Interface Dynamics for Quasi-Stationary Stefan Problem

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Abstract

We investigate the interface dynamics in Laplacian growth model, using the conformal mapping technique. Starting from the governing equation obtained by B.Shraiman and D.Bensimon, we derive intergro-differential evolution equation of interphase dynamics. It is shown that such representation of the conformal mapping technique is convenient for computer simulations of the quasi-stationary Stefan problem.

Key words: Laplacian growth, mathematical model, intergro-differential equations, conformal mapping, computer simulation.

1 Introduction

There are various physical processes in nature, such as crystallization, combustion, deposition etc. in which it is possible to single out a moving boundary having essentially different physical properties on either side of itself. The position of the given boundary is not known beforehand, and is determined by a self-consistent way of the distribution of its physical parameters in space. Such problems with moving boundaries are known as Stefan problems. In the given class of systems with free boundaries it is possible to single out a subclass where the distribution of the field in the space is stationary. For this type of systems the utilization of the theory of functions of a complex variable is very useful in the construction of effective surface dynamics of the interface. The central issue of the problem is the conformal mapping of the physical regions of the space onto some special kind of regions. A large number of models for
interfacial dynamics have been proposed. Reviews and some descriptions of such models may be found in [1–3]. One of the most famous of these was introduced by Shraiman and Bensimon [4]. It is widely known as the conformal mapping equation and is still under investigation today.

2 Mathematical model

Let us consider the problem of Laplacian growth which can be formulated in the following way. We investigate the growth of the two-dimensional region $D$ and suppose that its boundary $\partial D$, which represents the physical interface, is the analytic Jordan curve $\Gamma$ (right hand side of Fig. 1). The field $\varphi$ outside the region $D$, satisfies the Laplace equation

$$\Delta \varphi = 0.$$  

(1)

The boundary $\partial D$ grows at a rate that is proportional to the normal gradient of the field at the interface. Therefore the evolution of the interface is governed by the following equation

$$v_n = -\nabla_n \varphi|_\Gamma.$$  

(2)

The potential field $\varphi$ satisfies the following boundary condition on the interface

$$\varphi|_\Gamma = d_0 k|_\Gamma,$$  

(3)

where $k$ is the curvature of the interface $\Gamma$, and $d_0$ is the dimensionless surface tension parameter.

Equations (1)-(3) determine the boundary-value problem with moving interface, or the Stefan problem. The goal in solving a Stefan problem is to find the location of the moving boundary and to determine the distribution of the field $\varphi$. Application of the methods of conformal mapping [1,2,4–7,12], which are based on the Riemann mapping theorem, allows us to simplify the above task. The theorem states, that a conformal map exists, mapping the exterior of $D$ onto a standard domain, for example, the exterior of the unit disk, and the boundary $\partial D$ corresponds to unit circle. In this case one can parametrize the evolution of the interface in the $z$ plane by the time-dependent conformal map $F(w,t)$, which maps the exterior of the unit disk at each instant of time, $|w| = 1$, onto the exterior of $D$ (Fig. 1). Therefore the evolution of the interface can be represented in terms of

$$\Gamma (\theta, t) = \lim_{w \to e^{i\theta}} F(w,t), \quad 0 \leq \theta < 2\pi.$$  

(4)

As it is shown by Shraiman and Bensimon [4], the conformal map satisfies the
The following equation of motion

$$\frac{\partial \Gamma (\theta, t)}{\partial t} = -i \frac{\partial \Gamma (\theta, t)}{\partial \theta} S \left[ \frac{\partial \Gamma (\theta, t)}{\partial \theta} \right]^{-2} \bigg|_{w=e^{i\theta}},$$

(5)

where $S[\cdot]$ stands for the Schwartz operator. For the exterior of the unit circle with some boundary condition $f(\theta)$ the Schwartz operator can be represented in the following way

$$S[f(\theta)] = -\frac{1}{2\pi} \int_{0}^{2\pi} d\theta' e^{i\theta'} + w e^{i\theta'} - w f(\theta') + iC,$$

(6)

where $C$ is an arbitrary constant. Denoting the action of the Schwartz operator on $\left| \frac{\partial \Gamma (\theta, t)}{\partial \theta} \right|^{-2}$ as the limit of an analytic function $G(w, t)$ whose explicit form is unique [5], the equation of motion for $F(w, t)$ (5) is transformed to

$$\frac{dF(w, t)}{dt} = w \frac{dF(w, t)}{dw} G(w, t).$$

(7)

Now, consider the logarithmic derivative of (7) with respect to $w$

$$\frac{d}{dt} \ln \frac{dF(w, t)}{dw} = \left( \frac{dF(w, t)}{dw} \right)^{-1} \frac{d}{dw} \left( w \frac{dF(w, t)}{dw} G(w, t) \right).$$

(8)

Using the explicit forms of the conformal map $F(w, t)$ it is possible to reduce the partial differential equation (5), governing the interface dynamics, to a set of ordinary differential equations for the dynamics of the critical points of the conformal map [4]. This facilitates the analytic and numerical study of the complex behavior in the evolution of the interface [5,6].

In spite of this fact, it is convenient to introduce the following field [2,12]

$$\eta(\theta, t) = \left| \frac{dF(w, t)}{dw} \right|^{-1} \bigg|_{w=e^{i\theta}}.$$  

(9)

The conformality of $F(w, t)$ demands that all critical points of the map, i.e., zeroes of its derivative, lie within the unit disk. Therefore, $\ln \frac{dF(w, t)}{dw}$ is an
analytic function outside the region $D$. Taking into account that
\[ \ln \left[ \frac{dF(w, t)}{dw} \right]_{w=e^{i\theta}} = \ln \left| \frac{dF(w, t)}{dw} \right|_{w=e^{i\theta}} + i\psi(w, t) \left|\frac{dF(w, t)}{dw}\right|_{w=e^{i\theta}}, \tag{10} \]
and solving Schwartz problem outside the unit disc in the $w$ plane one can reconstruct the equation of motion for $F(w, t)$ from the equation of motion for $\eta(\theta, t)$, and from $\psi(\theta, t)$ one can represent the oriented angle between the tangent vectors $\tau_w$ and $\tau_{F(w)}$ (Fig. 1). To explain the sense of using (6) we should also write the relation between $\eta(\theta, t)$ and $\psi(\theta, t)$ [2,12]
\[ \psi(\theta, t) = -\frac{1}{2\pi} \int_0^{2\pi} d\theta' \cot \frac{\theta' - \theta}{2} \ln \eta(\theta', t). \tag{11} \]

In the present paper we investigate the properties of the equation of motion, obtained by Shraiman and Bensimon (5), and derive the equation of motion for $\eta(\theta, t)$, that is more suitable for numerical simulation. In this case let us note that
\[ \left| \frac{\partial \Gamma(\theta, t)}{\partial \theta} \right|^2 = \left| \frac{dF(w, t)}{dw} \right|^2 \left| \frac{dF(w, t)}{dw} \right|_{w=e^{i\theta}} = \eta^2(\theta, t). \tag{12} \]
Thus, $G(w, t) = S(\eta^2(\theta, t))$, where all constants in Schwartz operator are explicitly determinable [2,5]. Regarding the real part of the left side of the equation (8) as $\frac{d}{dt} \ln \left| \frac{dF(w, t)}{dw} \right|$ and simplifying the right side of the equation we obtain
\[ \frac{d}{dt} \ln \left| \frac{dF(w, t)}{dw} \right| = \Re \left[ G(w, t) + wG(w, t) \frac{dG(w, t)}{dw} + wG(w, t) \frac{d}{dw} \ln \frac{dF(w, t)}{dw} \right]. \tag{13} \]

Determining the real part of each term within square brackets of the right side of (13) and setting $w = e^{i\theta}$, we get
\[ \Re \left[ G(w, t) \right]_{w=e^{i\theta}} = \eta^2(\theta, t), \tag{14} \]
\[ \Re \left[ wG(w, t) \frac{d}{dw} \ln \left| \frac{dF(w, t)}{dw} \right| \right]_{w=e^{i\theta}} = \frac{\partial}{\partial \theta} \Im \left[ G(w, t) \right]_{w=e^{i\theta}}, \tag{15} \]
\[ \Re \left[ wG(w, t) \frac{d}{dw} \ln \left| \frac{dF(w, t)}{dw} \right| \right]_{w=e^{i\theta}} = \Re \left[ wG(w, t) \frac{d}{dw} \left( \ln \left| \frac{dF(w, t)}{dw} \right| + i\psi(w, t) \right) \right]_{w=e^{i\theta}} \]
\[ = -\eta^2(\theta, t) \frac{\partial \psi(\theta, t)}{\partial \theta} + \Im \left[ G(w, t) \frac{d}{d\theta} \ln \left| \frac{dF(w, t)}{dw} \right| \right]_{w=e^{i\theta}}. \tag{16} \]
Taking into account (9) and (12), we rewrite equation (13) in the form

\[
\frac{d}{dt} \ln \eta^{-1}(\theta, t) = \eta^2(\theta, t) + \frac{\partial}{\partial \theta} \left[ \text{Im} \left[ G(w, t) \right] \right]_{w=e^{i\theta}} - \eta^2(\theta, t) \frac{\partial \psi(\theta, t)}{\partial \theta} \\
+ \text{Im} \left[ G(w, t) \right]_{w=e^{i\theta}} \left( \frac{d}{d\theta} \ln \eta^{-1}(\theta, t) \right). \tag{17}
\]

Taking into account (11) and the fact that the imaginary part of the solution of the Schwartz problem (6) can be written in terms of

\[
\text{Im} \left[ G(w, t) \right]_{w=e^{i\theta}} = \frac{1}{2\pi} \int_0^{2\pi} d\theta' \cot \frac{\theta' - \theta}{2} \eta^2(\theta', t), \tag{18}
\]

we can simplify equation (17), and obtain the following equation of motion for \( \eta(\theta, t) \)

\[
\frac{\partial \eta(\theta, t)}{\partial t} = -\eta^3(\theta, t) - \eta(\theta, t) \frac{\partial}{\partial \theta} \frac{1}{2\pi} \int_0^{2\pi} d\theta' \cot \frac{\theta' - \theta}{2} \eta^2(\theta', t) \\
+ \eta^3(\theta, t) \frac{\partial}{\partial \theta} \frac{1}{2\pi} \int_0^{2\pi} d\theta' \cot \frac{\theta' - \theta}{2} \ln \eta(\theta', t) \\
+ \frac{\partial \eta(\theta, t)}{\partial \theta} \frac{1}{2\pi} \int_0^{2\pi} d\theta' \cot \frac{\theta' - \theta}{2} \eta^2(\theta', t). \tag{19}
\]

The last equation is similar to the one derived by using variational principles of conformal maps [2,12].

In order to derive the equation of motion for the field \( \eta(\theta, t) \) in the presence of surface tension \( (d_0 \neq 0) \) we need to obtain the corresponding form of \( G(w, t) \) for the nonzero surface tension [7]. Thus, one can rewrite the desired evolution equation for mapping (7)(see [7]) in the form

\[
\frac{dF(w, t)}{dt} = -w \frac{dF(w, t)}{dw} G_0(w, t), \tag{20}
\]

where \( G_0(w, t) \) is the analytic function of \( w \), whose real part on \( |w| = 1 \) is specified as follows [7]:

\[
\text{Re} \left[ G_0(w, t) \right]_{w=e^{i\theta}} = - \left[ \frac{1 - d_0 \text{Re} \left( w \frac{dS[k(w, t)]}{dw} \right)_{w=e^{i\theta}}}{|w| \frac{dF(w, t)}{dw} |^2_{w=e^{i\theta}}} \right]_{w=e^{i\theta}}. \tag{21}
\]
From equation (20) it follows that to derive the equation of motion for \( \eta(\theta, t) \), we have to define the real and imaginary parts of \( G_0(\omega, t)|_{\omega = e^{i\theta}} \)

\[
\Re [G_0(\omega, t)]|_{\omega = e^{i\theta}} = \eta^2(\theta, t) \left( 1 - d_0 \frac{\partial}{\partial \theta} \Im [S(k(\omega, t))]|_{\omega = e^{i\theta}} \right), \tag{22}
\]

\[
\Im [G_0(\omega, t)]|_{\omega = e^{i\theta}} = \frac{1}{2\pi} \int_0^{2\pi} d\theta' \cot \frac{\theta' - \theta}{2} \eta^2(\theta', t) \left( 1 - d_0 \frac{\partial}{\partial \theta'} \Im [S(k(\omega, t))]|_{\omega = e^{i\theta}} \right), \tag{23}
\]

where

\[
\Im [S(k(\omega, t))]|_{\omega = e^{i\theta}} = \frac{1}{2\pi} \int_0^{2\pi} d\theta' \cot \frac{\theta' - \theta}{2} k(\theta', t). \tag{24}
\]

After simplifications, we obtain the following equation of motion for \( \eta(\theta, t) \):

\[
\frac{\partial \eta(\theta, t)}{\partial t} = -\eta^2(\theta, t) v_n(\theta, t) - \\
- \frac{\partial}{\partial \theta} \frac{1}{2\pi} \int_0^{2\pi} d\theta' \cot \frac{\theta' - \theta}{2} \eta^2(\theta', t) v_n(\theta, t) + \\
+ \eta^2(\theta, t) v_n(\theta', t) \frac{\partial}{\partial \theta} \frac{1}{2\pi} \int_0^{2\pi} d\theta' \cot \frac{\theta' - \theta}{2} \ln \eta(\theta', t) + \\
+ \frac{\partial \eta(\theta, t)}{\partial \theta} \frac{1}{2\pi} \int_0^{2\pi} d\theta' \cot \frac{\theta' - \theta}{2} \eta(\theta', t) v_n(\theta', t). \tag{25}
\]

One can also notice [2,7,12] that

\[
v_n(\theta, t) = \eta(\theta, t) \left( 1 - d_0 \frac{\partial}{\partial \theta} \Im [S(k(\omega, t))]|_{\omega = e^{i\theta}} \right) = \eta(\theta, t) (1 - d_0 k(\theta, t)), \tag{26}
\]

where \( k(\theta, t) \) is the curvature of \( \Gamma(\theta, t) \):

\[
k(\theta, t) = -\eta(\theta, t) \frac{\partial \psi(\theta, t)}{\partial \theta} = \eta(\theta, t) \frac{\partial}{\partial \theta} \frac{1}{2\pi} \int_0^{2\pi} d\theta' \cot \frac{\theta' - \theta}{2} \ln \eta(\theta', t). \tag{27}
\]

Equations (25)-(26) represent the system of integro- differential equations that completely describes the evolution of the interface. Such representation of Shraiman-Bensimon equations is a new result in this area of research [8–11].
It should be noted that equations (25)-(26) in some way are similar to the equations of motion obtained in [2,12], but they are more convenient for computer simulation. The fact is that in the computational experiment we used the approach based on the approximation of the required functions by trigonometric polynomials [13]

\[ \tilde{f}_n(u) = \frac{1}{2\pi} \sum_{j=0}^{2n-1} \tilde{f}(u_j) \sin[n(u - u_j)] \cot \frac{u - u_j}{2} \]  

on the given system of internal nodes

\[ u_j = \frac{\pi j}{n}, \quad j = 0, 1, ..., 2n - 1. \]  

The decomposition of the function \( \tilde{f}(u) \) in terms of trigonometric polynomials allows us to use the quadrature formula [13] for calculating singular integrals with Hilbert kernels

\[ J(u_0^m) = \sum_{j=0}^{2n-1} \frac{1}{2n} \tilde{f}(u_j) \cot \frac{u_j - u_0^m}{2} \]  

defined on the system of external nodes

\[ u_0^m = \frac{\pi(2m - 1)}{2n}, \quad m = 0, 1, ..., 2n - 1. \]  

The representation of function \( \tilde{f}(u) \) in the form of (28) gave us the ability to apply the analytical expressions to practically any derivative \( \tilde{f}^{(v)} \) of the initial function during the calculations. It should be noted that the functional relationship (30) was derived in [14] also, but it was used for quadrature formula only.

Taking into account that all functions under the integral sign on the right side of equations (25)-(26) can be represented in the form (28), and accepting the following designations on the grid (31), we can write

\[ \eta_m(t, \theta) = \eta(t, \theta_0^m), \quad m = 0, 1, ..., 2n - 1. \]  

Consequently, from the equations of interface dynamics we obtain the system of ordinary differential equations:

\[ \frac{d\eta_m}{dt} = \tilde{F}_m \left\{ \eta_m, \sum_{j=0}^{2n-1} f_j \cot \frac{\theta_j - \theta_0^m}{2} \right\}, \]
where $\tilde{F}_m$ is a nonlinear operator determined by the right hand side of equations (25)-(26), and $f_j$ takes the values $v_n(\theta_j)\eta(\theta_j)$, $\ln(\eta(\theta_j))$, $\eta(\theta_j)$. To find the other derivatives, like $\eta'_m = \partial \eta / \partial \theta$ and $J'_i = \partial J_i / \partial \theta$ at a point $\theta^0_m$ and at transition from points $\theta_j$ to $\theta^0_m$, formulae (28), (30) were also used.

### 3 Numerical simulation of the interface dynamics

The system of equations (25)-(26) was reduced to a system of ordinary nonlinear differential equations with full Jacobian. Other peculiarity of the system is that for calculation we use internal and external nodes (29),(31). The integrals in the right hand side of equations we calculate in external nodes and integrand values in internal nodes.

The numerical integration of the specified systems was carried out using an ODE software based on the multistep backward differential formulae [15]. The choice of the value of discretization and local error of integration was made, so that the required accuracy of the solution and the stability of the calculations was provided.

It should be noted that such approach was used for investigation of the system derived in [2]. At the same time the above-mentioned characteristic features make system (25)-(26) more preferable for several reasons. In the system (25)-(26) the equation for radius $R$ and the additional integral term on the right-hand side are both absent. In the high-order systems with full Jacobian this provides a certain advantage for computer simulation. In addition, the computation of all the singular integrals is done on the basis of trigonometric polynomials, and the reduced term, as well as the equation for $R$, is of a
rather different type than singular integrals and this requires more computational resources. That is why the approach proposed here is better for practical modeling. We used approximation of the function on the right hand side in the outer nodes by the values in the internal nodes. This makes it possible to integrate the system more effectively with higher order and at the same time to get more accurate surface dynamics.

Fig. 3. Comparison of interface profiles for different values of surface tension $d_0$: a) $d_0 = 0.15$; b) $d_0 = 0.12$; c) $d_0 = 0.09$; d) $d_0 = 0.06$; e) $d_0 = 0.04$; f) $d_0 = 0.03$. Initial perturbations for all the figures are the same: $\eta(\theta, t_0 = 0) = 1.0 - 0.01 \cos(4\theta)$; $(t_{step} = 0.5 \cdot 10^3)$ 18 times are plotted, spaced between $0 < t \leq 0.9 \cdot 10^4$.

In numerical simulation of nonlinear integro-differential equations we find the evolution of variable $\eta(u_m^0)$ in time. In this case with the help of transformations of the expressions

\begin{align}
  dx &= \frac{d\theta}{\eta(\theta, t)} \cos \left( \theta + \psi(\theta) + \frac{\pi}{2} \right), \\
  dy &= \frac{d\theta}{\eta(\theta, t)} \sin \left( \theta + \psi(\theta) + \frac{\pi}{2} \right).
\end{align}

for the system of equations (25)-(26), it is possible to reproduce geometrically defined interfaces in parametric form.

In numerical experiments the evolution of the interface at different initial perturbations $\delta\eta|_{t=0}$ and surface energies $d_0$ was investigated. Figure 2 illustrates the characteristic form of non-uniform distributions of the variables $\eta, \psi$ and the corresponding interfaces for different initial perturbations. In the figure relatively smooth variations of the interface correspond to quite large variations in the field $\eta$ and, hence, also in the angle $\psi$. Thus, relatively small errors in determining the field $\eta$ do not cause large changes in the interface geometry. In addition, even relatively long parts of the interface contract when mapped
into the interval $[0, 2\pi]$. Conversely, pronounced changes in the contour $\Gamma$ are stretched out. In this sense such mappings are "adaptive" with respect to the information about the detailed geometric structure of the interface.

The structures obtained during computer simulation strongly depend on initial perturbations. The structures are strongly dependent on the value of the surface energy. An increase in the surface energy leads to a smoother structure (fig. a and b) in Fig. 3), while a decrease in the surface energy leads to a more diversified structures. This property corresponds to the analytical results obtained for the system.

References


