Existence of traveling wave solutions for Ginzburg-Landau-type problems in infinite cylinder

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EXISTENCE OF TRAVELING WAVE SOLUTIONS FOR GINZBURG-LANDAU-TYPE PROBLEMS IN INFINITE CYLINDERS

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Abstract. We study a class of systems of reaction-diffusion equations in infinite cylinders. These systems of equations arise within the context of Ginzburg-Landau theories and describe the kinetics of phase transformation in second-order or weakly first-order phase transitions with non-conserved order parameter. We use a novel variational characterization to study existence of traveling wave solutions under very general assumptions on the nonlinearities. These solutions are a special class of the traveling wave solutions which are characterized by a fast exponential decay in the direction of propagation. Our main result is a simple verifiable criterion for existence of these traveling waves. We also prove boundedness, regularity, and some other properties of the obtained solutions, as well as several sufficient conditions for existence or non-existence of such traveling waves, and give rigorous upper and lower bounds for their speed. In addition, we prove that the speed of the obtained solutions gives a sharp upper bound for the propagation speed of a class of disturbances which are initially sufficiently localized. We give a sample application of our results using a computer-assisted approach.

1. Introduction

This paper is concerned with the study of traveling wave solutions of reaction-diffusion systems of gradient type

$$u_t = \Delta u + f(u), \qquad f(u) = -\nabla_u V(u). \tag{1.1}$$

Here, $u = u(x,t) \in \mathbb{R}^m$, $V : \mathbb{R}^m \to \mathbb{R}$, $x = (y,z) \in \Sigma = \Omega \times \mathbb{R}$, $\Omega \subset \mathbb{R}^{n-1}$ is a bounded domain, so Σ is an infinite cylinder. Either Neumann or Dirichlet boundary conditions can be chosen:

$$(n \cdot \nabla u)|_{\partial\Sigma} = 0, \text{ or } u|_{\partial\Sigma} = 0, \tag{1.2}$$

where n is the outward normal to $\partial\Sigma$ (in fact, one could treat more complicated boundary conditions in a similar way).

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Equation (1.1) is a prototypical equation in the theory of phase transition kinetics. Systems undergoing second-order or weakly first-order phase transitions are characterized by the presence of a "soft mode" near the transition temperature. This allows to introduce the concept of the "order parameter" to describe the thermodynamic state of the system near the transition point [1]. The order parameter is generally a vector field and can physically describe, e.g., the magnitude of the spontaneous polarization, magnetization, or a structural change in a crystal. If the order parameter is a non-conserved quantity, as is the case in ferroelectrics and ferromagnetics, for example, the relaxation of the soft mode toward equilibrium may be modeled as a simple gradient flow down the Ginzburg-Landau free energy (see, for example, [2–4])

$$u_t = -\frac{\delta F}{\delta u}, \qquad F[u] = \int \left(\frac{1}{2} \sum_{i=1}^{m} |\nabla u_i|^2 + V(u) \right) dx. \tag{1.3}$$

Here $F[u]$ is a free energy functional, in which $V(u)$ is local thermodynamic potential, typically obtained via a Taylor expansion and symmetry arguments (see, for example, [5, 6]), and the gradient term penalizes spatial variations of the order parameter [1, 5] (for the effect of anisotropy, see the end of Section 6).

We note that equations of Ginzburg-Landau type can sometimes be systematically derived from the more "microscopic" theories, such as kinetic Monte Carlo models, etc. (see, e.g., [7–9]). For example, the scalar ($m = 1$) Ginzburg-Landau equation can be derived by performing a gradient expansion of the nonlocal evolution equation obtained for the long-range Ising model subject to Glauber dynamics near the phase transition point [7]. Let us also point out that the choice of the boundary conditions is also dictated by the physics at the surface and is, therefore, problem-dependent. For example, in the context of coarse-grained spin systems with long-range interactions mentioned above the Dirichlet boundary conditions will be more appropriate, as opposed to the more conventional choice of Neumann boundary conditions in Ginzburg-Landau-type problems.

As an example, if u_i are the three components of the magnetization vector in a ferromagnetic crystal with cubic symmetry near Curie temperature, and h_i are the components of the applied field, the kinetics of u may be described by the following Ginzburg-Landau equation:

$$\tau \frac{\partial u_i}{\partial t} = g \Delta u_i + h_i + a u_i - b u_i^3 - c u_i \sum_{i \neq j} u_j^2, \tag{1.4}$$

where a, b, g, τ are all positive constants, and $c > -\frac{b}{2}$, in three space dimensions [5]. Note that Ginzburg-Landau-type equations often arise as results of normal form expansions near bifurcation points for partial differential equations (see, for example, [10]). Let us also point out that scalar reaction-diffusion equations, which automatically fall into the category of gradient systems, arise in a wide variety of applications, most notably in biology [11].

Traveling wave solutions are special solutions of Eq. (1.1) of the form $u(y, z, t) = \bar{u}(y, z - ct)$ with $c \in \mathbb{R}$, which describe uniformly translating "phase change regions", moving with speed c. In the following, with no loss of generality we will only consider the solutions moving from left to right, so we assume $c > 0$ everywhere below. This is an important class of solutions of Eq. (1.1) which is believed to describe the long-time asymptotics of the solutions of the initial value problem for

Eq. (1.1) with sufficiently localized initial data (for recent developments, see [12–14]). In fact, it was recently shown that under certain assumptions only a special class of traveling wave solutions can be selected as the long-time asymptotic solution for the initial value problem [15]. These so-called *variational traveling waves* are characterized by a fast exponential decay ahead of the traveling wave solution and admit an interesting variational characterization which allows to establish a number of their properties. This paper will be concerned with the problem of existence of such traveling wave solutions.

Substituting the traveling wave ansatz into Eq. (1.1), we obtain the following elliptic problem for $\bar{u}$:

$$\bar{u}_{zz} + \Delta_y \bar{u} + c\bar{u}_z + f(\bar{u}) = 0, \tag{1.5}$$

with the boundary conditions from Eq. (1.2). Let us point out that this equation has attracted a great deal of attention, starting with the early works in [16, 17]. In particular, the case of scalar equations (i.e., $m = 1$) has been extensively analyzed (see [18–20] for reviews, and more recent work in [21–24]). Much less is known about the solutions of Eq. (1.5) for systems ($m > 1$). So far general existence results were limited to the case of monotone systems for which the maximum principle holds [25], and gradient systems with bistable nonlinearities in one space dimension [26–30]. Here we are going to establish existence of certain solutions for Eq. (1.5) with the gradient-type nonlinearity under very general assumptions. Our main existence result is contained in the following theorem (for definitions and statements of hypotheses, see Section 2).

Theorem 1.1. *Under hypotheses (H1)–(H3), there exists* $c^\dagger > 0$ *and* $\bar{u} \in H^1_{c^\dagger}(\Sigma; \mathbb{R}^m)$, $\bar{u} \not\equiv 0$, *satisfying Eq. (1.5) with* $c = c^\dagger$. *Furthermore,* $\bar{u}$ *is a classical solution,* $\bar{u}(x) \in \mathcal{K}$ *for all* $x \in \Sigma$, *and* $|\bar{u}(y, z)| \leq Ce^{-\lambda z}$ *for some* $C > 0$ *and* $\lambda > 0$.

Let us give a summary of our results here. In Section 2 we introduce the functional spaces, the exponentially weighted Sobolev spaces of vector-valued functions $H^1_c(\Sigma; \mathbb{R}^m)$, and the main variational problem, problem (P), to be analyzed. Here we present the three main hypotheses on the nonlinearity in Eq. (1.5) and discuss their significance. Then, in Section 3, under the assumption of existence, we establish a number of properties of the minimizers of problem (P). In particular, we establish boundedness, regularity, and global gradient estimates of the solutions, as well as uniqueness of the speed of the solutions. Going further, in Section 4 we introduce a constrained variational problem, problem (P'), which will be used to establish existence of minimizers for problem (P). Here we show that existence of solutions for problem (P') implies that for problem (P).

Then, in Section 5 we prove existence of minimizers for problem (P'). This result is established via a sequence of lemmas associated with the properties of the exponentially weighted Sobolev spaces $H^1_c(\Sigma; \mathbb{R}^m)$. We first obtain a uniform estimate that allows to get information on the exponential decay of functions obeying the constraint and uniform estimates on the $|| \cdot ||_{1,c}$-norm. The crucial piece of the proof is establishing lower semicontinuity of the considered functional. This is done by estimating the measure of "bad" sets, the sets $\Omega_+(z)$, for functions in balls in $H^1_c(\Sigma, \mathbb{R}^m)$, as $z \to +\infty$, via an application of Local Isoperimetric Inequality and the Co-Area Formula.

In Section 6 we establish several criteria of existence and non-existence of the considered type of the traveling waves. We also prove a number of properties of the minimizers, such as their one-dimensionality in the case of Neumann boundary conditions, or the fact that for the potentials V that depend only on the magnitude of the vector u the minimizers are essentially scalar (up to a constant vector). We conclude this section by proving that in a certain class of solutions of the original parabolic problem the speed of the minimizers is in fact a sharp upper bound on the speed of propagation of disturbances. Finally, in Section 7 we consider a two-variable Ginzburg-Landau model as a sample application, for which we explicitly verify various assumptions of the analysis using a computer-assisted approach.

Notations. Throughout the paper, u_i denote the components of $u \in \mathbb{R}^m$; C^k, C_0^∞, $C^{k,\alpha}$ denote the usual spaces of continuous functions with k continuous derivatives, smooth functions with compact support, continuously differentiable functions with Hölder-continuous derivatives of order k for $\alpha \in (0,1)$ (or Lipschitz-continuous when $\alpha = 1$), respectively. Unless it is otherwise clear from the context, "$\cdot$" denotes a scalar product and $|\cdot|$ the Euclidean norm in $\mathbb{R}^n$ (occasionally, when there can be no confusion, we use this notation to denote the same quantities in $\mathbb{R}^m$). The symbol ∇ is reserved for the gradient in $\mathbb{R}^n$, while ∇_y stands for the gradient in $\Omega \subset \mathbb{R}^{n-1}$ (we use ∇_u to denote the gradient in $\mathbb{R}^m$). Similarly, the symbol Δ stands for the Laplacian in $\mathbb{R}^n$, and Δ_y for the Laplacian in Ω. By a classical solution of Eq. (1.5) we mean a function $u \in (C^2(\Sigma) \cap C^1(\bar{\Sigma}))^m$ that satisfies this equation with a given value of $c \in \mathbb{R}$ and the boundary conditions in Eq. (1.2). For any domain $\omega \subseteq \Omega$, the quantity $|\omega|$ denotes the Lebesgue measure of $\omega \subseteq \mathbb{R}^{n-1}$ (with the convention that $|\Omega| = 1$ for $n = 1$), and $|\partial\omega|$ that of the boundary of ω. The numbers C, K, M, λ, etc., will denote generic positive constants.

2. Preliminaries and variational formulation

In this section, we introduce a few basic definitions and state our main assumptions. Throughout this paper it is assumed that Ω is a bounded domain with boundary of class C^2. We now list some assumptions on the regularity and growth of $V(u)$.

(H1): The function $V : \mathbb{R}^m \to \mathbb{R}$ satisfies

$$V \in C^0(\mathbb{R}^m), \qquad V(0) = \nabla_u V(0) = 0, \qquad |V(u)| \leq C|u|^2 \tag{2.1}$$

for some C.

(H2): There exists a convex compact set $\mathcal{K} \subset \mathbb{R}^m$ which contains the origin, such that $V \in C^{1,1}(\mathcal{K})$ and for all $u \notin \mathcal{K}$

$$V(u) \geq V(\Pi_{\mathcal{K}}(u)), \tag{2.2}$$

where $\Pi_{\mathcal{K}} : \mathbb{R}^m \to \mathbb{R}^m$ is the projection on the set $\mathcal{K}$, i.e., $\Pi_{\mathcal{K}}(u)$ is the closest point to u which lies in $\mathcal{K}$.

Naturally, the mapping $\Pi_{\mathcal{K}}$ is well-defined, since $\mathcal{K}$ is convex.

Remark 2.1. Our existence results remain valid if V depends explicitly on y, provided that $V(\cdot, u) \in C^0(\overline{\Omega})$ and hypothesis (H1) holds uniformly in $\overline{\Omega}$, or if instead of Eq. (2.2) in hypothesis (H2) one assumes

$$(\nu \cdot \nabla_u V)|_{\partial\mathcal{K}} \geq 0, \tag{2.3}$$

where ν is any outward normal to $\partial\mathcal{K}$.

For the latter, if V is defined only within $\mathcal{K}$, we can always consider the following continuous extension of $V(u)$ to the whole of $\mathbb{R}^m$:

$$\widetilde{V}(u) = V(\Pi_{\mathcal{K}}(u)) + \nabla_u V(\Pi_{\mathcal{K}}(u)) \cdot (u - \Pi_{\mathcal{K}}(u)). \tag{2.4}$$

Indeed, $\widetilde{V}(u)$ is Lipschitz continuous on the whole $\mathbb{R}^m$ and, furthermore, is continuously differentiable up to the boundary of $\mathcal{K}$ [31]. Clearly, by Eq. (2.3) hypothesis (H2) holds for $\widetilde{V}$. Also, since $\Pi_{\mathcal{K}}$ is 1-Lipschitz, $\widetilde{V}$ satisfies the growth condition $|\widetilde{V}(u)| \leq C|u|^2$, and so hypothesis (H1) is also met by $\widetilde{V}$.

We note that in the context of Eq. (1.1) the set $\mathcal{K}$, together with an assumption like the one in Eq. (2.3), plays the role of an invariant region, and its existence is generally required for global existence of solutions for the initial value problem associated with Eq. (1.1) (see, for example, [32, 33]).

We now introduce the definition of the exponentially weighted Sobolev spaces we will be working in:

Definition 2.2. *For $c > 0$, denote by $H^1_c(\Sigma; \mathbb{R}^m)$ the completion of the restrictions of $(C_0^\infty(\mathbb{R}^n))^m$ to Σ with respect to the norm*

$$||u||^2_{1,c} = ||u||^2_{L^2_c(\Sigma)} + ||\nabla u||^2_{L^2_c(\Sigma)}, \qquad ||u||^2_{L^2_c(\Sigma)} = \sum_{i=1}^m \int_\Sigma e^{cz} |u_i|^2 dx.$$

For Dirichlet boundary conditions, replace $C_0^\infty(\mathbb{R}^n)$ with $C_0^\infty(\Sigma)$ above.

The weight appearing in the definition of spaces $H^1_c(\Sigma; \mathbb{R}^m)$ arises quite naturally in the context of propagation for Eq. (1.1) [15, 34]. Indeed, Eq. (1.1) written in the reference frame moving with speed c loses a variational structure of Eq. (1.3) because of the appearance of the term containing a first derivative. However, by multiplying this equation by an appropriate weight (e^{cz}) we get an equation which again has a variational structure [15].

Let us mention an important general property of the spaces $H^1_c(\Sigma; \mathbb{R}^m)$ which is an analogue of the Poincaré inequality and will be needed to establish the existence result.

Lemma 2.3. *For all $u \in H^1_c(\Sigma; \mathbb{R}^m)$, we have*

$$\frac{c^2}{4} \int_\Sigma e^{cz} \sum_{i=1}^m u_i^2 dx \leq \int_\Sigma e^{cz} \sum_{i=1}^m \left(\frac{\partial u_i}{\partial z} \right)^2 dx. \tag{2.5}$$

Proof. The proof follows from the estimate in Eq. (5.1) of Lemma 5.1 below, in the limit $R \to -\infty$. □

For $u \in H^1_c(\Sigma; \mathbb{R}^m)$ define two functionals

$$\Phi_c[u] = \int_\Sigma e^{cz} \left(\frac{1}{2} \sum_{i=1}^m |\nabla u_i|^2 + V(u) \right) dx, \tag{2.6}$$

$$\Gamma_c[u] = \frac{1}{2} \int_\Sigma e^{cz} \sum_{i=1}^m \left(\frac{\partial u_i}{\partial z} \right)^2 dx. \tag{2.7}$$

Clearly, by hypothesis (H1) both functionals are well-defined for all $u \in H^1_c(\Sigma; \mathbb{R}^m)$.

At least formally, Eq. (1.5) describing the traveling wave solutions of Eq. (1.1) is the Euler-Lagrange equation associated with the functional Φ_c [15]. A major difficulty, however, is the fact that the speed c of the traveling wave is also part of the solution and must, therefore, be determined simultaneously. Our approach to this question is via the following variational problem:

(P): *Find a non-trivial minimizer* $\bar{u} \in H^1_{c^\dagger}(\Sigma;\mathbb{R}^m)$ *of* $\Phi_{c^\dagger}$ *for some* $c^\dagger > 0$.

Now the speed $c = c^\dagger$ is part of the solution of problem (P), and we have

Proposition 2.4. *Let* $\bar{u}$ *be a solution of problem (P), with* $\bar{u}(x) \in \mathcal{K}$ *for all* $x \in \Sigma$. *Then* $\bar{u}$ *satisfies Eq. (1.5) weakly in* $H^1_c(\Sigma;\mathbb{R}^m)$ *with* $c = c^\dagger$.

Proof. Observe that by hypotheses (H1) and (H2) for $u \in \mathcal{K}$ the functional $\Phi_{c^\dagger}[u]$ is of class C^1 on $H^1_{c^\dagger}(\Sigma;\mathbb{R}^m)$. Therefore, if $\bar{u}$ is a minimizer of $\Phi_{c^\dagger}$, then for any $\varphi \in H^1_{c^\dagger}(\Sigma;\mathbb{R}^m)$

$$\int_\Sigma e^{c^\dagger z} \sum_{i=1}^m \left(\frac{\partial \bar{u}_i}{\partial z}\frac{\partial \varphi_i}{\partial z} + \nabla_y \bar{u}_i \cdot \nabla_y \varphi_i + \frac{\partial V(\bar{u})}{\partial u_i}\varphi_i \right) dx = 0, \tag{2.8}$$

which is a weak version of Eq. (1.5) with $c = c^\dagger$. □

We point out that under hypotheses (H1) and (H2) we will further prove regularity of the solutions of problem (P) (see Section 3 below). So these solutions are classical solutions of Eq. (1.5). Let us also mention that several other variational approaches to traveling waves exist [19, 35–37].

Before turning to the analysis of problem (P), let us introduce the following two constants

$$\nu_0 = \mu_0 + \liminf_{|u|\to 0} \frac{2V(u)}{|u|^2}, \qquad \mu_- = \min_{u\in\mathcal{K}} \frac{2V(u)}{|u|^2}, \tag{2.9}$$

where μ_0 is the smallest eigenvalue of $-\Delta_y$, and the lim inf is taken over $u \in \mathcal{K}$. Clearly, in view of hypothesis (H1), both are well-defined. These quantities play a crucial role for the existence of solutions of problem (P), as we will show below. To motivate their introduction, let us consider in more detail the decay of the solutions of Eq. (1.5) at plus infinity (see also [15]). To this end, let us linearize Eq. (1.5) around $u = 0$ at large z. Then the solutions of Eq. (1.5) that decay as $z \to +\infty$ are expected to be approximately a superposition of functions $u_k(y,z) = e^{-\lambda_k z} v_k(y)$, where λ_k satisfy

$$\lambda_k^2 - c\lambda_k - \nu_k = 0, \tag{2.10}$$

and $v_k(y) \in \mathbb{R}^m$ and $\nu_k \in \mathbb{R}$ are the eigenfunctions and the eigenvalues defined by

$$-\Delta_y v_i + \sum_{j=1}^m H_{ij}(0) v_j = \nu_k v_i, \qquad H_{ij}(u) = \frac{\partial^2 V(u)}{\partial u_i \partial u_j}, \tag{2.11}$$

where $H_{ij}(u)$ is the Hessian of the potential $V(u)$ (here we assume that V is twice differentiable at the origin), provided $\operatorname{Re}\lambda_k > 0$. We note that ν_k can, in turn, be broken up into a sum of the eigenvalue μ_k of $-\Delta_y$ in Ω with the boundary conditions from Eq. (1.2), and the eigenvalues of a symmetric matrix $H_{ij}(0)$, implying that ν_k are all real, bounded from below, and increasing as $k \to \infty$.

Equation (2.10) can be trivially solved to give

$$\lambda_k^\pm(c) = \frac{c \pm \sqrt{c^2 + 4\nu_k}}{2}, \tag{2.12}$$

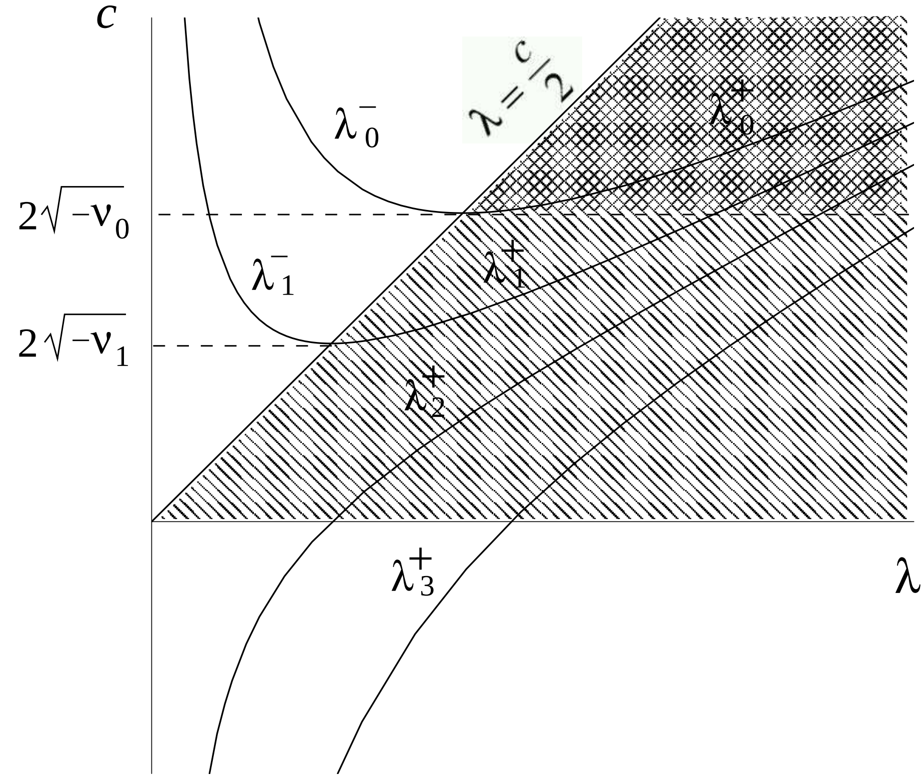

FIGURE 1. A qualitative form of the dependences $\lambda_k^{\pm}(c)$ for $\nu_0 < \nu_1 < 0 < \nu_2 < \ldots$

so for each $\nu_k \neq 0$ there is at least one solution with $\operatorname{Re}\lambda_k > 0$. Thus, in the case of twice differentiable $V(u)$ the value of ν_0 determines the slowest possible rate of decay of the solutions of Eq. (1.5) at plus infinity, corresponding to the plus sign in Eq. (2.12), while μ_- gives a lower bound for ν_0.

We now state the third assumption needed to establish existence of solutions of problem (P).

(H3): There exist $c > 0$ such that $c^2 + 4\nu_0 > 0$, and $u \in H_c^1(\Sigma; \mathbb{R}^m), u \not\equiv 0$, such that $\Phi_c[u] \leq 0$.

Let us explain the meaning of this assumption. The condition $c^2 + 4\nu_0 > 0$ ensures the weak lower semicontinuity of the functional Φ_c on $H_c^1(\Sigma; \mathbb{R}^m)$ (see Proposition 5.5), hence it is crucial to proving existence of minimizers for Φ_c. The condition $\Phi_c[u] \leq 0$ for some $u \not\equiv 0$, guarantees that the minimizer it is not identically equal to zero. Due to Proposition 3.5, this assumption is necessary in order to have traveling wave solutions of Eq. (1.5) lying in $H_c^1(\Sigma; \mathbb{R}^m)$.

Observe that, if $\nu_0 \geq 0$, the first condition in (H3) is automatically satisfies, and the second condition can be expressed only in terms of z-independent functions (see Proposition 6.2). On the other hand, if $\nu_0 < 0$, there exists a finite set of k's, for which $\nu_k < 0$. In turn, for those k's and $c^2 > -4\nu_0 \geq -4\nu_k$ there are two values of $\lambda_k = \lambda_k^{\pm} > 0$ that solve Eq. (2.10), with $\lambda_k^- < \frac{c}{2} < \lambda_k^+$, see Eq. (2.12). As an illustration, consider the case $\nu_0 < \nu_1 < 0 < \nu_2 < \ldots$, in Fig. 1. Here we show schematically the locations of the curves $\lambda_k^\pm$ as functions of c for the first four values of k. Since the solution of problem (P) belongs to $H^1_c(\Sigma;\mathbb{R}^m)$, it must decay faster than $e^{-cz/2}$ (hatched area in Fig. 1), and so all the solutions of Eq. (1.5) decaying asymptotically as $e^{-\lambda_k^- z}$ at plus infinity are automatically excluded. Hence, the hypothesis (H3), together with existence of solutions of Eq. (1.5), implies the existence of traveling waves with fast exponential decay. These are special solutions of Eq. (1.5), since generically one would expect the decay with the slower rate $e^{-\lambda_k^- z}$ at some k for which $\nu_k < 0$ (see also [10, 12, 15, 24, 38]). Under hypothesis (H3) we will be looking for the traveling waves moving with speed $c^\dagger \geq c > 2\sqrt{-\nu_0}$ when $\nu_0 < 0$ (see below), this region corresponds to the cross-hatched area in Fig. 1.

3. Properties of minimizers

Before proceeding to the construction of solutions to problem (P), we investigate a number of their properties. First, observe that both Φ_c and Γ_c transform similarly under translations.

Lemma 3.1. *Let $u \in H^1_c(\Sigma;\mathbb{R}^m)$ and $u_a(y,z) := u(y,z-a)$. Then, $u_a \in H^1_c(\Sigma;\mathbb{R}^m)$ also, and*

$$\Phi_c[u_a] = e^{ca}\Phi_c[u] \quad \text{and} \quad \Gamma_c[u_a] = e^{ca}\Gamma_c[u]. \tag{3.1}$$

From this Lemma, which is verified by direct inspection of the respective functionals, we get the following important

Proposition 3.2. *If $\bar u$ is a solution of problem (P), then $\Phi_{c^\dagger}[u] \geq 0$ for all $u \in H^1_{c^\dagger}(\Sigma;\mathbb{R}^m)$ and $\Phi_{c^\dagger}[\bar u] = 0$.*

Proof. The first statement is an obvious consequence of the fact that $\bar u$ is the minimizer, if the second statement holds. To prove the latter, we first note that $\inf_{u\in H^1_{c^\dagger}(\Sigma;\mathbb{R}^m)} \Phi_{c^\dagger}[u] \leq 0$, since zero is in $H^1_{c^\dagger}(\Sigma;\mathbb{R}^m)$. On the other hand, if $\Phi_{c^\dagger}[u] < 0$ for some $u \in H^1_{c^\dagger}(\Sigma;\mathbb{R}^m)$, then $\Phi_{c^\dagger}[u_a] < \Phi_{c^\dagger}[u]$, where $u_a \in H^1_{c^\dagger}(\Sigma;\mathbb{R}^m)$ is as in Lemma 3.1, with $a > 0$, hence there are no minimizers of $\Phi_{c^\dagger}$. □

In other words, the assumption about the existence of a non-trivial $u \in H^1_c(\Sigma;\mathbb{R}^m)$ such that $\Phi_c[u] \leq 0$ in hypothesis (H3) is in fact necessary, since the solution of problem (P) has this property for $c = c^\dagger$ by Proposition 3.2.

Next we establish *a priory* bounds on $\bar u$ and $\nabla \bar u$ for the solutions of problem (P).

Proposition 3.3. *If $\bar u$ is a solution of problem (P), then*
(i) *$\bar u(x) \in \mathcal{K}$ for all $x \in \Sigma$.*
(ii) *$\bar u \in (C^2(\Sigma) \cap C^1(\overline{\Sigma}))^m$ and $\nabla \bar u \in (L^\infty(\Sigma))^{mn}$.*
(iii) *For all $x = (y,z) \in \Sigma$ we have $|\bar u(y,z)| \leq Ce^{-\lambda z}$ for some $C > 0$ and $\lambda > 0$.*

Proof. (i) Let $\Pi_{\mathcal{K}} : \mathbb{R}^m \to \mathbb{R}^m$ be the projection on the convex set $\mathcal{K}$, as in hypothesis (H2). Recall that [31]

$$\Pi_{\mathcal{K}}(u) = u - d_{\mathcal{K}}(u)\nabla d_{\mathcal{K}}(u), \tag{3.2}$$

where $d_{\mathcal{K}}(u)$ is the distance of $u \in \mathbb{R}^m$ from the set $\mathcal{K}$. Then, if we replace $\bar u$ by $\tilde u := \Pi_{\mathcal{K}}(\bar u) \in H^1_c(\Sigma;\mathbb{R}^m)$, we have $V(\tilde u) \leq V(\bar u)$ by (2.2) and $\sum_{i=1}^m |\nabla \tilde u_i|^2 \leq \sum_{i=1}^m |\nabla \bar u_i|^2$ since Π_K is a 1-Lipschitz function. Let $W \subset \Sigma$ be defined as $W := \{x \in \Sigma : \ \bar u(x) \notin \mathcal{K}\}$ and assume, by contradiction, that W has positive measure.

Then, since the function $d_{\mathcal{K}}(\bar{u})$ is not constant on W, there exist a set $W' \subset W$ of positive measure and a constant $\delta > 0$ such that $|\nabla d_{\mathcal{K}}(\bar{u}(x))| \geq \delta$ a.e. on W'. Approximating $\bar{u}$ with smooth functions in $H^1_c(\Sigma;\mathbb{R}^m)$ and differentiating (3.2), we obtain

$$\int_{W'} e^{cz} \sum_{i=1}^{m} |\nabla \tilde{u}_i|^2 \, dx < \int_{W'} e^{cz} \sum_{i=1}^{m} |\nabla \bar{u}_i|^2 \, dx,$$

which implies $\Phi_c[\tilde{u}] < \Phi_c[\bar{u}]$ and contradicts the minimality of $\bar{u}$. So, $\bar{u}(x) \in \mathcal{K}$ for a.e. $x \in \Sigma$. Then, the statement of the Proposition follows from the regularity result below.

(ii) Since by the above result $\bar{u} \in (L^\infty(\Sigma))^m$ and f is continuous on the essential range of $\bar{u}$, we have $f_i \in L^p_{\text{loc}}(\Sigma)$, for any $p \geq 1$ and for all $1 \leq i \leq m$. So, choosing p sufficiently large and applying the De Giorgi-Nash theory to each component $\bar{u}_i$ of $\bar{u}$, we obtain that $\bar{u}_i \in C^{0,\alpha}(\Sigma)$, $1 \leq i \leq m$ with some $\alpha \in (0,1)$ (see, for example, [39, Theorem 8.22]). Then, since $f \in C^{0,1}(\mathcal{K})$ by hypothesis (H2), it follows from Schauder theory [39] that $\bar{u} \in (C^{2,\alpha}(\Sigma))^m$.

To obtain $C^{1,\alpha}$ regularity of $\bar{u}$ up to the boundary of Σ and a uniform estimate for $\nabla\bar{u}$, we apply to each component $\bar{u}_i$ the classical $W^{2,p}$ $(p > n)$ regularity theory (see, e.g., [40, 41]), which can be easily adapted to the case of a fixed slice of the cylinder Σ. We shall give the proof in detail in the case of Dirichlet boundary condition; for Neumann boundary conditions, using the the estimates of [40] (see also [41]) instead of the estimates of [39], and recalling that $\partial\Sigma$ is uniformly of class C^2, the same proof can be easily adapted.

By setting $v_i := \bar{u}_i e^{cz/2}$, one can see that after a change of variables Eq. (1.5) with $\bar{u} \in H^1_c(\Sigma;\mathbb{R}^m)$ is equivalent to

$$\Delta v_i - \frac{c^2}{4} v_i = f_i(\bar{u}) e^{cz/2} \qquad v_i \in H^1(\Sigma),\ v|_{\partial\Sigma} = 0, \tag{3.3}$$

where $H^1(\Sigma)$ is the usual Sobolev space.

For fixed $z_1, z_2, \tilde{z}_1, \tilde{z}_2 \in \mathbb{R}$, with $[\tilde{z}_1, \tilde{z}_2] \subset (z_1, z_2)$, consider the slices $\Sigma_0 := \Omega \times (z_1, z_2)$ and $\tilde{\Sigma}_0 := \Omega \times (\tilde{z}_1, \tilde{z}_2)$ of the cylinder Σ. Since $\bar{u}_i \in L^\infty(\Sigma)$ (by part (i)), the right-hand side of Eq. (3.3) is in $L^p(\tilde{\Sigma}_0)$ for all $p > 1$. By standard regularity theory (see, e.g., [39, Theorem 8.12 and Theorem 9.16]), we deduce $v_i \in W^{2,p}(\tilde{\Sigma}_0)$. Moreover, the apriori estimate given by [39, Theorem 9.13] to Eq. (3.3) on the domains $\tilde{\Sigma}_0$ and Σ_0 yields

$$\begin{aligned} \|v_i\|_{W^{2,p}(\Sigma_0)} &\leq C\left(\|v_i\|_{L^p(\tilde{\Sigma}_0)} + \left\|f_i(\bar{u})e^{cz/2}\right\|_{L^p(\tilde{\Sigma}_0)}\right) \\ &= C\left(\left\|u_i e^{cz/2}\right\|_{L^p(\tilde{\Sigma}_0)} + \left\|f_i(\bar{u})e^{cz/2}\right\|_{L^p(\tilde{\Sigma}_0)}\right) \end{aligned} \tag{3.4}$$

where C depends on the parameters n, p, c and the geometry of Σ_0, $\tilde{\Sigma}_0$. Since both $\bar{u}_i, f_i(\bar{u}) \in L^\infty(\Sigma)$, we can set $M = \max\{||\bar{u}_i||_{L^\infty(\Sigma)}, ||f_i(\bar{u})||_{L^\infty(\Sigma)}\}$ to obtain

$$\|v_i\|_{W^{2,p}(\Sigma_0)} \leq 2MC \left\|e^{cz/2}\right\|_{L^p(\tilde{\Sigma}_0)} \leq 2MC|\tilde{\Sigma}_0| e^{c\tilde{z}_2/2}.$$

By choosing $p > n/2$ and applying [39, Theorem 7.26], we deduce that

$$||v_i||_{C^1(\overline{\Sigma}_0)} \leq 2MCS|\tilde{\Sigma}_0|^{\frac{1}{n} - \frac{1}{p} + 1} e^{c\tilde{z}_2/2}, \tag{3.5}$$

where $S = S(n,p)$ is the Sobolev imbedding constant. Hence, coming back to the function u and using the inequality in Eq. (3.5) we get that $\bar{u}_i \in C^1(\overline{\Sigma}_0)$, and for any $(y,z) \in \overline{\Sigma}_0$

$$\begin{aligned} |\nabla \bar{u}_i(y,z)| &= |\nabla(v_i e^{-cz/2})| \leq |e^{-cz/2}\nabla v_i| + \frac{c}{2}|e^{-cz/2} v_i| \\ &\leq (2+c) MCS |\tilde{\Sigma}_0|^{\frac{1}{n}-\frac{1}{p}+1} e^{c(\tilde{z}_2-\tilde{z}_1)/2} = C', \end{aligned}$$

where the constant C' is invariant with respect to translations of the slice along z. So, translating the slices $\Sigma_0, \tilde{\Sigma}_0$ simultaneously along z, we obtain the estimate for all $x \in \Sigma$.

(iii) Now we prove the uniform exponential decay of u as $z \to +\infty$. Suppose, to the contrary, there exists a sequence $x_k = (y_k, z_k) \in \Sigma$, such that $z_k \to +\infty$ and $|\bar{u}(x_k)| e^{\lambda z_k} \to \infty$ for all $\lambda > 0$. Since $\partial\Omega$ is Lipschitz continuous, Σ satisfies the uniform interior cone property. So there exists a cone $\mathcal{C}_\Sigma$ (with finite height) such that each point $x_k \in \Sigma$ is the vertex of a cone $\mathcal{C}_k$ congruent to $\mathcal{C}_\Sigma$ that lies in $\bar{\Sigma}$. Up to a subsequence, we can further assume that $\mathcal{C}_i \cap \mathcal{C}_j = \varnothing$ for all $i \neq j$. By the previous result, we have $\nabla \bar{u} \in (L^\infty(\Sigma))^{mn}$, so $|\bar{u}(x)| \geq \frac{1}{2}|\bar{u}(x_k)|$ for all $x \in \tilde{\mathcal{C}}_k$, where $\tilde{\mathcal{C}}_k$ is a smaller cone similar to $\mathcal{C}_k$, with the same vertex and $|\tilde{\mathcal{C}}_k| = \min\{|\mathcal{C}_k|, \epsilon|\bar{u}(x_k)|^n\}$ for some $\epsilon > 0$ (recall that $n = \dim \Sigma$). By assumption we have $|u(x_k)| \geq e^{-\lambda z_k}$ for all $k \geq N$ for some integer N, and also we can choose N large enough that $|\tilde{\mathcal{C}}_k| \geq \epsilon e^{-n\lambda z_k}$. But this implies

$$\int_\Sigma e^{c^\dagger z} |\bar{u}|^2 dx \geq \sum_{k=1}^\infty \int_{\tilde{\mathcal{C}}_k} e^{c^\dagger z} |\bar{u}|^2 dx \geq \frac{\epsilon}{4} \sum_{k=N}^\infty e^{(c^\dagger - \lambda(2+n)) z_k} = \infty$$

for $\lambda = c^\dagger/(2+n)$, which contradicts the fact that $\bar{u} \in H^1_{c^\dagger}(\Sigma; \mathbb{R}^m)$. □

Let us point out that the obtained value of $\lambda = c^\dagger/(2+n)$ in the proof above is not at all sharp. It should be possible to obtain better estimates for λ by studying the asymptotic behavior of solutions of Eq. (1.5) at plus infinity. However, for $n \leq 3$ it is possible to get a sharp estimate on the rate of decay of the solution. Indeed, since the right hand side of (3.4) with $p = 2$ is uniformly bounded by the $L^2_{c^\dagger}$-norm of the solution (recall also hypothesis (H2)) on Σ, it follows that $\|v_i\|_{W^{2,2}(\Sigma)} \leq C$ for some constant $C > 0$. Hence from the Sobolev embedding Theorem we get $\|v_i\|_{C^0(\Sigma)} \leq C$, implying $|u(y,z)| \leq C e^{-c^\dagger z/2}$, which is sharp.

A crucial property of the considered variational problem is uniqueness of the speed $c^\dagger$ (this point was already briefly discussed in [15]).

Proposition 3.4. *The value of $c^\dagger$ in the solution of problem (P) is unique.*

Proof. Assume by contradiction that there exist $c_2^\dagger > c_1^\dagger$, together with $\bar{u}^{(1)} \in H^1_{c_1^\dagger}(\Sigma; \mathbb{R}^m)$ and $\bar{u}^{(2)} \in H^1_{c_2^\dagger}(\Sigma; \mathbb{R}^m)$ such that

$$\Delta \bar{u}^{(1,2)} + c_{1,2}^\dagger \frac{\partial \bar{u}^{(1,2)}}{\partial z} + f(\bar{u}^{(1,2)}) = 0.$$

Let us first show that $\bar{u}^{(2)} \in H^1_{c_1^\dagger}(\Sigma;\mathbb{R}^m)$. Since $c_2^\dagger > c_1^\dagger$, we have

$$\begin{aligned}\int_{-\infty}^{+\infty}\int_\Omega e^{c_1^\dagger z}|\bar{u}^{(2)}|^2 dydz &= \int_{-\infty}^{0}\int_\Omega e^{c_1^\dagger z}|\bar{u}^{(2)}|^2 dydz + \int_{0}^{+\infty}\int_\Omega e^{c_1^\dagger z}|\bar{u}^{(2)}|^2 dydz \\ &\le \frac{M|\Omega|}{c_1^\dagger} + \int_0^\infty \int_\Omega e^{c_2^\dagger z}|\bar{u}^{(2)}|^2 dydz \\ &\le \frac{M|\Omega|}{c_1^\dagger} + \int_\Sigma e^{c_2^\dagger z}|\bar{u}^{(2)}|^2 dx < \infty,\end{aligned}$$

for some $M > 0$, where we took into account that the solutions of problem (P) are uniformly bounded by Lemma 3.3. Since, in turn, by Proposition 3.3 we have $\nabla \bar{u}^{(1,2)} \in (L^\infty(\Sigma))^{mn}$ as well, this argument can be repeated for the gradient. So $\bar{u}^{(2)} \in H^1_{c_1^\dagger}(\Sigma;\mathbb{R}^m)$.

In order to get a contradiction, let us scalar multiply Eq. (1.5), with $\bar{u} = \bar{u}^{(2)}$ and $c = c_2^\dagger$, by $e^{c_1^\dagger z}\frac{\partial \bar{u}^{(2)}}{\partial z}$ and integrate over Σ. This is justified since $\bar{u}^{(2)}$ is a classical solution of Eq. (1.5) by Proposition 3.3, hence we can integrate the expression over the domain $\Sigma_R := \Omega \times (-R, R)$ and then let $R \to +\infty$ on a suitable sequence. After a number of integrations by parts, we obtain

$$\begin{aligned} 0 &= \int_\Sigma e^{c_1^\dagger z} \sum_{i=1}^m \frac{\partial \bar{u}_i^{(2)}}{\partial z}\left(\frac{\partial^2 \bar{u}_i^{(2)}}{\partial z^2} + c_2^\dagger \frac{\partial \bar{u}_i^{(2)}}{\partial z} + \Delta_y \bar{u}_i^{(2)} + f_i(\bar{u}^{(2)})\right) dx \\ &= \Phi_{c_1^\dagger}[\bar{u}^{(2)}] + 2(c_2^\dagger - c_1^\dagger)\Gamma_{c_1^\dagger}[\bar{u}^{(2)}], \end{aligned} \tag{3.6}$$

which implies that $\Phi_{c_1^\dagger}[\bar{u}^{(2)}] < 0$, contradicting Proposition 3.2. □

We now extend the result in Proposition 3.2 to any classical solution $\bar{u}$ of Eq. (1.5) which lies in $H^1_c(\Sigma;\mathbb{R}^m)$.

Proposition 3.5. *Let $\bar{u} \in H^1_c(\Sigma;\mathbb{R}^m)$ be a classical solution of Eq. (1.5). Then*

$$\Phi_c[\bar{u}] = 0. \tag{3.7}$$

Proof. We scalar multiply as above Eq. (1.5) by $e^{cz}\frac{\partial \bar{u}}{\partial z}$ and integrate over Σ. The result then follows exactly as in Eq. (3.6). □

Observe that $\bar{u}$ in Proposition 3.5 may or may not be a solution of problem (P). In the first case we have $c = c^\dagger$ and all the critical points of Φ_c in $H^1_c(\Sigma;\mathbb{R}^m)$ are solutions of problem (P). In the second case we have $c < c^\dagger$, and $\bar{u}$ is only a critical point of Φ_c, and not a minimizer. This means that the solutions of Eq. (1.5) obtained by solving problem (P) are the fastest moving traveling waves within a class of sufficiently rapidly decaying solutions. This also means that, under hypotheses (H1) and (H2), if there exists a traveling wave solution $\bar{u} \in H^1_c(\Sigma;\mathbb{R}^m)$ with speed c satisfying $c^2 + 4\nu_0 > 0$, then problem (P) has a solution. This follows from the fact that in this case $\bar{u}$ is the function whose existence is required by (H3) [15].

4. Constrained minimization problem

To proceed with establishing existence of solutions for problem (P), let us make a simple, but crucial observation about the translational invariance of Eq. (1.5) in the z-direction, which leads to a natural loss of compactness. From the variational viewpoint, under the assumption of existence of a non-trivial $u \in H_c^1(\Sigma;\mathbb{R}^m)$ such that $\Phi_c[u] \leq 0$ in hypothesis (H3), which is necessary for existence of solution of problem (P), one cannot expect any kind of coercivity for the functional $\Phi_c[u]$, since the sequence of $u_n(z,y) := \bar{u}(z-n,y)$ has the property that $\Phi_c[u_n] \leq 0$, while $||u_n||_{1,c} \to \infty$ by Proposition 3.2.

To deal with this issue, we introduce an auxiliary variational problem. Define

$$\mathcal{B}_c := \{u \in H_c^1(\Sigma;\mathbb{R}^m) : \Gamma_c[u] = 1\}. \tag{4.1}$$

Then consider the following constrained variational problem:

(P'): *Find $u_c \in \mathcal{B}_c$ satisfying:* $\Phi_c[u_c] = \inf_{\mathcal{B}_c} \Phi_c[u] \leq 0.$

It is easy to see that the constraint $\mathcal{B}_c$ gives a natural way to fix translations along the axis the cylinder. In particular, the functional Φ_c becomes coercive on $\mathcal{B}_c$ (see Lemma 5.2).

In the following, we will show that existence of solutions of problem (P') implies the same for problem (P). Let us begin by proving that the solutions of the problem (P') also lie within $\mathcal{K}$.

Lemma 4.1. *Let u_c be a solution of problem (P'). Then $u(x) \in \mathcal{K}$ for a.e. $x \in \Sigma$.*

Proof. We use the same projection argument as in Proposition 3.3. Namely, suppose that $u_c(x)$ is not in $\mathcal{K}$ on a set of non-zero measure. Then, repeating the arguments of Proposition 3.3, we get $\Phi_c[\tilde{u}] < \Phi_c[u_c] \leq 0$, where $\tilde{u} := \Pi_{\mathcal{K}}(u_c)$. Similarly, $\Gamma_c[u_c] \geq \Gamma_c[\tilde{u}] > 0$, where the last inequality follows from the fact that $\Phi_c[\tilde{u}] < 0$. So, by Lemma 3.1 there exists a constant $a \geq 0$ such that $\tilde{u}_a(y,z) := \tilde{u}(y,z-a)$ is in $\mathcal{B}_c$, and $\Phi_c[\tilde{u}_a] \leq \Phi_c[\tilde{u}]$. Therefore, u_c is not a minimizer of problem (P'), leading to contradiction. □

The following Proposition establishes the connection between the solutions of problems (P) and (P').

Proposition 4.2. *If u_c is a solution of problem (P'), then*

$$\bar{u}(y,z) = u_c(y, z\sqrt{1-\Phi_c[u_c]}) \quad \text{and} \quad c^\dagger = c\sqrt{1-\Phi_c[u_c]}, \tag{4.2}$$

are those for problem (P).

Proof. First of all, as in Proposition 2.4, we have Φ_c and Γ_c of class C^1 on $H_c^1(\Sigma;\mathbb{R}^m)$. Let $D\Gamma_c[u]v$ be the Fréchet derivative of Γ_c at u acting on v. Since

$$D\Gamma_c[u]u = \int_\Sigma e^{cz} \sum_{i=1}^m \left(\frac{\partial u_i}{\partial z}\right)^2 dx = 2 \qquad \forall u \in \mathcal{B}_c,$$

we get $D\Gamma_c[u] \not\equiv 0$ on the constraint. Thus, applying the Lagrange Multiplier Theorem (see, for example, [42, Section 3.5]) we obtain that

$$\int_\Sigma e^{c^\dagger z} \sum_{i=1}^m \left((1-\mu)\frac{\partial u_{c,i}}{\partial z}\frac{\partial \varphi_i}{\partial z} + \nabla_y u_{c,i} \cdot \nabla_y \varphi_i + \frac{\partial V(u_c)}{\partial u_i}\varphi_i \right) dx = 0. \tag{4.3}$$

where μ is the Lagrange multiplier.

Let us now show that $\mu < 1$. Indeed, suppose the opposite is true. Fix $a > 0$, and consider $u_a(x) := e^{-ca/2}u_c(x) \in \mathcal{K}$ a.e. (recall that $\mathcal{K}$ is convex, and $0 \in \mathcal{K}$). Then, for $v = u_a - u_c \in H_c^1(\Sigma;\mathbb{R}^m)$ the Fréchet derivatives of Φ_c and Γ_c on u_c satisfy

$$D\Phi_c[u_c]v = \mu D\Gamma_c[u_c]v \leq 2(e^{-ca/2}-1) < 0, \tag{4.4}$$

where we recalled that $u_c \in \mathcal{B}_c$. Therefore, since Φ_c is of class C^1, there exists a sufficiently small $a > 0$ such that $\Phi_c[u_a] < \Phi_c[u_c] \leq 0$. Now, consider $\tilde{u}_a(y,z) := u_a(y, z-a)$. A straightforward calculation then shows that $\tilde{u}_a \in \mathcal{B}_c$. However, by Lemma 3.1 and the fact that $\Phi_c[u_a] < 0$, we obtain that $\Phi_c[\tilde{u}_a] < \Phi_c[u_c]$, contradicting the fact that u_c is a minimizer.

So, $\mu < 1$, and from Lemma 4.1 and the argument of Proposition 3.3 we deduce that $u_c \in (C^2(\Sigma))^m$ and satisfies

$$(1-\mu)\left(\frac{\partial^2 u_c}{\partial z^2} + c\frac{\partial u_c}{\partial z}\right) + \Delta_y u_c + f(u_c) = 0. \tag{4.5}$$

Now, we scalar multiply Eq. (4.5) by $e^{cz}\frac{\partial \bar{u}}{\partial z}$ as in Proposition 3.4, and integrate over Σ to obtain

$$\begin{aligned} 0 &= \int_\Sigma e^{cz} \sum_{i=1}^m \left[(1-\mu)\frac{\partial u_{c,i}}{\partial z}\frac{\partial^2 u_{c,i}}{\partial z^2} + \nabla_y u_{c,i} \cdot \frac{\partial}{\partial z}\nabla_y u_{c,i} + \frac{\partial V(u_c)}{\partial u_i}\frac{\partial u_{c,i}}{\partial z} \right] dx \\ &= c(\mu - \Phi_c[u_c]), \end{aligned}$$

where we recalled that $\Gamma_c[u_c] = 1$. This means that

$$\mu = \Phi_c[u_c]. \tag{4.6}$$

To show that $\bar{u}$ and $c^\dagger$ are the solutions of problem (P), first fix $u \in H^1_{c^\dagger}(\Sigma;\mathbb{R}^m)$ and introduce $\tilde{u}(y,\zeta) = u\left(y, \frac{\zeta}{\sqrt{1-\mu}}\right)$, which is possible since $\mu < 1$. Then

$$\int_{-\infty}^{+\infty}\int_\Omega e^{c\zeta}\tilde{u}_i^2 dyd\zeta = \sqrt{1-\mu}\int_{-\infty}^{+\infty}\int_\Omega e^{c^\dagger z} u_i^2 dydz, \tag{4.7}$$

$$\int_{-\infty}^{+\infty}\int_\Omega e^{c\zeta}|\nabla_y \tilde{u}_i|^2 dyd\zeta = \sqrt{1-\mu}\int_{-\infty}^{+\infty}\int_\Omega e^{c^\dagger z}|\nabla_y u_i|^2 dydz, \tag{4.8}$$

$$\int_{-\infty}^{+\infty}\int_\Omega e^{c\zeta}\left(\frac{\partial \tilde{u}_i}{\partial z}\right)^2 dyd\zeta = \frac{1}{\sqrt{1-\mu}}\int_{-\infty}^{+\infty}\int_\Omega e^{c^\dagger z}\left(\frac{\partial u_i}{\partial z}\right)^2 dydz. \tag{4.9}$$

Therefore, $\tilde{u} \in H^1_c(\Sigma;\mathbb{R}^m)$, and

$$\begin{aligned} \Phi_{c^\dagger}[u] = \int_{-\infty}^{+\infty}\int_\Omega e^{c^\dagger z}\left(\frac{1}{2}\sum_{i=1}^m\left[\left(\frac{\partial u_i}{\partial z}\right)^2 + |\nabla_y u_i|^2\right] + V(u)\right) dydz = \\ \frac{1}{\sqrt{1-\mu}}\int_{-\infty}^{+\infty}\int_\Omega e^{c\zeta}\left(\frac{1}{2}\sum_{i=1}^m\left[(1-\mu)\left(\frac{\partial \tilde{u}_i}{\partial \zeta}\right)^2 + |\nabla_y \tilde{u}_i|^2\right] + V(\tilde{u})\right) dyd\zeta = \\ \frac{1}{\sqrt{1-\mu}}\left(\Phi_c[\tilde{u}] - \mu\Gamma_c[\tilde{u}]\right). \end{aligned} \tag{4.10}$$

Now we claim that if the solution of problem (P') exists, then

$$\Phi_c[\tilde{u}] \geq \mu\Gamma_c[\tilde{u}]. \tag{4.11}$$

Indeed, if $\Gamma_c[\tilde{u}] = 0$, then by Lemma 2.3 and hypothesis (H1) we have $\Phi_c[\tilde{u}] = 0$ also, so Eq. (4.11) holds trivially. On the other hand, if $\Gamma_c[\tilde{u}] > 0$, then there exists a constant $a \in \mathbb{R}$ such that the translated function $\tilde{u}_a(y,z) := \tilde{u}(y,z-a)$ of $\tilde{u}$ is in $\mathcal{B}_c$. Hence, $\Phi_c[\tilde{u}_a] \geq \Phi_c[u_c] = \mu\Gamma_c[\tilde{u}_a]$, and by Lemma 3.1 the inequality in Eq. (4.11) holds for $\tilde{u}$, with equality achieved when $\tilde{u} = u_c$. Hence, by Eq. (4.10) we have $\Phi_{c^\dagger}[u] \geq 0$ for all $u \in H^1_{c^\dagger}(\Sigma;\mathbb{R}^m)$, and $\bar{u}$ gives the solution of problem (P). □

5. Existence of constrained minimizers

To proceed in the proof of existence of constrained minimizers, we need to establish some compactness properties of the sublevel sets of Φ_c. Since we will work in the weak topology of $H^1_c(\Sigma;\mathbb{R}^m)$, which is a Hilbert space, it is enough to show that Φ_c has bounded sublevel sets (i.e. it is coercive). This, however, may be problematic for Φ_c in general, even after eliminating translations in some way. As a simple example, consider $\Phi_c[u] = \frac{1}{2}\int_{\mathbb{R}} e^{cz}(u_z^2 - u^2 + u^4)\,dz$, with $u : \mathbb{R} \to \mathbb{R}$. It is easy to see that this functional is not coercive on $H^1_c(\mathbb{R})$ when $c = 2$. Indeed, consider a sequence of functions $u_n \in H^1_c(\mathbb{R})$ defined as $u_n(z) = e^{-(1+1/n)z}$ for $z \geq 0$ and $u_n(z) = 1$ for $z < 0$. Clearly, the sequence (u_n) is not bounded in $H^1_c(\mathbb{R})$. However, a straightforward calculation shows that $\Phi_c[u_n] = (3n^2+5n+2)/(2n(n+2)) < \infty$. This difficulty in fact is not merely technical, and puts certain limitations on the applicability of our variational approach. In particular, as can be seen from the example just mentioned, it cannot be used to characterize the minimal speed traveling waves in systems with Fisher-type nonlinearities (see also Proposition 6.1 and [15]).

Even if coercivity of Φ_c may not hold in general, in Lemma 5.2 we show that it does hold if we consider the intersection of the sublevel sets of Φ_c with the set $\mathcal{B}_c$ defined in (4.1). So, establishing existence for problem (P') amounts to proving weak sequential lower semicontinuity of Φ_c. Here, again, there is a difficulty, since Σ is an unbounded domain and $V(u)$ is allowed to be negative, so the standard theory [43] does not apply. In the following we will establish sequential lower semicontinuity of the functional Φ_c under the assumption $c^2 + 4\nu_0 > 0$ from hypothesis (H3). This assumption is also essential, as it is possible to construct sequences in $H^1_c(\Sigma;\mathbb{R}^m)$ on which Φ_c "jumps up", if this condition is not satisfied. Consider again the functional in the example above, but now with $c = 1$, and take $u_n \in H^1_c(\mathbb{R})$ to be defined as $u_n(z) = \frac{1}{\sqrt{n}}e^{-z/2-z^2/n^2}$ for $z \geq 0$ and $u_n(z) = \frac{1}{\sqrt{n}}$ for $z < 0$. It is not difficult to see that the sequence (u_n) is bounded in $H^1_c(\mathbb{R})$, and furthermore u_n converge weakly to zero. However, a calculation shows that $\lim_{n\to\infty}\Phi_c[u_n] = -3\sqrt{\pi}/(16\sqrt{2}) < 0 = \Phi_c[0]$. It is also clear that for this example $c^2 + 4\nu_0 = -3 < 0$, and the assumption of hypothesis (H3) is violated.

We begin by proving the following lemma about a Poincaré-type inequality in the weighted Sobolev space $H^1_c(\Sigma;\mathbb{R}^m)$.

Lemma 5.1. *Let $u \in H^1_c(\Sigma;\mathbb{R}^m)$. Then*

$$\frac{c^2}{4}\int_R^{+\infty}\int_\Omega e^{cz}\sum_{i=1}^m u_i^2 dydz \leq \int_R^{+\infty}\int_\Omega e^{cz}\sum_{i=1}^m \left(\frac{\partial u_i}{\partial z}\right)^2 dydz, \tag{5.1}$$

$$\int_\Omega \sum_{i=1}^m u_i^2(y,R)dy \leq \frac{e^{-cR}}{c}\int_R^{+\infty}\int_\Omega e^{cz}\sum_{i=1}^m \left(\frac{\partial u_i}{\partial z}\right)^2 dydz, \tag{5.2}$$

for any $R\mathbb{R}$.

Proof. Let us first prove Eq. (5.1)

$$\frac{c}{2}\int_R^{+\infty}\int_\Omega e^{cz}u_i^2\,dydz = -\frac{1}{2}e^{cR}\int_\Omega u_i^2(y,R)dy - \int_R^{+\infty}\int_\Omega e^{cz}u_i\frac{\partial u_i}{\partial z}dydz$$
$$\leq \left(\int_R^{+\infty}\int_\Omega e^{cz}u_i^2\,dydz\right)^{1/2}\left(\int_R^{+\infty}\int_\Omega e^{cz}\left(\frac{\partial u_i}{\partial z}\right)^2 dydz\right)^{1/2},$$

which implies (5.1).

Turn to Eq. (5.2) now. Since

$$\int_R^{+\infty}\int_\Omega e^{cz}\left(\sqrt{c}u_i+\frac{1}{\sqrt{c}}\frac{\partial u_i}{\partial z}\right)^2 dydz\geq 0,$$

we get

$$\begin{aligned}\frac{1}{c}\int_R^{+\infty}\int_\Omega e^{cz}\left(\frac{\partial u_i}{\partial z}\right)^2 dydz &\geq -2\int_R^{+\infty}\int_\Omega e^{cz}u_i\frac{\partial u_i}{\partial z}dydz-c\int_R^{+\infty}\int_\Omega e^{cz}u_i^2dydz\\ &= e^{cR}\int_\Omega u_i^2(y,R)dy,\end{aligned}$$

which gives Eq. (5.2). □

The following Lemma estimates the norm of $\nabla_y u$ on $\mathcal{B}_c$ and, via Lemma 2.3, establishes coercivity of the functional Φ_c on $\mathcal{B}_c$.

Lemma 5.2. *Let V satisfy hypotheses (H1) and (H2), and let $u\in\mathcal{B}_c$. Then*

$$\int_\Sigma e^{cz}\sum_{i=1}^m|\nabla_y u_i|^2dx\leq 2\Phi_c[u]+\frac{8|\mu_-|}{c^2}, \tag{5.3}$$

where μ_- is defined in Eq. (2.9).

Proof. By Eqs. (2.9) and Lemma 2.3, we have

$$\int_\Sigma e^{cz}V(u)dx\geq -\frac{|\mu_-|}{2}\int_\Sigma e^{cz}\sum_{i=1}^m u_i^2dx\geq -\frac{2|\mu_-|}{c^2}\int_\Sigma e^{cz}\sum_{i=1}^m\left(\frac{\partial u_i}{\partial z}\right)^2dx.$$

Hence, for $\Gamma_c[u]=1$ we have

$$\frac{1}{2}\int_\Sigma e^{cz}|\nabla_y u|^2dx\leq\Phi_c[u]-\int_\Sigma e^{cz}V(u)\,dx\leq\Phi_c[u]+\frac{4|\mu_-|}{c^2},$$

which is equivalent to Eq. (5.3). □

We now turn to the question of lower semicontinuity. Let us introduce the following notation:

$$\Phi_c[u,(a,b)]=\int_a^b\int_\Omega e^{cz}\left(\frac{1}{2}\sum_{i=1}^m|\nabla u_i|^2+V(u)\right)dydz. \tag{5.4}$$

We will analyze the behavior of $\Phi_c[u,(-\infty,R)]$ and $\Phi_c[u,(R,+\infty)]$ on a weakly converging sequence and take the limit $R\to+\infty$. To this end, we first establish the sequential lower semicontinuity of $\Phi_c[u,(-\infty,R)]$ for all $R\in\mathbb{R}$, with respect to the weak topology of $H^1_c(\Sigma;\mathbb{R}^m)$.

Lemma 5.3. *Let V satisfy hypotheses (H1) and (H2), and let $u_n\rightharpoonup u$ in $H^1_c(\Sigma;\mathbb{R}^m)$. Then,*

$$\liminf_{n\to\infty}\Phi_c[u_n,(-\infty,R)]\geq\Phi_c[u,(-\infty,R)]$$

for any $R\in\mathbb{R}$.

Proof. This follows by standard semicontinuity results (see, for example, [43, Propositions 2.1, 2.2]) by considering $v:=e^{cz/2}u\in H^1(\Sigma;\mathbb{R}^m)$ and using the fact that by hypothesis (H2) $V(u)$ is bounded from below, and $\int_{-\infty}^R\int_\Omega e^{cz}dydz<\infty$. □

To proceed, we need to establish the following key estimate.

Lemma 5.4. *Let V satisfy hypotheses (H1) and (H2), and let $c^2 + 4\nu_0 > 0$. Then, for any $\epsilon > 0$ and $C > 0$ there exists $R = R(\epsilon, C)$ such that*

$$\Phi_c[u, (R, +\infty)] \geq -\epsilon,$$

for any $u \in H^1_c(\Sigma; \mathbb{R}^m)$ such that $||u||_{1,c} \leq C$.

Proof. Since $(C^\infty_0(\mathbb{R}^n))^m$ is dense in $H^1_c(\Sigma; \mathbb{R}^m)$, in the following arguments we can assume that $u \in (C^\infty_0(\mathbb{R}^n))^m$. We prove this Lemma via a sequence of steps.

Step 1. In view of Eq. (5.1) we have

$$\Phi_c[u, (R, +\infty)] \geq \int_R^{+\infty} \int_\Omega e^{cz} \left(\frac{1}{2} \left(\frac{c^2}{4} + \mu_0 \right) \sum_{i=1}^m u_i^2 + V(u) \right) dydz, \tag{5.5}$$

where, as in Eq. (2.9), $\mu_0 \geq 0$ is the smallest eigenvalue of $-\Delta_y$ in Ω with the corresponding boundary conditions. From the definition of ν_0 in Eq. (2.9), for any $\epsilon > 0$ there exists $\delta > 0$ such that for all $|u| < \delta$

$$V(u) \geq \frac{1}{2} \sum_{i=1}^m (\nu_0 - \mu_0 - \epsilon) u_i^2.$$

Therefore, if $c^2 + 4\nu_0 > 0$, the integrand in Eq. (5.5) is nonnegative for all $|u| < \delta$, with some positive δ.

Note that if $n = 1$, then from Eq. (5.2) follows that $u \to 0$ uniformly as $R \to +\infty$, so from the argument above immediately follows that $\Phi_c[u, (R, +\infty)] \geq 0$ for sufficiently large R, and the statement of the Lemma is proved (see also [24]). So, in the following we will assume that $n \geq 2$.

Step 2. Define

$$v(y, z) = -\frac{1}{2} \left(\frac{c^2}{4} + \mu_0 \right) \sum_{i=1}^m u_i^2(y, z) - V(u(y, z)) \tag{5.6}$$

and introduce

$$\Omega_+(z) = \{y \in \Omega : v(y, z) > 0\}. \tag{5.7}$$

By the result of Step 1, we have $|u(y, z)| \geq \delta$ whenever $y \in \Omega_+(z)$. Therefore

$$|\Omega_+(z)|\delta^2 \leq \sum_{i=1}^m \int_{\Omega_+(z)} u_i^2(y, z) dy \leq \sum_{i=1}^m \int_\Omega u_i^2(y, z) dy. \tag{5.8}$$

Combining this with Eq. (5.2) and taking into account that $\Gamma_c[u] = 1$, we obtain that

$$|\Omega_+(z)| \leq \frac{2e^{-cz}}{c\delta^2} \to 0 \quad \text{as} \quad z \to +\infty. \tag{5.9}$$

Step 3. Now we want to estimate the integral in Eq. (5.5). First, observe that $v(y, z) = 0$ whenever $y \in \partial\Omega_+(z) \cap \Omega$. From Eqs. (5.5) and (5.6) we have

$$\Phi_c[u, (R, +\infty)] \geq -\int_R^{+\infty} \int_{\Omega_+(z)} e^{cz} v \, dydz. \tag{5.10}$$

Let us introduce the level sets (for simplicity of notation, we suppress the z-dependence in the definition)

$$\omega(t) = \{y \in \Omega_+(z) : v(y, z) > t\}. \tag{5.11}$$

In view of Eq. (5.9) we have $|\omega(t)| \le |\Omega_+(z)| \le \frac{1}{2}|\Omega|$ for sufficiently large R. Then, by Local Isoperimetric Inequality [44] there exists a constant C_Ω which depends only on Ω and not on ω, such that (recall that $\dim\Omega = n-1$)

$$|\omega|^{\frac{n-2}{n-1}} \le C_\Omega |\partial\omega_0|, \qquad \partial\omega_0 = \partial\omega \cap \Omega. \tag{5.12}$$

Then, using the Cavalieri Principle and then the Co-Area Formula [44], we obtain

$$\begin{aligned} \int_{\Omega_+(z)} v dy &= \int_0^\infty |\omega(t)| dt \le |\Omega_+(z)|^{\frac{1}{n-1}} \int_0^\infty |\omega(t)|^{\frac{n-2}{n-1}} dt \\ &\le C_\Omega |\Omega_+(z)|^{\frac{1}{n-1}} \int_0^\infty |\partial\omega_0(t)| dt = C_\Omega |\Omega_+(z)|^{\frac{1}{n-1}} \int_{\Omega_+(z)} |\nabla_y v| dy. \end{aligned} \tag{5.13}$$

Let us now multiply the last integral in Eq. (5.13) by e^{cz} and integrate over $(R, +\infty)$. Then, using the definition of v in Eq. (5.6), Chain Rule, hypothesis (H1) and Schwarz inequality, we obtain

$$\begin{aligned} &\left(\int_R^{+\infty} \int_{\Omega_+(z)} e^{cz} |\nabla_y v| dydz\right)^2 \\ &= \left(\int_R^{+\infty} \int_{\Omega_+(z)} e^{cz} \left| \sum_{i=1}^m \left[\left(\frac{c^2}{4} + \mu_0\right) u_i + \frac{\partial V}{\partial u_i}\right] \nabla_y u_i \right| dydz\right)^2 \\ &\le M \left[\int_R^{+\infty} \int_{\Omega_+(z)} e^{cz} \left(\sum_{i=1}^m u_i^2\right)^{1/2} \left(\sum_{j=1}^m |\nabla_y u_j|^2\right)^{1/2} dydz\right]^2 \\ &\le M \int_R^{+\infty} \int_{\Omega_+(z)} e^{cz} \sum_{i=1}^m u_i^2 dydz \int_R^{+\infty} \int_{\Omega_+(z)} e^{cz} \sum_{i=1}^m |\nabla_y u_i|^2 dydz \\ &\le \frac{8M}{c^2} \int_{-\infty}^{+\infty} \int_\Omega e^{cz} \sum_{i=1}^m |\nabla_y u_i|^2 dydz, \end{aligned} \tag{5.14}$$

where M is a constant independent of R and u, and in the last step we used Eq. (5.1) and the fact that $u \in \mathcal{B}_c$. Combining this with Eqs. (5.9) and (5.13), obtain

$$\int_R^{+\infty} \int_{\Omega_+(z)} e^{cz} v\, dydz \le K e^{-\frac{cR}{n-1}} \left(\int_{-\infty}^{+\infty} \int_\Omega e^{cz} \sum_{i=1}^m |\nabla_y u_i|^2 dydz\right)^{1/2}, \tag{5.15}$$

where K is a constant independent of R and u.

Finally, by assumption the integral in the right-hand side of Eq. (5.15) is bounded by C, so its left-hand side can be made arbitrarily small by choosing large enough R. In view of Eq. (5.10), this proves the statement of the Lemma. □

Combining the two Lemmas above, we obtain the following

Proposition 5.5. *Let V satisfy hypotheses (H1) and (H2), and let $c^2 + 4\nu_0 > 0$. Then, the functional Φ_c is sequentially weakly lower semicontinuous on $H_c^1(\Sigma; \mathbb{R}^m)$.*

Proof. Let $u_n \rightharpoonup u$ in $H^1_c(\Sigma;\mathbb{R}^m)$. Hence, (u_n) is bounded in $H^1_c(\Sigma;\mathbb{R}^m)$, and by Lemmas 5.3 and 5.4

$$\begin{aligned}\liminf_{n\to\infty}\Phi_c[u_n] &\geq \liminf_{n\to\infty}\{\Phi_c[u_n,(-\infty,R)]\}+\liminf_{n\to\infty}\{\Phi_c[u_n,(R,+\infty)]\}\\ &\geq \Phi_c[u,(-\infty,R)]-\epsilon\\ &= \Phi_c[u]-\Phi_c[u,(R,+\infty)]-\epsilon, \end{aligned}\tag{5.16}$$

for large enough R. Now, by noting that $\Phi_c[u,(R,+\infty)]\leq\epsilon$ for sufficiently large R, Eq. (5.16) leads to

$$\liminf_{n\to\infty}\Phi_c[u_n]\geq\Phi_c[u]-2\epsilon,$$

and since $\epsilon>0$ is arbitrary, we conclude that $\Phi_c[u]\leq\liminf_{n\to\infty}\Phi_c[u_n]$. □

We are now ready to prove our existence result.

Proposition 5.6. *Let V satisfy hypotheses (H1) and (H2), and suppose that there exists $u\in\mathcal{B}_c$ such that $\Phi_c[u]\leq 0$, for some c satisfying $c^2+4\nu_0>0$. Then problem (P') has a solution.*

Proof. Let (u_n) be a minimizing sequence for problem (P'), i.e $u_n\in\mathcal{B}_c$ with $\Phi_c[u_n]\to\inf_{\mathcal{B}_c}\Phi_c$. By assumption, $\inf_{\mathcal{B}_c}\Phi_c\leq 0$, and without the loss of generality we may assume that $\Phi_c[u_n]\leq 0$. Since $\Gamma_c[u_n]=1$, from inequality (2.5) we get that $\int_\Sigma e^{cz}|u_n|^2dx\leq\frac{8}{c^2}$. Also, from Lemma 5.2 we get a similar bound for the norm of $\nabla_y u$. Thus, the sequence (u_n) is bounded in $H^1_c(\Sigma)$ and therefore, up to a subsequence, it converges weakly to some $u\in H^1_c(\Sigma)$.

If $\inf_{\mathcal{B}_c}\Phi_c=0$, we deduce that u in the assumption of this proposition is a minimizer. Therefore, let us assume that $\inf_{\mathcal{B}_c}\Phi_c<0$. Then, by lower semicontinuity of Φ_c established in Proposition 5.5 we have $\Phi_c[u]\leq\inf_{\mathcal{B}_c}\Phi_c<0$, so $u\not\equiv 0$. Also, since by standard semicontinuity results [43]

$$1=\liminf_{n\to\infty}\Gamma_c[u_n]\geq\Gamma_c[u]>0,$$

we can, by using Lemma 3.1, choose $a\geq 0$ such that

$$\Gamma_c[u_a]=1 \quad\text{ with }\quad u_a(y,z):=u(y,z-a).$$

Since $\inf_{\mathcal{B}_c}\Phi_c<0$ and $a\geq 0$, we derive

$$\Phi_c[u_a]=e^{ca}\Phi_c[u]\leq\Phi_c[u]\leq\inf_{\mathcal{B}_c}\Phi_c,$$

with the first inequality being strict when $a>0$. Therefore, $a=0$, meaning that $\Gamma_c[u]=1$ and $\Phi_c[u]=\inf_{\mathcal{B}_c}\Phi_c$, so u solves problem (P'). □

Let us point out that, for one-dimensional problems ($n=1$), in which the functional Γ_c generates an equivalent norm in $H^1_c(\mathbb{R})$, the minimizing sequence (u_n) converges to u strongly in $H^1_c(\mathbb{R})$.

6. Further properties of minimizers

In this section we analyze problem (P) and its solutions in more detail. Our first result, based on the application of Theorem 1.1, is a general non-existence result for the solutions of problem (P) with sufficiently large c (see also [15, 24]).

Proposition 6.1. *Let V satisfy hypotheses (H1) and (H2), and let $c^2+4(\mu_0+\mu_-)>0$, where μ_0 is the smallest eigenvalue of $-\Delta_y$ in Ω with boundary conditions from Eq. (1.2), and μ_- is given by Eq. (2.9). Then problem (P) has no solutions.*

Proof. Let $\bar{u} \in H^1_c(\Sigma;\mathbb{R}^m)$ be a solution of problem (P). By Propositions 3.3 and 2.4 we know that $\bar{u}(x) \in \mathcal{K}$ and $\bar{u}(\cdot,z) \in (C^2(\Omega)\cap C^1(\overline{\Omega}))^m$. Since

$$\int_\Omega |\nabla_y \bar{u}_i(y,z)|^2 dy \geq \mu_0 \int_\Omega \bar{u}_i^2(y,z) dy \qquad z \in \mathbb{R},$$

we obtain, using Lemma 2.3,

$$\begin{aligned} \Phi_c[\bar{u}] &\geq \frac{1}{2}\sum_{i=1}^m \int_{-\infty}^{+\infty}\int_\Omega e^{cz}\left[\left(\frac{\partial \bar{u}_i}{\partial z}\right)^2 + (\mu_0+\mu_-)\bar{u}_i^2\right] dydz \\ &\geq \frac{1}{2}\left(\frac{c^2}{4}+\mu_0+\mu_-\right)\int_{-\infty}^{+\infty}\int_\Omega e^{cz}\sum_{i=1}^m \bar{u}_i^2 dydz > 0, \end{aligned}$$

unless $\bar{u} = 0$. But this contradicts Proposition 3.2. □

Naturally, in view of the discussion at the end of Section 3 this implies that under the assumptions of Proposition 6.1 there are *no* traveling wave solutions lying in $H^1_c(\Sigma;\mathbb{R}^m)$. A simple example of such a situation is the Fisher's equation in one space dimension, for which it is known that all the traveling wave solutions decay at infinity with the rate $e^{-\lambda_- z}$ (see Eq. (2.10) with $\nu_k = 0$) and, therefore, cannot lie in $H^1_c(\mathbb{R})$ [24, 45].

Let us point out that we will have $\mu_- \geq 0$ if $V(u) \geq 0$ throughout $\mathcal{K}$, so a necessary condition for existence of solutions of problem (P), which is familiar from the analysis of the one-dimensional scalar problem [20], is that $V(u) < 0$ somewhere in $\mathcal{K}$. In that case, if also $\mu_0 + \mu_- < 0$, problem (P) may have solutions only with $c \leq c_{\max}$, where $c_{\max} = 2\sqrt{-\mu_0 - \mu_-}$ [15].

Next we establish the following necessary and sufficient condition for existence of traveling wave solutions for potentials with linearly stable equilibrium at $u = 0$. Let us introduce the functional

$$E[v] := \int_\Omega \left(\frac{1}{2}\sum_{i=1}^m |\nabla_y v_i|^2 + V(v)\right) dy \qquad v \in H^1(\Omega;\mathbb{R}^m), \tag{6.1}$$

where $H^1(\Omega;\mathbb{R}^m)$ is the Sobolev space of functions with values in $\mathbb{R}^m$ (for Dirichlet boundary conditions, take $H^1_0(\Omega;\mathbb{R}^m)$ instead). Under the hypotheses (H1) and (H2), this functional is known to have a minimizer $\bar{v} \in H^1(\Omega;\mathbb{R}^m)$ (see [43]) which satisfies the corresponding boundary conditions and such that

$$\Delta_y \bar{v} + f(\bar{v}) = 0. \tag{6.2}$$

Observe that for Neumann boundary conditions $\bar{v}$ is constant and is simply a minimum of the potential V. It turns out that this functional can be used to characterize the existence of solutions of Eq. (1.5).

Proposition 6.2. *Let V satisfy (H1) and (H2), and assume $\nu_0 \geq 0$. Then Eq. (1.5) has a solution $\bar{u} \in H^1_c(\Sigma;\mathbb{R}^m)$ if and only if*

$$\inf E[v] < 0, \tag{6.3}$$

where the inf *is taken over the functions $v \in H^1(\Omega;\mathbb{R}^m)$ that satisfy the boundary conditions in Eq. (1.2).*

Proof. Let us first prove that this assumption is sufficient. If Eq. (6.3) is satisfied, then choose a trial function

$$u_\lambda(y,z) = \begin{cases} \bar v(y), & z < 0, \\ \bar v(y) e^{-\lambda z}, & z \geq 0, \end{cases} \tag{6.4}$$

where $\bar v$ is a minimizer of E. Clearly, $u_\lambda \in H^1_c(\Sigma; \mathbb{R}^m)$ if $\lambda > \frac{c}{2}$. Substituting this into the definition of Φ_c, we find that

$$\Phi_c[u_\lambda] \leq \frac{1}{c} E[\bar v] + \frac{1}{2(2\lambda - c)} \int_\Omega \sum_{i=1}^m \left((\lambda^2 + C) \bar v_i^2(y) + |\nabla_y \bar v_i|^2 \right) dy, \tag{6.5}$$

where we used hypothesis (H1). Noting that $E[\bar v] < 0$, for fixed λ it is then possible to choose c so small that the right-hand side of this expression is negative. Then, u_λ will satisfy hypothesis (H3), which assures the existence of a solution $\bar u$ of problem (P) by Theorem 1.1.

Let us prove that the assumption (6.3) is also necessary. Suppose on the contrary that $E[v] \geq 0$ for all $v \in H^1(\Omega; \mathbb{R}^m)$. Then also

$$\int_\Sigma e^{cz} \left(\frac{1}{2} \sum_{i=1}^m |\nabla_y u_i|^2 + V(u) \right) dx \geq 0$$

for all $u \in H^1_c(\Sigma; \mathbb{R}^m)$. Using this and Lemma 2.3, we then obtain

$$\Phi_c[u] \geq \int_\Sigma e^{cz} \sum_{i=1}^m \left(\frac{\partial u_i}{\partial z} \right)^2 dx \geq \frac{c^2}{8} \int_\Sigma e^{cz} \sum_{i=1}^m u_i^2 \, dx.$$

Therefore, from Proposition 3.5 we conclude that there are no nontrivial solutions of Eq. (1.5) which lie in $H^1_c(\Sigma; \mathbb{R}^m)$. □

In the case of Neumann boundary conditions, we have the following result.

Proposition 6.3. *Let $\bar u$ be a solution of problem (P) with Neumann boundary conditions. Then $\bar u$ depends only on the variable z.*

Proof. Let us consider the function $g: \Omega \to \mathbb{R}$ defined as

$$g(y) := \int_\mathbb{R} e^{cz} \left(\frac{1}{2} \sum_{i=1}^m |\nabla \bar u_i|^2 + V(\bar u) \right) dz,$$

so that by Proposition 3.2 we have $\Phi_c[\bar u] = \int_\Omega g(y) dy = 0$. Assume first that the function g is not constant a.e. in Ω. Hence, we can choose $\bar y \in \Omega$ such that $g(\bar y) < 0$. By Fubini's Theorem, we can also assume that the function $\tilde u(y,z) := \bar u(\bar y, z)$ belongs to $H^1_c(\Sigma; \mathbb{R}^m)$. However, clearly $\Phi_c[\tilde u] \leq g(\bar y)|\Omega| < 0$, contradicting Proposition 3.2.

If the function g is constant a.e. on Ω but $\bar u$ depends on y, then we can choose $\bar y \in \Omega$ such that

$$\int_\mathbb{R} e^{cz} |\nabla_y \bar u(\bar y, z)|^2 \, dz > 0.$$

Defining $\tilde u$ as above, we get $\Phi_c[\tilde u] < \Phi_c[\bar u] = 0$, which gives again a contradiction. □

Next we establish the fact that the solutions of problem (P) are essentially scalar functions, if the potential V depends only on the modulus of u.

Proposition 6.4. *Assume $V(u) = V(|u|)$, i.e. the function V depends only on the modulus of u, and let $\bar{u} \in H^1_c(\Sigma;\mathbb{R}^m)$ be a solution of problem (P). Then, there exists a vector $v \in \mathbb{R}^m$ and a function $\eta \in C^2(\Sigma) \cap C^1(\overline{\Sigma})$, $\eta(x) > 0$, such that $\bar{u}(x) = \eta(x)v$ for any $x \in \Sigma$.*

Proof. Consider the non-empty open set $\Sigma' \subseteq \Sigma$ on which $|\bar{u}| > 0$. Introduce $\eta(x) = |\bar{u}(x)|$ on Σ and $n(x) = \bar{u}(x)/|\bar{u}(x)|$ on Σ'. The latter has the meaning of the director of the vector field u, and so we have $|n| = 1$. From these definitions $\bar{u} = \eta n$ in Σ' and $\nabla \bar{u} = 0$ a.e. in $\Sigma \backslash \Sigma'$. So a straightforward calculation shows that

$$\sum_{i=1}^{m} |\nabla \bar{u}_i|^2 = |\nabla \eta|^2 + \eta^2 \sum_{i=1}^{m} |\nabla n_i|^2 \geq |\nabla \eta|^2. \tag{6.6}$$

Now consider $\tilde{u}(x) = (\eta(x), 0, \ldots, 0) \in H^1_c(\Sigma;\mathbb{R}^m)$. If the last inequality in Eq. (6.6) is strict, then

$$\Phi_c[\tilde{u}] < \Phi_c[\bar{u}],$$

since by assumption $V(\tilde{u}) = V(\eta) = V(|\bar{u}|) = V(\bar{u})$, and this contradicts the minimality of $\bar{u}$. So, $\nabla n = 0$ in Σ' and $\tilde{u}$ is also a minimizer, and, therefore, is regular by Proposition 3.3. Therefore, η is a classical solution of the scalar equation

$$\Delta \eta + c\eta_z - V'(\eta) = 0, \tag{6.7}$$

and, furthermore, $\eta(x) \geq 0$. Then, we have in fact $\eta(x) > 0$ everywhere in Σ, and so $\Sigma' = \Sigma$. Indeed, define the function $c^{\pm}(x) = \left[\frac{V'(\eta(x))}{\eta(x)}\right]^{\pm}$, where $[v]^- = -\min\{v, 0\}$ and $[v]^+ = \max\{v, 0\}$, for all $x \in \Sigma'$, and set $c^{\pm}(x) = 0$ otherwise. Note that by hypothesis (H2) we have $c^{\pm} \in L^\infty(\Sigma)$. Then Eq. (6.7) can be rewritten as

$$\Delta \eta + c\eta_z - c^+(x)\eta = -c^-(x)\eta \leq 0.$$

So, by Strong Maximum Principle [39, Theorem 3.5], we conclude that $\eta(x) > 0$ for all $x \in \Sigma$. It then follows that n is a constant vector throughout Σ, which concludes the proof. □

In other words, to find the solution of problem (P) under the above assumption, one only needs to consider the scalar equation whose solutions lie in the considered exponentially weighted Sobolev spaces. Notice that for constant sign solutions of Eq. (6.7) precise estimates of the decay of the solution as $z \to +\infty$ can be obtained [46]. Since, in addition, our solutions lie in spaces $H^1_c(\Sigma;\mathbb{R}^m)$, it follows that $u = O(e^{-\lambda_0^+ z})$, where λ_0^+ is defined in Eq. (2.12), for large positive z. Thus, generally these solutions are special in the sense that they have a non-generic fast exponential decay at $+\infty$ (see also [24]).

Our next group of results concerns the behavior of solutions of problem (P) as $z \to -\infty$. Our main tool here is the familiar energy estimate for gradient systems.

Lemma 6.5. *Let $\bar{u} \in H^1_{c^\dagger}(\Sigma;\mathbb{R}^m)$ be a solution of problem (P), then $\bar{u}_z \in (L^2(\Sigma))^m$.*

Proof. Scalar multiplying Eq. (1.5) by $\bar{u}_z$ and integrating over $\Sigma_R := \Omega \times (-R, R)$, $R > 0$, we get

$$\begin{aligned} 0 &= \sum_{i=1}^m \int_{\Sigma_R} \frac{\partial \bar{u}_i}{\partial z} \left(\frac{\partial^2 \bar{u}_i}{\partial z^2} + \Delta_y \bar{u}_i + c^\dagger \frac{\partial \bar{u}_i}{\partial z} + f_i(\bar{u}) \right) dx \\ &= c^\dagger \int_{\Sigma_R} \sum_{i=1}^m \left(\frac{\partial \bar{u}_i}{\partial z} \right)^2 dx \\ &\quad + \left[\int_\Omega \left(\frac{1}{2} \sum_{i=1}^m \left(\frac{\partial \bar{u}_i}{\partial z} \right)^2 - \frac{1}{2} \sum_{i=1}^m |\nabla_y \bar{u}_i|^2 - V(\bar{u}) \right) dy \right]_{-R}^{R}, \end{aligned} \tag{6.8}$$

where we used the boundary conditions in Eq. (1.2) to erase the boundary term

$$\int_{\partial\Omega \times (-R \times R)} (\nabla_y \bar{u}_i \cdot n_{\partial\Sigma}) \frac{\partial \bar{u}_i}{\partial z} \, d\sigma dz.$$

Recalling that by Proposition 3.3 we have $\bar{u}_i \in W^{1,\infty}(\Sigma)$, passing to the limit in the equality (6.8) for $R \to +\infty$, we obtain the thesis. □

For any $R \in \mathbb{R}$, let $\widetilde{\Sigma}_R := \Omega \times (R, R + 1)$. By the results of part (ii) of Proposition 3.3 we have that the functions $\bar{u}_i$ are uniformly bounded in $W^{2,p}(\widetilde{\Sigma}_R)$, with $p > n$, independently of R. It then follows that $\bar{u}_z$ is bounded and uniformly continuous on Σ, hence from Lemma 6.5 we get

$$\lim_{z \to \pm\infty} \bar{u}_z(y, z) = 0 \quad \text{uniformly in } y \in \Omega. \tag{6.9}$$

On the other hand, by Proposition 3.3 we know that $\bar{u}(z, y) \to 0$ uniformly in $y \in \Omega$ as $z \to +\infty$. Then, by the same $W^{2,p}(\widetilde{\Sigma}_R)$ estimate and Sobolev imbedding theorem we get

$$\lim_{z \to +\infty} |\nabla_y \bar{u}(y, z)| = 0 \quad \text{uniformly in } y \in \Omega. \tag{6.10}$$

In the following Proposition we characterize the possible limits (i.e. the α-limit set) of $\bar{u}(\cdot, z)$ for $z \to -\infty$ (we refer the reader also to [47] for related results using dynamical systems techniques).

Proposition 6.6. *Let $\bar{u} \in H^1_{c^\dagger}(\Sigma; \mathbb{R}^m)$ be a solution of problem (P), then there exists a sequence $z_n \to -\infty$ and a function $v \in (C^2(\Omega) \cap C^1(\overline{\Omega}))^m$, satisfying the same boundary conditions as $\bar{u}$, such that*

$$\begin{aligned} \lim_{n \to +\infty} \bar{u}(\cdot, z_n) &= v \qquad \text{in } (C^1(\overline{\Omega}))^m. \\ \Delta_y v + f(v) &= 0 \qquad \text{in } \Omega. \end{aligned} \tag{6.11}$$

Conversely, let v be any function such that $\lim_{n\to\infty} \bar{u}(\cdot, z_n) = v$ in $(C^1(\overline{\Omega}))^m$, for some sequence $z_n \to -\infty$. Then $v \in (C^2(\Omega) \cap C^1(\overline{\Omega}))^m$, v satisfies the same boundary conditions as $\bar{u}$, and Eq. (6.11) holds.

Proof. Let $\phi \in H^1(\Omega; \mathbb{R}^m)$ be a test function (we further assume $\phi \in H^1_0(\Omega; \mathbb{R}^m)$ if we have Dirichlet boundary conditions). Scalar multiplying Eq. (1.5) by $\phi(y)$ and integrating over $\widetilde{\Sigma}_R$, we get

$$\begin{aligned} 0 &= \left[\sum_{i=1}^m \int_\Omega \phi_i \frac{\partial \bar{u}_i}{\partial z}\, dy\right]_R^{R+1} + c\sum_{i=1}^m \int_{\widetilde{\Sigma}_R} \phi_i \frac{\partial \bar{u}_i}{\partial z}\, dx \\ &\quad - \sum_{i=1}^m \int_{\widetilde{\Sigma}_R} \left(\nabla_y \bar{u}_i \cdot \nabla_y \phi_i - f_i(\bar{u})\phi_i\right)\, dx. \end{aligned} \tag{6.12}$$

Since $\bar{u}_z \to 0$ in $C^0(\widetilde{\Sigma}_R)$ for $R \to -\infty$, we have

$$\lim_{R\to-\infty} \sum_{i=1}^m \left(\left[\int_\Omega \phi_i \frac{\partial \bar{u}_i}{\partial z} dy\right]_R^{R+1} + c\int_{\widetilde{\Sigma}_R} \phi_i \frac{\partial \bar{u}_i}{\partial z}\, dx \right) = 0. \tag{6.13}$$

Note that the family of functions $\bar{u}(y, z+R)$ is equibounded in $(C^1(\overline{\Sigma}_0))^m$, where $\Sigma_0 := \Omega \times (0,1)$. Indeed, from the estimates of Proposition 3.3, we get a uniform bound on $\bar{u}_i(y, z+R)$ in $W^{2,p}(\Sigma_0)$, with $p > n$. So, by Ascoli-Arzelà Theorem there exists a sequence $R_n \to -\infty$ and a function $\tilde{v}$ such that $\bar{u}(y, z+R_n) \to \tilde{v}$ in $(C^1(\overline{\Sigma}_0))^m$. Moreover, since $\lim_{R\to-\infty} \bar{u}_z(y, z+R) = 0$ uniformly on $\overline{\Sigma}_0$, we obtain $\tilde{v}_z = 0$, i.e. the function $\tilde{v}$ depends only on y. Setting $v(y) := \tilde{v}(y, z)$, we then obtain that $\lim_{n\to\infty} \bar{u}(\cdot, z_n) = v$ in $(C^1(\Omega))^m$, e.g., for $z_n = R_n$.

From Eqs. (6.12) and (6.13), it then follows

$$\begin{aligned} 0 = \lim_{n\to+\infty} \sum_{i=1}^m \int_{\widetilde{\Sigma}_{R_n}} \left(\nabla_y \bar{u}_i \cdot \nabla_y \phi_i - f_i(\bar{u})\phi_i\right)\, dx \\ = \sum_{i=1}^m \int_\Omega \left(\nabla_y v_i \cdot \nabla_y \phi_i - f_i(v)\phi_i\right)\, dy, \end{aligned} \tag{6.14}$$

for any $\phi \in H^1(\Omega; \mathbb{R}^m)$ (resp. for any $\phi \in H^1_0(\Omega; \mathbb{R}^m)$), which implies $v \in (C^2(\Omega) \cap C^1(\overline{\Omega}))^m$, v satisfies the same boundary conditions as $\bar{u}$ on $\partial\Omega$, and $\Delta_y v + f(v) = 0$ in Ω.

Conversely, let us assume that there exists a function v such that $\lim_{n\to\infty} \bar{u}(\cdot, z_n) = v$ in $(C^1(\overline{\Omega}))^m$, for some sequence $z_n \to -\infty$. Then, reasoning exactly as above with $R_n = z_n$ we obtain that $v \in (C^2(\Omega) \cap C^1(\overline{\Omega}))^m$, v satisfies the same boundary conditions as $\bar{u}$ on $\partial\Omega$, and $\Delta_y v + f(v) = 0$ in Ω. □

We note that by regularity of $\bar{u}$ a weaker form of convergence (such as weak $(L^2(\Omega))^m$, e.g.) implies a stronger $(C^1(\Omega))^m$ convergence of the second part of Proposition 6.6.

Let $E[v]$ be the functional defined in (6.1) and introduce

$$W := \left\{ v \in H^1(\Omega; \mathbb{R}^m) : \ v(y) \in \mathcal{K} \text{ for all } y \in \Omega, \text{ and } E[v] < 0 \right\}.$$

Again, in the case of the Dirichlet boundary conditions replace $H^1(\Omega; \mathbb{R}^m)$ with $H^1_0(\Omega; \mathbb{R}^m)$. Taking $R = -z_n$ in (6.8) and letting $n \to +\infty$, from Proposition 6.6 and Eqs. (6.9) and (6.10) we obtain

Corollary 6.7. *Let v be as in Proposition 6.6. Then $v \in W$, in particular, $v \neq 0$.*

Under some extra assumptions on the critical points of $E[v]$ it is possible to give more precise information on the asymptotic behavior of the solutions of problem (P) at $z = -\infty$.

Corollary 6.8. *Assume that any critical point of E in W is isolated in the strong topology in $H^1(\Omega;\mathbb{R}^m)$. Then the limit in Proposition 6.6 is a full limit, i.e.*

$$\lim_{z\to-\infty}\bar{u}(\cdot,z)=v \quad \text{in } (C^1(\overline{\Omega}))^m,$$

with $v\in W$.

Proof. Note that the mapping $z\mapsto\bar{u}(\cdot,z)$ is a continuous mapping from $\mathbb{R}$ to $H^1(\Omega;\mathbb{R}^m)$. Suppose that the full limit of $\bar{u}(\cdot,z)$ does not exist. By continuity of this mapping, Proposition 6.6 and Corollary 6.7, there exists $\epsilon>0$ and a sequence $z_n'\to-\infty$ such that $\epsilon\le||\bar{u}(\cdot,z_n')-v||_{H^1(\Omega;\mathbb{R}^m)}\le 2\epsilon$, where $v\in W$ is some limit from Proposition 6.6, and the 2ϵ-neighborhood of v does not contain any other elements of W. By regularity of $\bar{u}$ we can pass to a subsequence, still labeled (z_n') that converges strongly in $H^1(\Omega;\mathbb{R}^m)$. Therefore, if $v'=\lim_{n\to\infty}\bar{u}(\cdot,z_n')$, then $\epsilon\le||v'-v||_{H^1(\Omega;\mathbb{R}^m)}\le 2\epsilon$, too. But, by Proposition 6.6 and Corollary 6.7 every convergent sequence in $(C^1(\Omega))^m$ has a limit that is in W, which contradicts the assumption that there are no elements of W in the 2ϵ-neighborhood of v that are distinct from v. □

Note that a sufficient condition for a critical point of E to be isolated is that it is non-degenerate (i.e., that the second variation of E does not contain zero eigenvalues). Also note that in the case of Neumann boundary conditions we know from Proposition 6.3 that the function $\bar{u}$ is independent of $y\in\Omega$, which implies that the function v is a constant. Therefore, we get the full limit in Proposition 6.6 simply if we assume that any critical point of V in the open set $\big\{u\in\mathbb{R}^m : V(u)<0\big\}\subset\mathbb{R}^m$ is isolated.

We conclude this section by showing that under suitable assumptions the solutions of Eq. (1.1), having compact support at $t=0$, propagate along Σ with asymptotic speed bounded by $c^\dagger$.

Proposition 6.9. *Suppose that problem (P) has a solution, and let $u(x,t)\in\mathcal{K}$ be a solution of Eq. (1.1) such that $u(\cdot,t)\in H^1_c(\Sigma;\mathbb{R}^m)$ and $u_t(\cdot,t)\in(L^2_c(\Sigma)\cap L^\infty(\Sigma))^m$, with some $c>c^\dagger$, for all $t>0$. Then, for any $c'>c^\dagger$, $u(y,z+c't,t)\to 0$ as $t\to\infty$ uniformly on compact subsets of Σ.*

Proof. Fix a constant c'' such that $c^\dagger<c''<\min\{c,c'\}$. Notice that, since $u(\cdot,t)\in(L^\infty(\Sigma))^m$ we have $\nabla u(\cdot,t)\in(L^\infty(\Sigma))^{mn}$ uniformly for any $t\ge t_0$, with $t_0>0$ (see [41]), hence $u(\cdot,t)\in H^1_{c''}(\Sigma;\mathbb{R}^m)$. Differentiating $\Phi_{c''}[u(y,z+c''t,t)]$ in t and integrating by parts, we get

$$\frac{d\Phi_{c''}[u(y,z+c''t,y)]}{dt}=-\int_\Sigma e^{c''z}\sum_{i=1}^m\left(\Delta u_i+c''\frac{\partial u_i}{\partial z}+f_i(u)\right)^2dx\le 0. \quad (6.15)$$

Since also $c''>c^\dagger$, we have $0\le\Phi_{c''}[u(y,z+c''t,t)]\le\Phi_{c''}[u(y,z,0)]$, and by Lemma 3.1 we get

$$\Phi_{c''}[u(y,z+c't,t)]=e^{-c''(c'-c'')t}\Phi_{c''}[u(y,z+c''t,t)]\to 0 \quad (6.16)$$

as $t \to \infty$. On the other hand, letting $\tilde{u}(y, \zeta, t) := u\left(y, \frac{c^\dagger}{c''}\zeta, t\right)$ and retracing the arguments of Eqs. (4.7) – (4.9), we get $\tilde{u} \in H^1_{c^\dagger}(\Sigma; \mathbb{R}^m)$ and

$$\begin{aligned}
\Phi_{c''}[u] &= \int_\Sigma e^{c''z} \left(\frac{1}{2} \sum_{i=1}^m \left[\left(\frac{\partial u_i}{\partial z} \right)^2 + |\nabla_y u_i|^2 \right] + V(u) \right) dydz \\
&= \frac{c^\dagger}{c''} \int_\Sigma e^{c^\dagger \zeta} \left(\frac{1}{2} \sum_{i=1}^m \left[\left(\frac{c''}{c^\dagger} \right)^2 \left(\frac{\partial \tilde{u}_i}{\partial \zeta} \right)^2 + |\nabla_y \tilde{u}_i|^2 \right] + V(\tilde{u}) \right) dyd\zeta \\
&= \frac{c''^2 - c^{\dagger 2}}{c'' c^\dagger} \Gamma_{c^\dagger}[\tilde{u}] + \frac{c^\dagger}{c''} \Phi_{c^\dagger}[\tilde{u}] \geq \frac{c''^2 - c^{\dagger 2}}{c'' c^\dagger} \Gamma_{c^\dagger}[\tilde{u}] = \frac{c''^2 - c^{\dagger 2}}{c''^2} \Gamma_{c''}[u],
\end{aligned}$$

since $\Phi_{c^\dagger}[u] \geq 0$ for all $u \in H^1_{c^\dagger}(\Sigma; \mathbb{R}^m)$ by Proposition 3.2. But then, using Lemma 2.3 we have

$$\Phi_{c''}[u(y, z + c't, t)] \geq \frac{c''^2 - c^{\dagger 2}}{8} \int_\Sigma e^{c''z} \sum_{i=1}^m u_i^2(y, z + c't, t)^2 dydz. \tag{6.17}$$

Therefore, $u(y, z + c't, t) \to 0$ in $(L^2_{c^\dagger}(\Sigma))^m$ as $t \to \infty$, hence uniformly on compact subsets of Σ by the uniform bound on $|\nabla u_i|$. □

Notice that the assumptions of Proposition 6.9 are easily satisfied, for example whenever (2.3) holds and the solution $u(x, t)$ is such that $u(\cdot, 0)$ takes values in $\mathcal{K}$ and has compact support.

Let us emphasize that the result in Proposition 6.9 implies that the speed $c^\dagger$ obtained in problem (P) has a special significance for the solutions of the original parabolic problem. Indeed, $c^\dagger$ is the maximum speed with which solutions may propagate (e.g., in the sense of the speed of the leading edge [15]). On the other hand, observe that this is also a sharp upper bound, since existence of solutions of problem (P) obviously implies existence of solutions of Eq. (1.1) which propagate with speed $c^\dagger$.

Finally, let us note that in general the free energy functional in Eq. (1.3) may include the effect of anisotropy [1, 5], i.e. the gradient square term in $F[u]$ can be replaced by a quadratic form generated by a symmetric positive-definite constant matrix G. Then the analogue of Eq. (1.1) becomes

$$u_t = \nabla \cdot (G \nabla u) - \nabla_u V(u). \tag{6.18}$$

Similarly, the boundary conditions for this equation should be modified from Eq. (1.2) and become

$$(\nu \cdot G \, \nabla u)|_{\partial\Sigma} = 0 \quad \text{or} \quad u|_{\partial\Sigma} = 0. \tag{6.19}$$

One can naturally ask whether the above problem admits traveling wave solutions, too. Indeed, it is not difficult to see that Eq. (6.18) with the boundary conditions from Eq. (6.19) can be reduced to Eq. (1.1), with the boundary conditions from Eq. (1.2), by the linear change of variables

$$x' = G^{-1/2} x.$$

In this way we obtain a problem of the type considered above on a modified cylinder Σ', which can then be treated in the same fashion.

7. An application

In this section, we consider a sample application problem, for which various assumptions of the theorems above can be explicitly verified, and demonstrate the practical utility of our methods. For a particular example we will use a computer-assisted approach to obtain the necessary estimates for existence. Note that with a bit of extra work these types of results can be made completely rigorous. This, however, is beyond the purpose of this section, which is to illustrate our theorems. Below we will adopt the term *variational traveling wave* [15] to denote the traveling wave solutions of Eq. (1.5) with speed c, that lie in $H^1_c(\Sigma;\mathbb{R}^m)$.

As a sample problem, we will consider Eq. (1.4) with $\tau = 1$, $g = 1$, $a = 3$, $b = 1$, $c = \frac{3}{2}$, $h_1 = \frac{11}{20}$, and $h_2 = 0$. For simplicity, we will consider the case $m = 2$ and $n = 1$ (implying that $\Sigma = \mathbb{R}$), so that the vector character of the problem is preserved. Let us mention that in one space dimension existence of traveling wave solutions in gradient systems can be also studied by topological techniques [27–29].

Thus, with $u = (u_1, u_2)$, this problem has the following expression for the potential V:

$$V(u_1, u_2) = -\tfrac{11}{20}u_1 - \tfrac{3}{2}(u_1^2 + u_2^2) + \tfrac{1}{4}(u_1^4 + u_2^4) + \tfrac{3}{4}u_1^2u_2^2. \tag{7.1}$$

The plot of the level curves of V is presented in Fig. 2. An inspection of this figure shows that V has one local maximum $O(p_0, 0)$, four local minima $P_\pm(p_\pm, 0)$ and $Q_\pm(p_1, \pm q_1)$, and four saddle points $R_\pm(p_2, \pm q_2)$ and $S_\pm(p_3, \pm q_3)$, respectively (see Fig. 2). It is easy to see that the set $\mathcal{K} := \{(u_1, u_2) \in \mathbb{R}^2 : V(u_1, u_2) \leq \frac{1}{4}\}$ has the required properties, being convex and satisfying Eq. (2.3). There is also a rectangle $\mathcal{K}_+ = \{(u_1, u_2) \in \mathbb{R}^2 : p_3 \leq u_1 \leq p_+,\ 0 \leq u_2 \leq q_3\}$ which is also convex and satisfies Eq. (2.3).

We are going to study existence of several types of traveling waves which connect to different equilibria, namely to O, P_-, and S_+. Each such case leads to a different variational problem, since in order to satisfy hypothesis (H1), one needs to subtract from V its value at the equilibrium point reached at $z = +\infty$. So, we will consider each such problem separately and establish existence and non-existence of variational traveling waves, as well as the upper and lower bounds for the speed. To simplify the notation, we will still say that u lies in $H^1_c(\Sigma;\mathbb{R}^m)$, tacitly assuming that the equilibrium point is properly subtracted from u.

Let us point out that if one sets $u_2 = 0$, then the problem becomes scalar, and existence of traveling waves connecting P_+, P_-, and O is well-known (see, e.g., [19,45]). These are the heteroclinic orbits P_-O, P_+O, and P_+P_-, respectively, and there exists a continuous family of solutions monotonically connecting P_+ and P_- with O, and a unique solution going monotonically from P_+ to P_-. Furthermore, an exact solution for the traveling wave P_+P_- can be found [48], giving $c = 0.393419$ for this wave. These are natural candidates for the solutions of the variational problems under consideration, so, in particular, we need to see whether we can discriminate between them and the solutions of the vector problem.

We start by studying the case of the waves connecting to O. To begin, we compute the value of ν_0, which in all considered cases is simply the smallest eigenvalue of the Hessian at the equilibrium approached at $z = +\infty$. A straightforward calculation shows that at O we have $\nu_0 = -2.94841 < 0$. So, in order to be able to apply Theorem 1.1, we need to find a trial function that makes the functional Φ_c

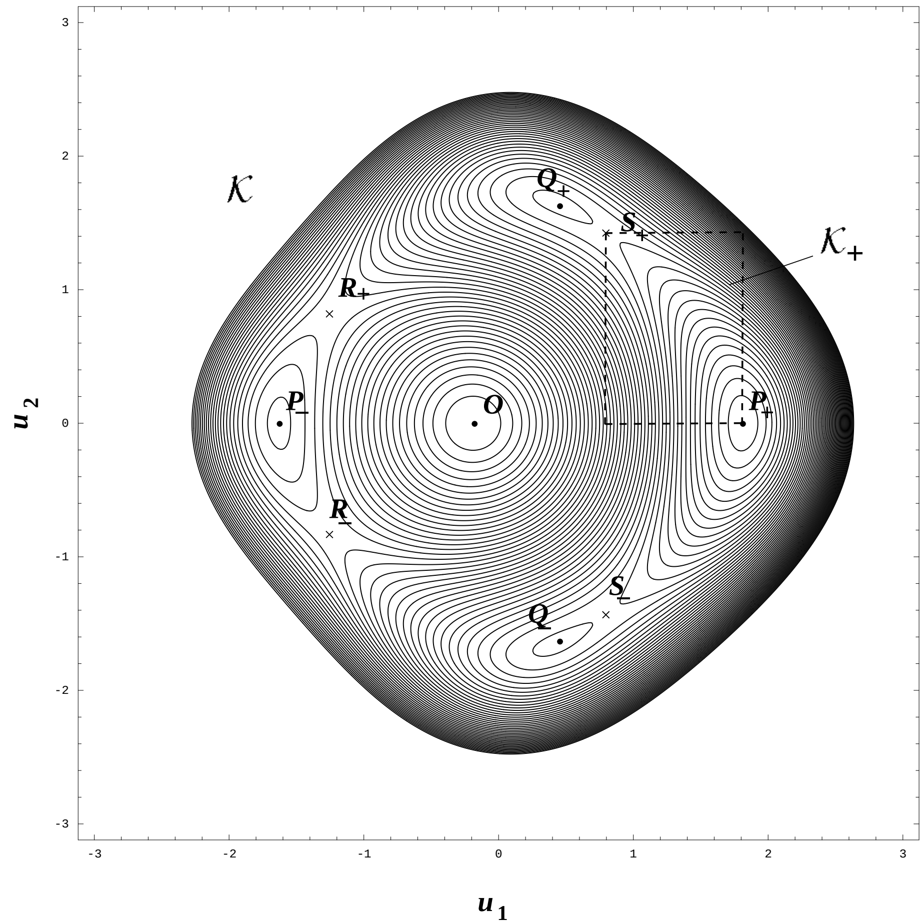

FIGURE 2. The level curves of the potential V in Eq. (7.1). The outermost contour corresponds to $V = \frac{1}{4}$ and shows the boundary of the set $\mathcal{K}$. The set $\mathcal{K}_+$ is enclosed by the dashed lines.

nonpositive for $c > c_0 = 2\sqrt{-\nu_0} = 3.43419$. We were not able to find such a trial function.

On the other hand, at O we can estimate the value of μ_- to be slightly greater than -3. By Proposition 6.1, there are no variational traveling waves for $c \geq c_1 = 3.4641$. Therefore, our method can give solutions only in a narrow range of $3.43419 < c < 3.4641$, if any. Since also for $c < 2\sqrt{-\nu_1} = 3.40401$ the solution will approach O in an oscillatory fashion (see the discussion in [15, Section 3]), it will not be expected to lie in $H^1_c(\mathbb{R})$, either. This suggests that there are no variational traveling waves that connect to O. In fact, we can prove that there are no variational traveling waves satisfying hypothesis (H3) that lie entirely to the left of O (that is, for which $u_1 \leq p_0$). Indeed, applying the Taylor formula, we have

$$V(u_1, u_2) = V(p_0, 0) + \tfrac{1}{2}\{(-3 + 3\tilde{u}_1^2 + \tfrac{3}{2}\tilde{u}_2^2)(u_1 - p_0)^2 + 6\tilde{u}_1\tilde{u}_2(u_1 - p_0)u_2 + (-3 + \tfrac{3}{2}\tilde{u}_1^2 + 3\tilde{u}_2^2)u_2^2\},$$

where $u_1 \leq \tilde{u}_1 \leq p_0 < 0$ and $\tilde{u}_2$ lies between 0 and u_2. Clearly, the coefficients of the first and the third terms in the curly brackets are greater or equal to $-\nu_0 = -3+\frac{3}{2}p_0^2$. Furthermore, since $(u_1 - p_0)\tilde{u}_1 \geq 0$ and $\tilde{u}_2 u_2 \geq 0$, we then have $V(u_1, u_2) \geq V(p_0, 0)+\frac{1}{2}\nu_0((u_1-p_0)^2+u_2^2)$, which implies that $\Phi_c[u] = 0$ if and only if $u = (p_0, 0)$ for all $u \in H^1_c(\mathbb{R})$ with $c^2 + 4\nu_0 > 0$, so Theorem 1.1 cannot be applied. Then, in view of Proposition 3.5, this means non-existence of variational traveling waves with these speeds. Note also that this argument can be strengthened to show that all the solutions P_-O with $u_2 = 0$ are not variational traveling waves (see also [24]). This is not unusual for the traveling waves invading an unstable equilibrium.

Let us now consider the waves that connect to P_-. Here we get that $\nu_0 = 0.994441 > 0$, and we know from the case $u_2 = 0$ that problem (P) has a solution. The question is whether this solution is in fact one-dimensional, and what the bounds on the speed are. To begin, we first find that for P_- the value of μ_- is slightly greater than -0.34. Again, by Proposition 6.1 this means that the variational traveling waves connecting to P_- may exist only for $c < c_1 = 2\sqrt{-\mu_-} < 1.1662$. To see whether there are variational traveling waves that move *faster* than in the case $u_2 = 0$, we construct the trial function $u = (u_1, u_2)$ defined as

$$\begin{aligned} u_1(z, a, b) &:= p_- + \tfrac{1}{2}(p_+ - p_-)(1 - \tanh az), \\ u_2(z, a, b) &:= b \operatorname{sech}^2 az. \end{aligned}$$

Next we evaluate Φ_c on u and minimize with respect to a and b. We then find a (large enough) value of c at which the minimum value of Φ_c is still negative. We found that the choice of $a = 0.5876, b = 1.6301$ works with $c = 0.5240$. So now, applying Theorem 1.1, we can conclude that there exists a traveling wave solution connecting to P_- that lies in $\mathcal{K}$ and has speed $0.5240 < c < 1.1662$. Observe that this speed is higher than that of the scalar solution obtained earlier, so the latter is in fact not a solution of problem (P). Also, by Corollary 6.8 the solution is a heteroclinic orbit from P_- to either $Q_\pm$, $S_\pm$, or P_+ (the equilibria O and $R_\pm$ have higher potential than P_-). Let us point out that our arguments can be easily made rigorous (with a slightly smaller value of c) by performing a linear interpolation of the above trial function, over finitely many intervals, then rationalizing the values of u at the interpolation nodes, and then carrying out some simple, albeit tedious analysis.

Finally, we turn to the solutions that connect to S_+ and lie in $\mathcal{K}_+$. For S_+, we obtain that $\nu_0 = -0.588022$, so in order for hypothesis (H3) to be satisfied, we need to find a trial function for which $\Phi_c < 0$ with $c > 1.53365$. We use the following trial function $u = (u_1, u_2)$:

$$\begin{aligned} u_1(z, a, b) &:= p_3 + \frac{p_+ - p_3}{1 + e^{az}}, \\ u_2(z, a, b) &:= q_3 - \frac{q_3}{(1 + e^{bz})^{3/2}}. \end{aligned}$$

Once again, we fix c and minimize $\Phi_c[u]$ with respect to a and b. As a result, we find that the functional is negative for $a = 1.1536$, $b = 0.8778$, and $c = 1.61 > 1.53365$. Therefore, the assumptions of Theorem 1.1 are satisfied in $\mathcal{K}_+$, and we obtain a traveling wave solution connecting to S_+ that lies in $\mathcal{K}_+$. On the other hand, we find μ_- to be slightly greater than -0.91, implying an upper bound for the speed of the traveling wave to be $c < 2\sqrt{-\mu_-} < 1.91$. Thus, the obtained solution will have speed $1.61 < c < 1.91$. Again, by Corollary 6.8 this is a heteroclinic orbit from S_+ to P_+.

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