}] \}, \]
\[ B = \{ \tilde{x}^* \in [\tilde{x}_{i}, \tilde{x}_{i+1}] \}, \]
\[ \bar{B} = \{ \tilde{x}^* \in [\tilde{x}_{i}, \tilde{x}_{i+1}] \}, \]
\[ \mathcal{A} = \{ x \in [\tilde{x}_{i}, x_{(i)}] \}, \]
\[ \bar{\mathcal{B}} = \{ \tilde{x}^* = \tilde{x}_{i+1} \}. \]
If \( x^* \in \bar{A}, \) then \( x_{(i)} \leq x^* \leq x_{(i+1)}. \) Therefore,
\( \tilde{x}_{i} \leq x^* \leq \tilde{x}_{i+1}. \) Thus, \( \tilde{x}^* \in \bar{B}. \) It means that the event \( \bar{A} \) implies the event \( \bar{B}. \) Therefore,
\[ p(\bar{A}) = p(A) \leq p(\bar{B}), \]
\[ \gamma(\tilde{x}_{i}) + \frac{1}{n+1} \leq p(\bar{B}), \] (8)
as far as
\[ A = \{ x^* \in [x_{(i)}, x_{(i+1)}] \} \cup \ldots \]
\[ \cup \{ x^* \in [x_{(i+1)}, x_{(i+2)}] \} \cup i \]
\[ \cup \{ x^* \in [x_{(i+1)}, x_{(i+2)}] \} \]
p(\bar{A}) = \frac{\lambda}{n+1} + \frac{1}{n+1}.

On the other hand, if \( \tilde{x}^* \in \bar{B}, \) then
\( \tilde{x}_{i} \leq x^* \leq \tilde{x}_{i+1}. \) This implies that \( x^* \in \mathcal{A} \cup [x_{(i)}, \tilde{x}_{i+1}]. \) So, \( x^* \in \mathcal{A} \cup \bar{A}. \) Thus,
\[ p(\bar{B}) \leq p(\mathcal{A}) + p(\bar{A}) = \]
\[ = p(\mathcal{A}) + p(A). \]
(9)

It is easy to see that from the condition \( x \in \mathcal{A} \) it follows that \( x \in [x_{(i)} - \delta, x_{(i)}], \) where \( \delta = 10^{-a} \) is a rounding error. By the above lemma,
\[ p(\mathcal{A}) \leq p(x_{(i)} - \delta \leq x^* \leq x_{(i)}) \leq \]
\[ \leq K \delta(n-i+1), \]
and it follows that
\[ p(\bar{B}) \leq \gamma(\tilde{x}_{i}) + \frac{1}{n+1} + K \delta(n-i+1). \]
Clearly,
\[ p(\mathcal{B}) = p(\bar{B}) \quad \text{and} \quad \text{p}(\tilde{x}^* = \tilde{x}_{i+1}) \]
\[ \text{p}(\tilde{x}^* = \tilde{x}_{i+1}) \leq p(\mathcal{B}) \leq 2K(n-i-\lambda) \delta. \]
Thus,
\[ \gamma(\tilde{x}_{i}) + \frac{1}{n+1} - 2K(n-1+\lambda) \delta \leq \]
\[ \leq p(\mathcal{B}) \leq \gamma(\tilde{x}_{i}) + \frac{1}{n+1} + K \delta(n-i+1), \]
since \( p(\mathcal{B}) \leq p(\bar{B}). \) This completes the prove of the theorem.

Corollary.
\[ p\left(\tilde{x}^* \in [\tilde{x}_{i}, \tilde{x}_{i+1}]\right) = \]
\[ = \gamma(\tilde{x}_{i}) + \frac{1}{n+1} \]
with precision up to the rounding error.

The estimate (10) implies that the probability
\[ p\left(\tilde{x}^* \in [\tilde{x}_{i(j)}, \tilde{x}_{i(j+1)}]\right), \quad i < j, 1 \leq i, j \leq m \]
can be calculated by the formula
\[ p_{ij} = p\left(A_{ij}\right) = p\left(\tilde{x}^* \in [\tilde{x}_{i(j)}, \tilde{x}_{i(j+1)}]\right) = \]
\[ = \gamma_i + \gamma_{i+1} + \ldots + \gamma_{j-1} + \frac{j-i}{n+1}, \]
(11)
where \( \gamma_i = \gamma(\tilde{x}_{i(j)}), \quad A_{ij} = \{\tilde{x}^* \in [\tilde{x}_{i(j)}, \tilde{x}_{i(j+1)}]\}. \)

Note that when the sample value \( \tilde{x}_{i(j)}, i \leq l \leq j-1 \) is not an atom, then \( \gamma_i = 0. \)

Consequently, if the sample does not contain any atoms, then the formula (10) transforms into the well-known formula
\[ p_{ij} = p\left(A_{ij}\right) = p\left(\tilde{x}^* \in [\tilde{x}_{i(j)}, \tilde{x}_{i(j)}]\right) = \]
\[ = \frac{j-i}{n+1}. \]

Denote by \( H \) the hypothesis that the continuous distribution functions \( F_G(u) \) and \( F_{G'}(u) \), of the respective general populations \( G \) and \( G' \), are equivalent. Suppose that
\[ x = (x_1, \ldots, x_n) \in G, \quad x' = (x'_1, \ldots, x'_m) \in G' \]
and let \( x_1 \leq \ldots \leq x_n, \quad x'_1 \leq \ldots \leq x'_m \) be their variational series. Assume that
Let is the frequency of the event. Similarly, let by the following statistics and .

If the null hypothesis be the probability of this event is calculated by formula (11). Let

\[
\begin{align*}
\hat{p}_y^{(1)} &= \frac{h_y^{(n)} m + g^2}{2 - g \sqrt{h_y^{(n)} (1-h) m + g^2}} / m + g^2, \\
\hat{p}_y^{(2)} &= \frac{h_y^{(n)} m + g^2}{2 + g \sqrt{h_y^{(n)} (1-h) m + g^2}} / m + g^2,
\end{align*}
\]

(12)

where \( h_y^{(n)} \) is the frequency of the event \( A_y^{(n)} \) in \( m \) trials. The value \( g \) determines the significance level of the confidence interval \( I_y^{(n,m)} = \left( \hat{p}_y^{(1)}, \hat{p}_y^{(2)} \right) \). By the 3σ-rule when \( g = 3 \), the significance level of this interval does not exceed 0.05.

Denote by \( N \) the number of all confidence intervals \( I_y^{(n,m)} = \left( \hat{p}_y^{(1)}, \hat{p}_y^{(2)} \right) \), \( N = n(n-1)/2 \) and by \( L \) the number of the intervals \( I_y^{(n,m)} \) that contains probabilities \( p_y^{(n)} \). Let

\[
h^{(n,m)} = \rho\left(F^*, F'\right) = \rho\left(x, x'\right) = \frac{L}{N}.
\]

Since \( h^{(n,m)} \) is the frequency of the random event \( B = \left\{ p_y^{(n)} \in I_y^{(n,m)} \right\} \) with probability \( p(B) = 1 - \beta \), substituting in equation (11)

\[
h_y^{(n,m)} = h_y^{(n)}, \ m = N \ \text{and} \ g = 3\ \text{we obtain the confidence interval} \ I_y^{(n,m)} = \left( \hat{p}_y^{(1)}, \hat{p}_y^{(2)} \right) \ \text{for the probability} \ p(B). \ \text{We shall call the statistics} \ h^{(n)} \ \text{modified p-statistics}. \ \text{It is the proximity measure} \ \rho\left(x, x'\right) \ \text{between} \ x \ \text{and} \ x'.
\]

3. Proximity measure between discrete samples. Let samples \( x = (x_1, x_2, ..., x_n) \) and \( y = (y_1, y_2, ..., y_m) \) be obtained by simple random sampling from discrete general populations \( G_x = \{x_1, x_2, ..., x_n\} \) and \( G_y = \{y_1, y_2, ..., y_m\} \) with distribution functions \( F_x \) and \( F_y \).

Now, let us assume that all values of \( x \) and \( y \) are exact and let the null hypothesis be

\[
H : F_x = F_y
\]

and the alternative hypothesis be

\[
H' : F_x \neq F_y.
\]

Denote by \( A_x^{(i)} \) the random event that some sample value of \( x \) is equal to \( \tilde{x}_i \), and by \( h_x^{(i)} \) the frequency of this event in \( x \). Similarly, let \( B_y^{(i)} \) be the random event that some sample value is equal to \( \tilde{y}_i \) and denote by \( h_y^{(i)} \) the frequency of this event in \( y \). Consider these events as results of a series of independent random experiments that create the Bernoulli schemes \( \{E_i^{(1)}\}_{i=1}^k \) and \( \{E_i^{(2)}\}_{i=1}^m \), respectively.

Using some of the results from [2-5], let us construct the confidence interval for \( h_y^{(i)} \)

\[
I_y^{(i)} = \left( h_y^{(i)} - 2\tilde{\delta}, h_y^{(i)} + 2\tilde{\delta} \right),
\]

where

\[
\tilde{\delta} = \sqrt{\frac{h_y^{(i)} \left(1 - h_y^{(i)}\right)}{n}} + \sqrt{\frac{h_y^{(i)} \left(1 - h_y^{(i)}\right)}{m}}.
\]

This interval contains the frequency \( h_y^{(i)} \) with probability that exceeds 0.95. Thus, the significance level of this interval, i.e. the value \( p\left(h_y^{(i)} \notin I_y^{(i)}\right) \), does not exceed 0.05.

Let us introduce the random variable

\[
\chi_i = \begin{cases} 1, & \text{if} \ h_y^{(i)} \in I_i^{(i)}, \\ 0, & \text{if} \ h_y^{(i)} \notin I_i^{(i)}, \ i = 1, 2, ..., k. \end{cases}
\]

We shall define the proximity measure between \( x \) and \( y \) by the following statistics

\[
\rho\left(x, y\right) = \frac{1}{k} \sum_{i=1}^k \chi_i.
\]

If the general populations \( G_x \) and \( G_y \) are equivalent, then the proximity measure
\( \rho(x, y) \) with probability greater than 0.95 is greater than 0.95, i.e.

\[
p \left( \rho(x, y) \geq 0.95 \right) \geq 0.95.
\]

Thus, the significance test for the test of hypothesis \( H \) about the equivalence of \( G_x \) and \( G_y \) may be formulated as follows:

1. If \( \rho(x, y) \geq 0.95 \) then \( x \) and \( y \) do not contradict \( H \).
2. If \( \rho(x, y) < 0.95 \) then \( H \) is rejected.

By the central limit theorem the statistics

\[
\rho(x, y) = \frac{1}{k} \sum_{i=1}^{k} \chi_i
\]

has asymptotic normal distribution and the 2s-rule is valid in that case. Hence, the significance level of this test for the Bernoulli schemes \( \{E_i^{(1)}\}_{i=1}^{m} \) and \( \{E_i^{(2)}\}_{i=1}^{m} \) does not exceed 0.05.

**Remark.** The constructed proximity measure is not symmetric. To obtain a symmetric proximity measure we must swap \( x \) and \( y \), calculate \( \rho(y, x) \) and compute the value

\[
\rho_{xy} = \frac{\rho(x, y) + \rho(y, x)}{2}.
\]

**4. Reference**

1. Van der Waerdeb, B. L. Mathematische Statistik. Springer-Verlag, 1957.