STRONG VERSIONS OF THE DFR PROPERTY

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CAMS Report 0405-18, Spring 2005

Center for Applied Mathematics and Statistics
NJIT

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Abstract

This paper investigates two district subclasses of DFR distributions that exhibit strong forms of non-aging behaviour. Our results characterize such distributions through several representation theorems and consider associated consequences such as closure properties and reliability bounds for distributions which are DFR in the strong sense considered. The usefulness and prevalance of the strogn DFR properties are illustrated by examples and several applications.

Keywords and phrases. Decreasing failure rate, completely monotone functions, representation theorems.
AMS 2001 Subject classifications. Primary 60K10; Secondary 60N05.

1. Introduction and Summary

[1]work done during the author's sabbatical leave at the Indian Statistical Institute.

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In statistical reliability theory, various notions of *aging*, or, degradation, such as the increasing failure rate (IFR) property and its weaker variants have been extensively researched, as they provide a meanigful framework for modeling and studying equipment degradation over time for most *physical / hardware* systems. By contrast, survival distributions with a *decreasing failure rate* (DFR) property, whichn is conceptually dual to the IFR property, does not appear to have received the same level of attention, perhaps because in traditional reliability theory as applied to hardware systems or, to biomedical settings for survival studies, there is an implicit emphasis on the relevance of 'aging notions' as a guideline for the choice of appropriate models of failure time / survival distributions.

A 'negatively aging' behavior over time, such as the DFR property, can be an important determinant in specifying the plausible choices of survival models in other settings. In particular, this is true in the context of software reliability applications, such as many non-homogenious Poisson Process (NHPP) *software reliability growth models*, where the 'intensity function' which corresponds to the software-fault detection rate is decreasing in time (Yamada [13]). A negatively aging behavior, such as the DFR property, can be induced in hardware systems under appropriate conditions (see Cozzolino [5] , Keilson [11]). These observations and the fact that the most familiar models of DFR distributions, such as the *Gamma* and *Weibull*, *when they are DFR* for appropriate shape parameter values, *have failure rates that are decreasing in a much stronger sense* than what the definition of DFR property requires, provided the motivation for the present work.

We introduce and investigate two nonempty proper subclasses of DFR survival distributions, which are natural strengthenings of the decreasing failure rate property. In section 2, these notions ('strongly DFR' and 'completely DFR' properties) and the corresponding nonparametric survival distribution classes are defined and then shown to be probabilistically meaningful. Examples of such distributions are given. Section 3 studies closure properties and reliability bounds for such distributions. Finally, in section 4 we derive representation theorems for these distribututrions, which shows how they can be constructed, starting with exponentials as basic building blocks.

Let $\{Exp\}$ denote the set of nondegenerate exponential distributions, $\{DFR\}$ the set of DFR distributions; any set of distributions identified by a defining property being denoted similarly. Although we are mostly interested in 'honest' failure distributions, i.e., which assign all of its mass on $[0, \infty)$; it will be occassionally convenient to consider distributions / measures which *may* have an atom at infinity. If $\{x\}$ is an atom of a distribution or measure M, $M\{x\} > 0$ denotes the mass of the atom ($x \leq \infty$). All failure distributions considered are on the half line $[0, \infty)$ and are honest, unless otherwise specified. The symbol $\overset{d}{=}$ denotes equality in distributions; f denotes density of a *cdf* F with tail (reliability) function $\overline{F} = 1 - F$. Given any set A of 'honest' distributions, $\overline{A}$ denotes the set of possibly defective distributions F such that $F(t)$ possesses the defining property of A for $0 < t < \infty$ and a possible atom $0 \leq F\{\infty\} < 1$. Integrals $\int_0^\infty$ will be understood to be over the domain $[0, \infty)$. If $\overline{R} = [0, \infty]$ and $h \geq 0$, set $\int_{\overline{R}} hdM = M\{\infty\} \lim_{x \to \infty} h(x) + \int_0^\infty hdM$. Recall [7] that a function $h(x)$ is *completely monotone* (henceforth abbreviated as c.m.) if on $(0, \infty)$ all derivatives $h^{(n)}$ exist and $(-1)^n h^{(n)}(x) \geq 0,\ n \geq 1$.

2. Strengthenings of the DFR property

Let F be a nondiscrete failure distribution, and $\overline{F} = 1 - F$ the corresponding reliability function.

Definition.

1. F is '*strongly DFR*' (SDFR) if the reliability function $\overline{F}(t)$ is c.m.

2. F is '*completely DFR*' (CDFR) if $F(0) = 0$ and the failure rate $r(t) = f(t)/\overline{F}(t)$ is c.m.

Remark. If $\overline{F}$ (r respectively) is c.m., but $F\{\infty\} \geq 0$, then $F\epsilon\{\overline{SDFR}\}$ ($\{\overline{CDFR}\}$ respectively), equivalently $F_0(t) \equiv F(t)/1 - F\{\infty\}$ (i.e., F conditioned on being finite) is SDFR (CDFR). A non-negative r.v. X is called SDFR (CDFR) if its *cdf* F is SDFR (CDFR).

Theorem 2.1 (i) (c.f.Keilson [10])*F is SDFR iff it has a representation,*

$$F(t) = \int_{\overline{R}} (1 - e^{-\lambda t}) G(d\lambda) \tag{2.1}$$

for some unique and possibly defective distribution G continuous at 0. F is absolutely continuous iff G is honest.

(ii) *F is CDFR iff (2.1) holds and the mixing distribution G is honest and infinitely divisible.*

Proof. (i) $\overline{F}$ restricted to $(0, \infty)$ is c.m. iff, by Bernstein's theorem (p. 416 [6]), it has a representation

$$\begin{aligned} \overline{F}(t) &= \int_0^\infty e^{-\lambda t} G(d\lambda), \quad t > 0 \\ &= G\{0\} + \int_{(0,\infty)} e^{-\lambda t} G(d\lambda) \end{aligned} \tag{2.2}$$

for some unique and possibly substochastic measure G on $[0, \infty)$. Since $\overline{F}(t) \to 0$ at $t \to \infty$, we must have $G\{0\} = 0$, i.e., G is continuous at 0. Also $G[0, \infty) = \overline{F}(0+)$ from (2.2). Thus the measure G is either honest or, substochastic. The corresponding *cdf* $G(x)$ is honest or defective according as $F\{0\} = 1 - \overline{F}(0+) = 1 - G[0, \infty) =$ or > 0. Extending the measure G to include a possible atom at infinity by setting $G\{\infty\} = 1 - G[0, \infty) = F\{0\}$, we see that f has a jump at 0 iff G is substochastic. This together with (2.2) yields (2.1).

(ii) If F is CDFR, the hazard function $R(t) = \int_0^t r(x)dx$ has a c.m. derivative $\Rightarrow \overline{F}(t) = s^{-R(t)}$ is c.m. by a well known composition property of c.m. functions. This and $F(0) \equiv F(0+) = 0$ implies that if F is CDFR, then F must be SDFR and absolutely continuous. Thus(2.1) holds with a honest G and we can write the Laplace transform of G as

$$\int_0^\infty e^{-\lambda t} G(d\lambda) = \overline{F}(t) = e^{-R(t)}, \ t > 0. \tag{2.3}$$

Since $R'(t) = r(t)$ in c.m. and $R(0) = 0$, a well known theorem of Feller (p. 425, [6]) shown G must be infinitely divisible. Conversely if G is any infinitely distribution on the half line and continuous at 0, its Laplace transform must be of the form $\exp(-\psi(t))$ where ψ' is c.m., $\psi(0) = 0$. Hence by (2.2), we have $R(t) = \psi(t)$, $t > 0$; *afortiori* $r(t) = \psi'(t)$ is c.m. □

In a somewhat restricted context, the absolutely continuous subclass of {SDFR} corresponding to honest mixing distributions has been noted by Keilson [10] who calls them 'completely monotone densities', and Goldie [8], Steutel [15] proved their infinite divisibility. By allowing G to be substochastic in (2.1), the possibility of an 'infant mortaity' effect (measured by $F\{0\} = G\{\infty\}$) is included and this does not affect the infinite divisibility conclusion. Reparametrizing (2.1) such that instead of the failure rate, the mean λ^{-1} is randomized; one can reinterpret (2.1) to say that a r.v. X is SDFR iff it has a representation

$$X \stackrel{d}{=} UW \tag{2.4}$$

where U, W are independent (note $0 \leq W < \infty$ a.s.), U is exponential and $\stackrel{d}{=}$ denotes equality in distribution. Finally if we consider the set of extended DFR distribution $\{\overline{DFR}\}$ as those F with $\log \overline{F}$ convex on $(0, \infty) \cap \{0 < F < 1\}$ and a possibly positive probability of 'immortality' $0 \leq F\{\infty\} < 1$, then one can drop the continuity condition at 0 in (2.1) to define $\{\overline{SDFR}\}$.

Feller's [7] representation of infinitely divisible Laplace-Stieltjes transform implies the following

Corollary 2.1. *F is $CDFR$ iff it has a representation*

$$\overline{F}(t) = \exp\left\{-\int_0^\infty \frac{1 - e^{-ty}}{y} M(dy)\right\},\ t \geq 0 \tag{2.5}$$

where M is a measure satisfying

$$\int_1^\infty \frac{M(dy)}{y} < \infty,\ \int_0^\infty \frac{M(dy)}{y} = \infty.$$

Call M the canonical measure associated with F. We note the following necessary modification of Feller's argument (p. 426, [7]) to allow for the extra condition $\int_0^\infty y^{-1} M(dy) = \infty$ and a direct proof. Sufficiency follows by direct computation. Note that (2.5) implies

$$-\log \overline{F}(t) \leq tM[0, 1) + \int_1^\infty \left(1 - e^{-ty}\right) y^{-1} M(dy)$$

Hence $\int_1^\infty y^{-1} M(dy) < \infty$ guarantees $F(0+) = 0$. The other condition $\int_0^\infty y^{-1} M(dy) = \infty$ implies F is honest. For the , converse, Bernstein's representation of the c.m. failure rate function and Fubini's theorem yield

$$-\log \overline{F}(t) = \int_0^t r(x)dx = \int_0^\infty \left\{\left(1 - e^{-ty}\right)/y\right\} M(dy). \qquad (2.6)$$

Since CDFRs being exponential mixtures are necessarily DFR, we have $F(t) < 1$, all $t > 0$. Hence bounding the integrand in (2.6),

$$\infty > -\log \overline{F}(t) \geq t \int_0^\infty \frac{M(dy)}{1+ty} \geq t \int_0^\infty \frac{M(dy)}{1+y},$$

choosing $t < 1$ for the last inequality. Hence

$$\int_1^\infty \frac{M(dy)}{y} \leq \int_1^\infty \frac{2}{1+y} M(dy) < \infty.$$

The remaining condition $\int_0^\infty y^{-1}M(dy) = \infty$ now follows from (2.4) since $\exp\left(-\int_0^\infty y^{-1}M(dy)\right) \leq \overline{F}(t) \to 0$, as $t \to \infty$. □

Theorem 2.1 also implies :

Corollary 2.2. $\{CDFR\} \subset \{SDFR\} \subset \{DFR\}$. *The inclusions are strict.*

A direct alternative argument can also be obtained using the closure properties of c.m. functions under composition. Strict inclusions are illustrated by the following counterexamples.

Counterexamples (i) Consider the d.f.F_0 with the tail probability function $\overline{F_0}$, defined by

$$\overline{F_0}(t) = \begin{cases} e^{-t/\mu}, & \text{if } t \leq \mu \\ (\mu/t)e^{-1}, & \text{if } t > \mu \end{cases}$$

Then F_0 is DFR with mean μ, but is *not* SDFR since $\overline{F}_0^{(2)}(\mu)$ does not exist ($\overline{F}_0$ is in fact a sharp upper bound for DFR tails with a mean μ [1]).

(ii) The reliability function $\overline{F}(t) = (1 - e^{-t})/t, t > 0$ is c.m. (viz., $\overline{F}(t)$ is the Laplace transform of the uniform distribution on$(0, 1)$), but the failure rate $r(t) = t^{-1} - (e^t - 1)^{-1}$ is not (since, $r''(t)$ is clearly negative). Thus F is SDFR, but is not CDFR.

The most commonly known DFR distributions are members of the $\{SDFR\}$, or $\{CDFR\}$ classes, as the following examples show.

Examples (i) *Exponential* : $\overline{F}(t) = e^{-\lambda t}$ is CDFR. Mixing distribution is degenerate at $\lambda > 0$.

(ii) *Power distribution* : $\overline{F}(t) = (c/c+t)^{\alpha}, c > 0, \alpha > 0$ is CDFR corresponding to gamma mixing with scale parameter c and shape parameter α. Corresponding to $c = 1$, the canonical measure M is $M(dy) = \alpha e^{-y} dy$. Note $\alpha = 1$ (exponential mixing) yields 'Zipf's law' whose canonical measure is also exponential.

(iii) *DFR gamma* : $\overline{F}(t) = [\Gamma(\alpha)]^{-1} \int_{\lambda t}^{\infty} e^{-x} x^{\alpha-1} dx, 0 < \alpha < 1$ is SDFR. To check that $\overline{F}$ is c.m., set $A(t) \equiv \Gamma(\alpha)\overline{F}(t)$. The functions $e^{-x}, x^{\alpha-1}$ are both c.m., hence so is their product $-A'(t)$ which is in turn equivalent to to $A(t)$ being c.m. The associated mixing distribution has U-shaped density

$$g(x) = \frac{\sin \alpha\pi}{\pi} x^{-1} (x - \lambda)^{-\alpha}, \ x \geq \lambda, \ \lambda > 0$$

(iv) *DFR Weibull* : $\overline{F}(t) = \exp(-t^{\alpha}), \ 0 < \alpha < 1$ is CDFR. The corresponding mixing distributions G of (2.1) is the family of stable laws with index $\alpha \in (0, 1)$. The canonical measure is

$$M(dy) = \alpha y^{-\alpha} dy / \Gamma(1 - \alpha)$$

This link of the DFR Weibulls with the family of *stable laws* which are not known in closed form except for $\alpha = \frac{1}{2}$, probably helps to explain why estimation of α and the Weibull reliability with good small sample properties is a difficult problem!

For a r.v. X with *cdf* F, let F^{*n} be the n-fold self-convolution F, and let X_n denote a r.v. with *cdf* $F_n := 1 - \overline{F}^{1/n}$; $n = 1, 2, \ldots$

Corollary 2.3 : (i) *X is CDFR iff each X_n is SDFR, $n = 1, 2, \ldots$*

(ii) If X is CDFR, then so is X^β for any $\beta \geq 1$.

Proof (i) By theorem 2.1, X with d.f. F is CDFR with mixing law G iff for each n, there exists an honest *cdf* G_n on $[0, \infty)$ with $G_n^{*n} = G$, i.e., if and only if

$$\left\{\overline{F}_n(t)\right\}^n = \overline{F}(t) = \int_0^\infty e^{-\lambda t} G_n^{*n}(d\lambda) = \left\{\int_0^\infty e^{-\lambda t} G_n(d\lambda)\right\}^n, \;\; t > 0,\; n \geq 1,$$

i.e., iff each X_n is SDFR, since $0 = G\{0\} = (G_n\{0\})^n$.

(ii) From (i), there exists *cdfs* G_n on $(0, \infty)$ such that the the CDFR d.f.F of satisfies

$$\overline{F}(t) = \left\{\int_0^\infty e^{-tx} G_n(dx)\right\}^n = E^n(e^{-tX_n}),\; n \geq 1.$$

For any $\beta \geq 1$, set $\alpha = \beta^{-1} \in (0, 1]$. Then

$$\overline{F}^{1/n}(t^\alpha) = \int_0^\infty e^{-t^\alpha x} G_n(dx) = \int_0^\infty \int_0^\infty e^{-\lambda(tx^\beta)} H_\alpha(d\lambda) G_n(dx)$$

where H_α is '*stable*' with index $\alpha \in (0, 1)$. If Y is any r.v. distributed as H_α, then

$$\overline{F}^{1/n}(t^\alpha) = \int_0^\infty E_{H_\alpha}(e^{-tx^\beta Y}) G_n(dx) = \int_0^\infty e^{-tz} K_{n,\alpha}(dz)$$

for some distribution $K_{n,\alpha}$ on $(0, \infty)$, since mixtures of Laplace transforms (i.e., c.m. functions) are Laplace transforms (c.m. functions). Thus

$$P(X^\beta > t) = \overline{F}(t^\alpha) = \int_0^\infty e^{-tz} K_{n,\alpha}^{*n}(dz),\; \text{all } n \geq 1,$$

which proves X^β is CDFR. □

While exponentials with constant failure rates are in the boundary of IFR and DFR as well as in the boundary between every known ageing characteristic such as (IFRA, NBU, DMRL etc.) and its non-aging dual, the extent of strong non-aging behavior induced by mixing them is evident from the limiting behavior of the resulting failure rate. If F is SDFR as in (2.1) with failure rate $r(t)$, then $r(t) \to \lambda_0$ as $t \to \infty$ where

$$\lambda_0 = \inf\{\lambda > 0 : G\{\lambda\} > 0\}$$

is the least 'point of increase' (i.e. atom) of the mixing distribution G in theorem 2.1. In particular with any continuous mixing, $r(t) \to 0$.

The next theorem shows that with a norming factor $n^{\frac{1}{2}}$ and a condition on the density, series systems of absolutely continuous SDFR components asymptotically differ from a similar system of exponential components by a Weibull probability factor.

Theorem 2.2 *Let $X_1, X_2, \ldots$ be iid absolutely continuous SDFR and suppose $r'(0+) > -\infty$. Let $Z_1, Z_2, \ldots$ be iid exponential with rate $\lambda = 1/f(0+)$. Call $X_{(1),n} = \min(X_1, \ldots, X_n)$, $Z_{(1),n}$ being defined similarly. Then*

$$e^{-ct^2} P(n^{\frac{1}{2}} X_{(1),n} > t) \sim P(n^{\frac{1}{2}} Z_{(1),n} > t),$$

where $c = 1/2\sigma^2$ and σ^2 is the variance of the mixing distribution of X_1.

Proof. The r.v.s $X_1, X_2, \cdots$ are *i.i.d.* SDFR with a common distribution $\overline{F}(t) = \int_0^\infty e^{-\lambda t} G(d\lambda)$, where G is on $(0, \infty)$ by the requirement $P(X_1 = 0) = 0$. Write $P(X_n > t) = E \exp(-tY_n)$, where $Y_1, Y_2, \ldots$ are *iid* with cdf G and second moment

$$\int_0^\infty \lambda^2 G(d\lambda) = -f'(0+) < \infty,$$

since $r'(0+) > -\infty$. Thus G has a finite positive variance σ^2, *afortiori* the mean $\mu := \int_0^\infty \lambda G(d\lambda) = f(0+) = \mu^{-1} \in (0, \infty)$. Hence,

$$\begin{aligned} \frac{P(\sigma n^{\frac{1}{2}} X_{(1),n} > t)}{P(\sigma n^{\frac{1}{2}} Z_{(1),n} > t)} &= \exp(n^{\frac{1}{2}} \mu t/\sigma) \overline{F}^n (t/\sigma n^{\frac{1}{2}}) \\ &= \exp(n^{\frac{1}{2}} \mu t/\sigma) E(\exp\left\{ -\frac{t}{\sigma\sqrt{n}} (Y_1 + \ldots + Y_n) \right\}) \\ &= E \exp\left\{ -t \sum_{i=1}^{n} (Y_i - \mu)/n^{\frac{1}{2}} \sigma \right\} \to e^{\frac{1}{2}t^2}. \end{aligned}$$

by central limit theorem. □

Since a Laplace transfer is uniquely determined i) by its values over any integral $(a, \infty), a > 0$, and also ii) by its values for any sequence a_n of arguments such that $\Sigma a_n^{-1} = \infty$ by the c.m. property, (p. 439, [7]) :

Corollary 2.4. *Let F_1, F_2 be SDFR.*

(i) *If F_1, F_2 agree for some t_0 onwards ($t_0 > 0$), then $F_1 = F_2$.*

(ii) *If $F_1(cn) = F_2(cn), n \geq n_0$ for some $c > 0$, some integer n_0, then $F_1 = F_2$.*

The following three results of Goldie [8], Steutel [15], Keilson [11] and Heyde [9] on mixtures of exponentials are of direct interest, since by corollary 2.2, they apply to {CDFR} and {SDFR} or their subsets with a moment condition, as noted below :

(i) Goldie [8] and Steutel [15] have shown all r.v.s with representation (2.3) (i.e., SDFR distributions) are infinitely divisible.

(ii) For the subclass of SDFR with a finite variance, Keilson and Steutel [12] has given a measure distance from exponentiality. If $F \in L^2$ with mean μ and variance σ^2, Keilson and Steutel's result implies that

$$\rho(F, Exp) = (\sigma/\mu)^2 - 1 \geq 0$$

is a metric on $\mathcal{T} = \{F : F \in L^2, F \text{ is SDFR}\}$ and thus is a distance from the exponential *cdf* $(1 - e^{-x})$.

(iii) For the subclass (with $\mu = 1$), Heyde [9] and Keilson [10] have given an estimate of the Levy distance :

$$\sup_{F \in \mathcal{T}} |\overline{F}(t) - e^{-t}| \leq C(\sigma^2 - 1)^{1/4},$$

where $C \leq 8\sqrt{3}\, 2^{-\frac{1}{4}} \pi^{-1}$ is an absolute constant. The minimal value of C is unknown.

Examples of SDFR distributions in Stochastic Processes : The following illustrative examples are drawn from known results and applications. They serve to demonstrate the prevalence of the SDFR property by showing that such distributions often arise naturally in several areas of stochastic processes. Except for the first example which has an empirical base, the SDFR property in each of the other examples is a provable consequence. Details and proofs can be found in the appropriate references.

(i). In a stochastic theory of *manpower planning*, Bartholomew [2] has used mixtures of exponentials to model the length of time employees stay with a given organization and applied it to British empirical data. In this context, he calls the associated distribution of the time to failure as CLS (*completed length of service*) distribution. The failure rate is then a measure of the propensity of the employees to leave. He also gives an approximate solution of the renewal equation corresponding to CLS distributions which are SDFR.

(ii). For any *birth-and-death process* on the non-negative integers, let T_{ij} be the passes time from state i to state j. Keilson [10] shows $T_{n,n+1}$ and $T_{n-1,n}$ have c.m. densities; i.e., *passage times to adjacent states* are SDFR.

(iii). For the *Esary-Marshall-Proschan maximum shock model* [6] of a device subject to a Poisson stream of shocks with intensity $\lambda > 0$ such that the device fails only when a critical shock threshold x (with *cdf* H) is reached in any particular shock, the threshold being random (*cdf* G) and independent of shock magnitudes; let f be the density of the time to failure. Esary, Marshall and Proschan show ([6], theorem 6.2) that for $t > 0, f(t)$ is a mixture of exponential densities. The following equivalent calculation of the reliability is simpler :

$$\overline{F}(t) = \sum_{j=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!} \int_0^{\infty} H^j(x) G(dx) = \int_0^{\infty} e^{-\lambda \overline{H}(x) t} G(dx)$$

and hence is SDFR without a jump at 0.

(iv). Consider a complex repairable system of independent Markov components with a binary structure [1] function ϕ and exponential repair times. If the 'perfect state' corresponds to all compoents working and the state space is partitioned into the 'bad' set $B = \{\phi = 0\}$ and the 'good' set $B^c = \{\phi = 1\}$, Keilson ([10], [11]) shows that each of the following random variables :

- a) sojourn time on the 'good' set, and
- b) the ergodic exit time,
- c) starting from the 'perfect state', time to last exit from the 'perfect state' before hitting the'bad' set$B$,

have densities which are mixtures of exponentials and hence absolutely continuous SDFR.

3. Closure properties and reliability bounds

Lemma 3.1. **a)** $\{SDFR\}$ *is closed under*

(i) *formation of series systems of independent components ,*

(ii) *mixtures,*

(iii) *transformation* $T : F(x) \to TF(x) = \mu^{-1} \int_0^t \overline{F}(x)dx$*, where* F *has mean* μ*,*

(iv) *proper (i.e., honest) limits in distribution* $\{\text{CDFR}\}$ *is closd under (i) and (iv).*

b) $\{CDFR\}$ *is closed under mixing and formation of series systems of independent components.*

Proof. (i) If X_1 are independent SDFR (*cdf* F_i) with mixing measures $G_i (i = 1, 2, \ldots, n)$, then $X_{1:n} = \min_{1 \le i \le n} X_i$ has *cdf* F,

$$\overline{F}(t) = \int_0^\infty e^{-\lambda t}(G_1 * \ldots * G_n)(d\lambda), \quad t > 0$$

where $*$ denotes convolution $G_1 * \ldots * G_n$ is continuous at 0 since each G_i is. Note $X_{1:n}$ has a jump at 0 iff some G_i is defective. If each G_i is infinitely divisible (each X_i CDFR), then so is their convolution, implying$X_{i:n}$ is CDFR.

(ii) Let $\{F_\alpha \ : \ \alpha \in \mathcal{A}\}$ be an indexed family of SDFR distributions, so that their exist for each $\alpha \in \mathcal{A}$, (possibly substochastic) d.f.s G_α on $[0, \infty)$, such that

$$\overline{F_\alpha}(t) = \int_0^\infty \ e^{-\lambda t} G_\alpha(d\lambda), \quad t > 0, \ \ \alpha \in \mathcal{A}.$$

If F is the mixture $F := \int_{\mathcal{A}} F_\alpha P(d\alpha)$, for some probability measure P on $\mathcal{A}$, then clearly,

$$\overline{F}(t) \ \ = \ \ \int_0^\infty \ e^{-\lambda t)Q(d\lambda), \quad t>0}$$

$$\text{where } Q(\lambda) \;=\; \int_{\mathcal{A}} G_\alpha(\lambda) P(d\alpha).$$

Thus the mixture F is SDFR. Note, $\{CDFR\}$ is not closed under mixing since mixtures of infinitely divisible laws need not be infinitely divisible.

(iii) If F is SDFR as in (2.1), then $\overline{TF}(t) = \int_0^\infty e^{-\lambda t} Q(d\lambda)$, $t > 0$, where $Q(d\lambda) = G(d\lambda)/\lambda\mu$. Since $\mu = \int_0^\infty \overline{F}(x)dx = \int_0^\infty \lambda^{-1} G(d\lambda)$ and G is continuous at 0, we conclude Q is continuous at 0 and honest even if G is not.

(iv) Suppose F_n is SDFR with mixing distribution $G_n, G_n[0,\infty) \le 1$, $n = 1, 2, \ldots$ F is honest, and $F_n \to F$ weakly, i.e., $F_n(t) \to F(t)$ for $t > 0$ since F_n being continuous on $(0, \infty)$, so is the limit. Since each $\overline{F}_n$ is c.m., the extended continuity theorem of Laplace transforms implies that $\overline{F}(t)$ must be the Laplace transform of a possibly defective distribution G such that $G_n \to G$ weakly. Since F is honest, by (2.2) we have $G\{0\} = \lim_{t\to\infty} \overline{F}(t) = 0$. Thus F is SDFR. If each G_n is honest and infinitely divisible, so is G; in which case F is CDFR. □

4. CDFR representation theorems

We obtain additional representations of the CDFR laws by exploiting the infinite divisibility of mixing distributions. The results again focus on exponentials as the basic elements, which under various compounding operations yield $\{SDFR\}$ and $\{CDFR\}$.

If A is any set of life distributions and S is any operation which performed on elements of A can generate now distributions, call A^S the set of all distributions which can be so generated. Let A^{LD} be the *limits in distribution* of A, A^{CP} the *compound Poisson distributions* generated by A and $A^{m(CP)}$ the set of *exponential mixturess* with compound Poisson mixing :

$$\begin{aligned} A^{LD} &:= \{F : \exists F_n \in A \text{ with } F_n \to F \text{ in distribution }\} \\ A^{CP} &:= \{F : F(t) = \sum_{j=0}^{\infty} e^{-\lambda}\frac{\lambda^j}{j!}H^{*j}(t),\ \text{some } H \in A,\ \lambda > 0\} \\ A^{m(CP)} &:= \{F : \overline{F}(t) = \int_0^{\infty} e^{-\lambda t}G(d\lambda),\ \text{some } G \in A^{CP}\} \end{aligned} \tag{1}$$

and finally for any indexed family $A = \{F_\alpha\}$ of *cdfs*, let

$$A^{CL} = \{F : F(t) = \int F_\alpha(t)G(d\alpha); F_\alpha \in A,\ G \text{ is honest }\} \tag{4.2}$$

and, $A^{\overline{CL}}$ the set obtained by allowing G to be possibly defective in (4.2). Note, (i) $A^{m(CP)} \subset \{\overline{SDFR}\}$, and (ii)$LD$ is a closure operation [3], as are $CL, \overline{CL}$, if A^{CL} is 'identifiable' [14] (i.e., distinct mixing laws G in (4.2) induce distinct elements F); the others are not. If A contains 'defective' distributions, let A^{LD^o} be the set of honest weak limits of A-elements F_n such that $F_n\{\infty\} \to 0$ (free, if A has no defective elements). For any set A of honest *cdfs* such that any *substochastic* $F \in \overline{A} \Rightarrow F_0(t) := F(t)/1 - F\{\infty\} \in A$, we note $\overline{A}^{LD^o} = A^{LD^o}$. Both $\{SDFR\}$ and $\{CDFR\}$ have this property. Choosing $A = \{Exp\}$ indexed by its mean, (2.1) with an obvious reprarametrization and lemma 3.1 (iv) yields

$$\{SDFR\}^{LD^o} = \{SDFR\} = \{Exp\}^{CL},\ \{CDFR\}^{LD^o} = \{CDFR\}. \tag{4.3}$$

A trivial modification of lemma 3.1 (iv) further implies $\{\overline{SDFR}\}^{LD} = \{\overline{SDFR}\}$.

For any *iid* sequence $Z_i, i = 1, 2, \cdots$ and a Poisson r.v. N independent of the $\{Z_i, i \geq 1$, let

$$Y = \begin{cases} 0, & \text{if } N = 0, \\ \sum_{i=1}^{N} Z_i, & \text{if } N > 0. \end{cases} \tag{2}$$

If the distribution of Z_1 ranges over A, the resulting set of possible distributions of Y is A^{CP}.

Theorem 4.1 (i) $\{SDFR\} = \{Exp\}^{CL}$,

$$\text{(ii) } \{CDFR\} = \{Exp\}^{CL,m(CP),LD^o} = \{SDFR\}^{m(CP),LD^o}.$$

Proof. The first claim is equivalent to a restatement of theorem 2.1(i), in view of (4.2). For the remaining claim, we argue as follows.

If F is CDFR with canonical measure M, set

$$I_n = \int_{n^{-1}}^{\infty} y^{-1} M(dy), \quad n = 1, 2, \ldots$$

Using corollary 2.1 for the representation of CDFR tails, we can write,

$$\overline{F}(t) = \lim_{n\to\infty} \exp\left\{-\int_{n^{-1}}^{\infty} \left(1 - e^{-ty}\right) y^{-1} M(dy)\right\},$$

such that $I_1 < \infty$ and $I_n \to \infty$. Set

$$H_n(x) \stackrel{def}{=} 1 - I_n^{-1} \int_x^{\infty} y^{-1} M(dy),\ x \geq n^{-1},\ n \geq 1$$

$$D_n(t) \stackrel{def}{=} \int_0^{\infty} e^{-tx} H_n(dx).$$

Then,D_n is SDFR with honest mixing distribution supported by $[n^{-1}, \infty)$. Given n, define a r.v. Y_n as in (4.4) choosing $EN = I_n$ and Z_1 distributed as J_n. Let G_n be the distribution of Y_n and consider the d.f. $F_n \in \{SDFR\}^{m(CP)}$ defined with mixing law G_n. Then

$$\begin{aligned} \overline{F}_n(t) &= \int_0^{\infty} e^{-tx} G_n(dx) = E(e^{-tY_n}) = s^{-I_n[1-J_n(t)]} \\ &= \exp\left\{-\int_{n^{-1}}^{\infty} \left(1 - e^{-ty}\right) y^{-1} M(dy)\right\} \to \overline{F}(t), \end{aligned}$$

as $n \to \infty$. Thus,

$$\{CDFR\} \subset \{SDFR\}^{m(CP),LD^o}. \tag{3}$$

Conversely, for any Y as in (4.4) with $EN = \theta > 0$, the extended r.v. $X \in (0, \infty]$ defined by

$$
\begin{aligned}
P(X > t) \stackrel{def}{=} E(e^{-tY}) &= P(N = 0) + \sum_{n=1}^{\infty} E(\exp\left\{-t \sum_{i=1}^{n} Z_i\right\}) P(N = n) \\
&= \exp\left\{-\theta + \theta E(e^{-tZ_1})\right\} \\
&= \exp\left\{-\int_0^{\infty} (1 - e^{-ty}) y^{-1} P(dy)\right\},
\end{aligned}
$$

is a defective CDFR tail, where P is the measure $P(B) = \theta \int_B yG(dy)$, and G denotes the distribution of Z_1. Thus $\int_0^{\infty} y^{-1} P(dy) = \theta P(Z_1 \leq 1) < \infty$. If the distribution of Z_1 is in A, then that of X belongs to $A^{m(CP)}$. Thus if A is any set of d.f.s on the half line, we have

$$
A^{m(CP)} \subset \{\overline{CDFR}\}
$$

Hence choosing $A = \{SDFR\}$ and combining with (4.5),

$$
\{CDFR\} \subset \{SDFR\}^{m(CP),LD^o} \subset \{\overline{CDFR}\}^{LD^o} \{CDFR\}^{LD^o} = \{CDFR\},
$$

using (4.3) and remarks immediately preceeding it. □

The class of stopped Poisson partial sums can be further exploited to yield the following result.

Theorem 4.2. *Every CDFR distribution can be represented as the limit in distribution of a sequence of Poisson mixtures of series systems of iid extended SDFR components.*

Proof. F is $CDFR \Leftrightarrow \overline{F}(t)$ is the Lalace transform of an infinitely divisible d.f. G; i.e, iff there is a sequence of constants $0 < \lambda_n \to \infty$ and honest *cdfs*

H_n on $[0,\infty)$ such that

$$\sum_{j=0}^{\infty} e^{-\lambda_n} \frac{\lambda_n^j}{j!} H_n^{*j}(t) \to G(t)$$

at continuity points of G (note, $H_n^{*0}(t) := 1$ for $t \geq 0$). The d.f. G is continuous at 0 since $\lambda_n \to \infty$. Define G_n such that $(1 - G_n)$ is the Laplace transform of H_n. Then $G_n \in \{SDFR\}$, and

$$\begin{aligned} \overline{F}(t) &= \int_0^\infty e^{-tx} G(dx) = \lim_{n\to\infty} e^{-\lambda_n[1-G_n(t)]} \\ &= \lim_{n\to\infty} \left\{ e^{-\lambda_n} + \sum_{j=1}^{\infty} e^{-\lambda_n} \frac{\lambda_n^j}{j!} \left[\overline{G}_n(t)\right]^j \right\}. \end{aligned} \tag{4.6}$$

If,

$$p_j(\lambda) := (1 - e^{-\lambda})^{-1}(e^{-\lambda}\lambda^j/j!), \quad j \geq 1, \;\; \lambda > 0$$

denotes the Poisson distribution conditioned to positive values, then (4.6) shows that the immortal component can be neglected in the limit, since

$$\begin{aligned} \overline{F}(t) &= \lim_{n\to\infty} \left\{ e^{-\lambda_n} + (1 - e^{-\lambda_n}) \sum_{j=1}^{\infty} p_j(\lambda_n)[\overline{G}_n(t)]^j \right\} \\ &= \lim_{n\to\infty} \sum_{j=1}^{\infty} p_j(\lambda_n) \int_0^\infty e^{-tx} H_n^{*j}(dx), \end{aligned} \tag{4.7}$$

which is the desired representation. □

Corollary 4.1. $\{\text{CDFR}\} = \{\overline{SDFR}\}^{CP,LD^o} = \{Exp\}^{\overline{CL},CP,LD^o}$

Proof. The first equality is theorem 4.2. the next assertion follows since the representation (2.1) without the continuity condition at 0 generates the set of extended SDFR distributions and by reparametrizing (2.1) such that the family {exp} is indexed by its mean. □

In (4.7), instead of allowing each subsystem of size $j (1 \leq j < \infty)$ to have a positive probability of immortality, note that we can write $F = \lim F_n$, where the system with reliability $\overline{F}_n(t)$ is *either*, a series system of $j \geq 1$ *iid* honest SDFR components with probability

$$c_{nj} = p_j(\lambda_n) H_p^{*j}(0+)$$

or, is an immortal component with probability $0 < c_{n0} = 1 - \sum_{j=1}^{\infty} c_{nj} \to 0$.

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