A Stretch-Temperature Model for Flame-Flow Interaction

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CAMS Report 0506-26, Spring 2006
Center for Applied Mathematics and Statistics
NJIT
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Abstract

The classical relation between the flame speed and the stretch, employed in modeling flame-flow interactions, is valid only for positive Markstein lengths (high Lewis numbers). At negative Markstein lengths (low Lewis numbers) the corresponding dynamical system suffers short-wavelength instability, making the associated initial value problem ill-posed. In this study the difficulty is resolved by incorporation of higher-order effects. As a result one ends up with a reduced model based on a coupled system of second-order dynamic equations for the flame interface and its temperature. As an illustration the new model is applied for description of diffusively unstable stagnation-point flow flames.

Keywords: flame instability, flame stretch, surface dynamics

PACS: 47.20.Ky; 47.20.Ma; 82.40 Py

1 Introduction

When the hydrodynamic length is much larger than the flame thickness, the premixed flame may be regarded as an interface, convected by the unsteady flow field while advancing towards the unburned gas at a normal velocity $v_F$. In the constant density limit when the flow-field $u(x,t)$ is prescribed (passive propagation) the flame dynamics is governed by the equation,

$$v_n - u \cdot N = v_F \hspace{1em} (\nabla \cdot u = 0)$$

(1)

Here $v_n$ is the normal velocity of the interface relative to the laboratory frame of reference, $N$ is a unit normal to the interface directed towards the unburned gas, and $v_F$ is the normal velocity of the interface relative to a local fluid element; $v_F$ depends on the local curvature of the interface as well as the local strain field.

For weakly curved flames evolving through large-scale flow fields [1][2],

$$v_F = v_L(1 - l_M\mathcal{K}),$$

(2)

where $v_L$ represents the laminar flame velocity in the absence of strain and curvature effects. $\mathcal{K}$ is the stretch, the combined effect of curvature and strain.

For simplicity of the mathematical expressions involved, yet without much detriment to general understanding, the further discussion is restricted to the planar geometry only. The relation specifying the stretch may then be written as,

$$v_L\mathcal{K} = -v_L\kappa + u_I \cdot T$$

(3)

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Here $l$ is the arc-length, $\mathbf{T}$ is the unit tangential vector, $\kappa = -\nabla \cdot \mathbf{N}$ is the curvature of the interface, $l_M$ is the Markstein length whose value depends on physico-chemical parameters of the system. In the constant density, high activation energy, near-equidiffusive approximation [3],

$$l_M = l_{th}(1 + \frac{1}{2}\beta(Le - 1))$$

where $l_{th}$ is the planar flame thickness; $\beta, Le$ are the Zeldovich and Lewis numbers, respectively.

Eqs. (1),(2) define a second-order equation. For positive $l_M$ (high Lewis numbers) the short-wavelength disturbances decay exponentially ensuring the well-posedness of the associated initial-value problem. However, if $l_M$ is negative (small Lewis numbers) the short-wavelength disturbances undergo exponential growth, making the initial-value problem ill-posed, and the whole model incapable of securing a sensible dynamical picture.

The short-wavelength instability is a trace of the well-known flame cellularity occurring at low Lewis numbers. An adequate description of cellular flames requires incorporation of higher-order effects absent in the first-order model based on Eqs. (1) (2). Such higher-order models have long been developed for the flames evolving in quiescent mixtures ($u = 0$). Here, depending on the asymptotic strategy adopted, one ends up either with a single fourth-order equation for the flame-interface [4] [5] or with a system of second-order equations for the flame interface and its temperature [6] [7]. The objective of the present study is an extension of the higher-order models to incorporate effects due to the background flow-field, which, to our knowledge, has not previously been undertaken.

2 Formulation

As a starting point we adopt the conventional constant density, high activation energy, near equidiffusive formulation where the reaction rate is modeled by the surface $\delta$ -function, $\delta_S$. In suitably chosen units, the corresponding system of equations reads [3],

$$\theta_r + \mathbf{U} \cdot \nabla \theta = \nabla^2 \theta + \exp(\psi)\delta_S,$$

$$\psi_r + \mathbf{U} \cdot \nabla \psi = \nabla^2 (\psi - \alpha \theta).$$

Far ahead of the flame interface $\mathcal{S}$, the quantities $\theta, \psi \to 0$, whereas far behind $\psi$ grows no faster than polynomially. Moreover, $\theta \equiv 1$ in the wake of the flame front.

$$\theta = \lim \left( \frac{(T - T_u)}{(T_b - T_u)} \right), \quad \psi = \lim \left( \frac{1}{2}\beta \Delta \mathcal{H}/c_p (T_b - T_u) \right)$$

are the reduced non-dimensional temperature and the excess enthalpy evaluated in the limit $\beta \to \infty$, $Le \to 1$, while the product $\alpha = \frac{1}{2}\beta(1 - Le)$ remains finite. $T, T_u, T_b$ are the local, initial and final temperatures, respectively. $\Delta \mathcal{H} = c_p (T_b - T_u) [(T - T_b) / ((T_b - T_u) + C/C_u)]$ is the excess enthalpy; $C, C_u$ are the local and the initial concentrations of the deficient reactant; $c_p$ is the specific heat, and $\mathbf{U} = \mathbf{u}/u_L$, scaled flow velocity. The scaled spatio-temporal coordinates are referred to $l_{th}$ and $l_{th}/u_L$, respectively.

For further discussion it is convenient to introduce a new quantity: $\Theta = \psi|_{\mathcal{S}}$. Since at the flame interface the deficient reactant is fully consumed ($C = 0$),

$$\Theta = \lim \left( \frac{1}{2}\beta (T - T_b) / (T_b - T_u) \right).$$

The latter therefore may be regarded as a scaled excess temperature of the interface.
3 Higher-order model

An extension of Eq. (2) accounting for higher-order effects may be either derived using an appropriate gradient expansion in the intrinsic coordinates, or may be constructed heuristically as a geometrically invariant extrapolation of the results obtained for freely evolving flames. In this paper we adopt the latter strategy as less formalistic and more appealing physically. The advantage of a suitably arranged heuristic approach is that it straightforwardly exposes the interplay between the different mechanisms involved. In the perturbative derivation the origin of the individual ingredients constituting the final outcome is often difficult to trace due to the heavy algebra involved (see also Sec. 5).

We begin with the case of the quiescent mixture ($\mathbf{U} = 0$) where the system (5) (6) allows for a planar traveling wave (in the direction, say, $\mathbf{N}$ with unit velocity) solution:

$$
\begin{align*}
\theta^0 &= \begin{cases} 
\exp(-y) & y > 0 \\
1 & y < 0,
\end{cases} \\
\psi^0 &= \begin{cases} 
-\alpha y \exp(-y) & y > 0 \\
0 & y < 0,
\end{cases} \\
\phi^0 &= 0,
\end{align*}
$$

(9)

where $y$ is the (signed) distance from the moving interface. The conventional linear stability analysis of the traveling wave solution yields the following set of relations between the perturbations of the interface $\delta F$, and its temperature $\delta \Theta$, valid in the long-wavelength limit,

$$
(\delta F)_\tau = (\delta F)_{ss} + \delta \Theta, \\
(\delta \Theta)_\tau = (\delta \Theta)_{ss} - \frac{\alpha}{4}(\delta F)_{ss} - \frac{1}{4}\delta \Theta.
$$

(10) (11)

One can easily check that the instability sets in at $\alpha > \alpha_{cr} = 1$.

Note that for infinitesimally small perturbations of the traveling wave the coordinate $s$ can be interpreted as either the usual Cartesian coordinate along the flat interface or the arc-length of the curved interface. Based on this observation and the coordinate invariance of the original free-interface problem (5) (6) the system (10) (11) suggests the following coordinate-free model for the flame interface dynamics,

$$
V_n = 1 + \Theta + \kappa, \\
\mathcal{D}_\tau \Theta = \Theta_{ss} - \frac{\alpha}{4}\kappa - \frac{1}{4}\Theta,
$$

(12) (13)

where $V_n = v_n/v_L$, $\kappa = \kappa_{th}$ are the scaled normal velocity and curvature, respectively. $\mathcal{D}_\tau$ is the temporal derivative along the normal $\mathbf{N}$; there is a simple connection between the normal time-derivative and the conventional partial time-derivative at constant $s$ [8],

$$
\mathcal{D}_\tau \Theta = \Theta_\tau + \left(\mathcal{D}_\tau s\right) \Theta_s,
$$

(14)

where

$$
\mathcal{D}_\tau s = -\int_0^s \kappa(\sigma, \tau) V_n(\sigma, \tau) d\sigma.
$$

(15)

In the presence of the background flow $\mathbf{U}$ the normal velocity in Eq. (12) should naturally be replaced by the flame speed relative to the fluid element $V_n - \mathbf{U} \cdot \mathbf{N}$, and $\mathcal{D}_\tau \Theta$ augmented by $(\mathbf{U} \cdot \mathbf{T}) \Theta_s$ reflecting advection of the interfacial temperature by the tangential flow field. The curvature term $-\kappa$ is a part of the total stretch (3), and, therefore, should be replaced by

$$
K = l_{th} \mathcal{K} = -\kappa + \mathbf{U}_s \cdot \mathbf{T} = -(1 + \mathbf{U} \cdot \mathbf{N}) \kappa + (\mathbf{U} \cdot \mathbf{T})_s
$$

(16)
As a result one ends up with the following flame-flow model valid for general Lewis numbers,

\[ V_n = U \cdot N + 1 + \Theta - K, \quad (17) \]

\[ D_r \Theta + (U \cdot T) \Theta_s = \Theta_{ss} + \frac{\alpha}{4}K - \frac{1}{4}\Theta. \quad (18) \]

The right of Eq.(2) therefore acquires a new term proportional to the interfacial excess temperature, governed by its own equation.

4 Cellular flames in the stagnation-point flow

As an illustration, in this section, the new \( K - \Theta \) model is employed to describe the classical system of the diffusively unstable flame (\( \alpha > 1 \)) held in the stagnation-point flow \( U(x, y) = (q(x - L/2), -qy) \) \cite{11, 12, 13} (Fig. 1). Here \( q \) is the appropriately scaled flow intensity. The system is considered over half-strip \( (0 < x < L, y < 0) \) with the zero-gradient boundary conditions for the flame interface \( y = F(x, \tau) \) and its excess temperature \( \Theta(x, \tau) \),

\[ F_x(0, \tau) = \Theta_x(0, \tau) = 0, \quad F_x(L, \tau) = \Theta_x(L, \tau) = 0. \quad (19) \]

The problem is solved numerically for \( \alpha = 2.6, L = 800, 0.01 \leq q \leq 0.02 \). It is found, as expected

![Diagram of premixed planar flame (bold line) stabilized in stagnation-point flow field. The upstream flow corresponds to the fresh mixture, the downstream to the burned gas.](image1)

![Figure 2: Temporal evolution of the cellular flame.](image2)
near $y = -1/q$. However, if $q$ is small enough ($q = 0.014$) the central part of the flame becomes cellular with the cells in a state of permanent irregular self-motion similar to that occurring in freely propagating cellular flames. The concentration of the cells near the central stream-line is a new effect, which appears to be beyond the scope of the weakly nonlinear model explored previously [13].

Figure 2 shows the temporal evolution of the cellular flame at $q = 0.014$, figure 3 the instantaneous profiles of the flame interface and figure 4 its temperature.

![Figure 3: An instantaneous profile of the flame interface.](image)

![Figure 4: Temperature distribution for the profile in Fig.3](image)

5 Concluding remarks

The offered $K - \Theta$ model (17) (18) is a product of successive coordinate-free extrapolations based on several special cases studied previously. For all that, the model is not merely a plausible suggestion. An independent perturbative analysis, assuming the background flow-field to be slowly varying both in space and time, yields largely the same results. The basic difference appears only in $\Theta$ terms of Eqs. (17) (18), which are now replaced by $\exp \Theta - 1$ and $1/4 \Theta \exp \Theta$, respectively. When $\Theta$ is small, which is normally the case in adiabatic flames, both strategies lead to identical results, thereby vindicating the heuristic approach. Details of the perturbative derivation, extended from the 2D over the 3D flows, will be presented elsewhere. Note, that the exponential terms become important in problems of flame extinction by radiative heat losses. In this case the right of the modified Eq. (18) acquires the term $-1/4 \nu \exp(-\Theta)$ [10] with $\nu$ being the appropriately scaled Stefan-Boltzmann constant.

The $K - \Theta$ model (16) (17) and its various versions are not the only reduced models capable of mimicking the flame-flow interaction of diffusively unstable flames. Another possibility is suggested
by the geometrically invariant extension of the KS equation [4] [5] constituting a single fourth-order equation for the interface dynamics,

\[ V_n = 1 - (\alpha - 1)\kappa - 4\kappa_{ss} \]  

(20)

Again, replacing \( V_n \) by \( V_n - \mathbf{U} \cdot \mathbf{N} \), and \( \kappa \) by \(-K\) one obtains,

\[ V_n = \mathbf{U} \cdot \mathbf{N} + 1 - (\alpha - 1)K + 4K_{ss} \]  

(21)

Equation (21), for all its appeal, is less general than the system (17) (18), and may prove to be less rich both structurally and dynamically. As mentioned earlier, the weakly nonlinear version of Eq. (21), applied to the stagnation-point flow [13], does not capture localization of the cellular structure clearly seen in Fig.2, 3 and 4.

Acknowledgments The work of MLF was supported in part by the NSF Grant DMS-0207308, PG was partially supported by the NSF Grant DMS-0554775, and GIS was supported by the US-Israel Binational Science Foundation under Grant 2002-008, the Israel Science Foundation under Grants 350/05 and 278/03, and the European Community Program RTN-HPRN-CT-2002-00274.

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