Predictive Inference for Future Responses from Two Component Systems

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By

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ABSTRACT

Predictive distributions for bivariate future responses are derived for systems whose components are connected in parallel or in series under the assumption that the lifetimes of the components are exponentially distributed. The predictive reliability, moment generating and hazard rate functions are derived for the former case. Illustrative examples are provided for each type of system.

KEY WORDS

Parallel connections, components connected in series, reliability, moment generating function, hazard rate, one parameter exponential distribution, predictive distribution.

1. INTRODUCTION

Systems that are composed of several components connected either in series or in parallel, have a duration that depends on the lifetimes of the components. One must know the distribution of the lifetimes of such components in order to measure the efficiency of such systems. In a parallel connection, a system of k components remains operational whenever at least one component is working, and the system fails whenever all of the components fail. For example, the useful life of a pair of kidneys belonging to a given individual can be determined by monitoring them until both fail. In a series system comprising k components, the system fails whenever one of the components fails. For example, if one resistor fails in a circuit system where the resistors are connected in series, the whole system fails.

Various distributional problems related to parallel or series connection systems have been discussed in the literature. For instance, Peña and Gupta (1990) obtained estimators of the parameters of the Marshall-Olkin exponential distribution when random samples from series and parallel systems are available; Smith (1982) extended an asymptotic result in connection with a certain series-parallel system; Harris and Soms (1973) developed procedures for testing hypotheses and obtaining confidence intervals for the reliability of systems of independent parallel components; Harris and Soms (1974) studied a family of distributions useful in statistical inference on the reliability of systems of independent parallel components; and Gertsbakh (1982) derived some confidence limits useful in reliability analysis on the basis of pooled data from parallel and series-parallel systems.

The exponential distribution has been used in a number of reliability studies. For example, Draper and Guttman (1972) discussed in detail the reliability of a system wherein the components are connected in series and the failure times of the components are independently and exponentially distributed. Butler and Huzurbazar (1997) approximated Bayesian predictive density functions of first passage times between states of a semi-Markov process and pointed out applications in survival analysis. To our knowledge, our paper constitutes a first attempt to apply predictive inference to systems of components connected in series or in parallel.

More specifically, we are concerned with predictive inference with respect to future responses as well as their reliability function. We consider the cases of dependent as well as independent exponential probability models for the components of the systems. Our derivations are based on the Bayesian framework. This novel approach makes use of prior information and therefore should produce more precise results. A non-informative prior distribution for the parameters has been utilized in order to simplify the derivations. Our methodology also applies to other priors; however the resulting density functions might then be very complicated. The results would be applicable in reliability studies in industry. Some numerical examples illustrate the results.

The organization of this paper is as follows: Section 2 discuss about the predictive distribution when the components are connected in parallel. The predictive distribution when the components are connected in series is given in Section 3. Finally, some discussion have been added in Section 4.

2. DEPENDENT COMPONENTS CONNECTED IN PARALLEL

Let X and Y be the random variables representing the failure times of the two components C_1 and C_2. It is assumed that the failure times for both components have exponential densities with parameters λ and η respectively. We assume that both components are dependent because the failure of either component may change the parameter of the life distribution of the other component. Let the parameter for component C_1 change from λ to $\lambda^{'}$, and the parameter for component C_2 change from η to $\eta^{'}$ while the system is operational. Then following Freund (1961), the bivariate density function of X and Y is given by

$$p(x, y|\lambda, \lambda^{'}, \eta, \eta^{'}) = \begin{cases} \lambda\eta^{'} \ \exp\left\{-\eta^{'} y - (\lambda + \eta - \eta^{'})x\right\}, \text{ for } 0 < x < y, \\ \\ \eta\lambda^{'} \ \exp\left\{-\lambda^{'} x - (\lambda + \eta - \lambda^{'})y\right\}, \text{ for } 0 < y < x, \end{cases} \tag{1}$$

where λ, $\lambda^{'}$, η, and $\eta^{'}$ are non-negative unknown constants. Accordingly, one has the following marginal densities:

$$p(x|\lambda, \lambda^{'}, \eta, \eta^{'}) \ = \ \frac{(\lambda - \lambda^{'})(\lambda + \eta) \ \exp\{-(\lambda + \eta)x\}}{\lambda + \eta - \lambda^{'}} + \frac{\lambda^{'}\eta \ \exp\left\{-\lambda^{'} x\right\}}{\lambda + \eta - \lambda^{'}}, \quad 0 < x < \infty,$$

provided that $\lambda + \eta - \lambda^{'} \neq 0$, and

$$p(y|\lambda, \lambda^{'}, \eta, \eta^{'}) \ = \ \frac{(\eta - \eta^{'})(\lambda + \eta) \ \exp\{-(\lambda + \eta)y\}}{\lambda + \eta - \eta^{'}} + \frac{\lambda\eta^{'} \ \exp\left\{-\eta^{'} y\right\}}{\lambda + \eta - \eta^{'}}, \quad 0 < y < \infty,$$

provided that $\lambda + \eta - \eta^{'} \neq 0$.

2.1 Joint predictive density function

In order to derive the likelihood function from model (1), consider a random sample of size n from the model. Let the sums of the first r failure times of the components C_1 and C_2 be respectively denoted by $\sum x$ and $\sum y$, and the sums of the next $(n-r)$ failure times of the components C_1 and C_2 be respectively denoted by $\sum{}'x$ and $\sum{}'y$. The likelihood function of the sample then becomes

$$\begin{aligned} \mathrm{L}(\lambda,\lambda^{'},\eta,\eta^{'}|\mathbf{x},\mathbf{y}) &= (\lambda\eta^{'})^{r}(\lambda^{'}\eta)^{n-r} \\ &\quad\times\exp\left\{-(\lambda+\eta-\eta^{'})\sum x-\eta^{'}\sum y-\lambda^{'}\sum{}'x-(\lambda+\eta-\lambda^{'})\sum{}'y\right\}. \end{aligned}$$

Following Jeffreys (1961), let a non-informative prior for the parameters $\lambda,\lambda^{'},\eta,\eta^{'}$ be

$$p(\lambda,\lambda^{'},\eta,\eta^{'})\propto 1/(\lambda\,\lambda^{'}\,\eta\,\eta^{'})\,.$$

Then the posterior density of $\lambda,\lambda^{'},\eta,\eta^{'}$ given the data $(\mathbf{x},\mathbf{y})$ is

$$\begin{aligned} p(\lambda,\lambda^{'},\eta,\eta^{'}|\mathbf{x},\mathbf{y}) &= \Psi(\mathbf{x},\mathbf{y})(\lambda\eta^{'})^{r-1}(\lambda^{'}\eta)^{n-r-1} \\ &\quad\times\exp\left\{-(\lambda+\eta-\eta^{'})\sum x-\eta^{'}\sum y-\lambda^{'}\sum{}'x-(\lambda+\eta-\lambda^{'})\sum{}'y\right\}, \end{aligned}$$

where the normalizing constant is

$$\begin{aligned} \Psi(\mathbf{x},\mathbf{y})^{-1} &= \int_{\lambda=0}^{+\infty}\int_{\lambda^{'}=0}^{+\infty}\int_{\eta=0}^{+\infty}\int_{\eta^{'}=0}^{+\infty}(\lambda\eta^{'})^{r-1}(\lambda^{'}\eta)^{n-r-1} \\ &\quad\times\exp\left\{-(\lambda+\eta-\eta^{'})\sum x-\eta^{'}\sum y-\lambda^{'}\sum{}'x-(\lambda+\eta-\lambda^{'})\sum{}'y\right\}d\eta^{'}d\eta d\lambda^{'}d\lambda \\ &= \frac{\Gamma(n-r)\Gamma(n-r)\Gamma(r)\Gamma(r)}{(\sum x+\sum{}'y)^{n}(\sum y-\sum x)^{r}(\sum{}'x-\sum{}'y)^{n-r}}\,. \end{aligned}$$

Let w and z be two future responses from the components C_1 and C_2 respectively; then the predictive density of $w,\ z$ given $\mathbf{x},\ \mathbf{y}$ is

$$p(w,z|\mathbf{x},\mathbf{y})=\begin{cases} \int_{\lambda=0}^{+\infty}\int_{\lambda^{'}=0}^{+\infty}\int_{\eta=0}^{+\infty}\int_{\eta^{'}=0}^{+\infty}\lambda\eta^{'}\ \exp\left\{-\eta^{'}z-(\lambda+\eta-\eta^{'})w\right\} \\ \times p(\lambda,\lambda^{'},\eta,\eta^{'}|\mathbf{x},\mathbf{y})d\eta^{'}d\eta d\lambda^{'}d\lambda,\ \text{for } 0<w<z, \\ \\ \int_{\lambda=0}^{+\infty}\int_{\lambda^{'}=0}^{+\infty}\int_{\eta=0}^{+\infty}\int_{\eta^{'}=0}^{+\infty}\eta\lambda^{'}\ \exp\left\{-\lambda^{'}w-(\lambda+\eta-\lambda^{'})z\right\} \\ \times p(\lambda,\lambda^{'},\eta,\eta^{'}|\mathbf{x},\mathbf{y})d\eta^{'}d\eta d\lambda^{'}d\lambda,\ \text{for } 0<z<w\,. \end{cases} \tag{2}$$

Thus for $0 < w < z$,

$$p(w, z|\mathbf{x}, \mathbf{y}) = \frac{1}{\frac{(\sum x+\sum' y)}{r}\frac{(\sum y-\sum x)}{r}}\left[1+\frac{w}{(\sum x+\sum' y)}\right]^{-(n+1)}\left[1+\frac{z-w}{(\sum y-\sum x)}\right]^{-(r+1)},$$

and for $0 < z < w$,

$$p(w, z|\mathbf{x}, \mathbf{y}) = \frac{1}{\frac{(\sum' x-\sum' y)}{n-r}\frac{(\sum x+\sum' y)}{n-r}}\left[1+\frac{z}{(\sum x+\sum' y)}\right]^{-(n+1)}\left[1+\frac{w-z}{(\sum' x-\sum' y)}\right]^{-(n-r+1)}.$$

There is no closed form representation of $p(w, z|\mathbf{x}, \mathbf{y})$. This predictive density function can be evaluated for a given set of data.

2.2 Predictive reliability, moment generating and hazard rate functions

Predictive reliability function

One might be interested in making inferences about a system's reliability. Let the random variables t_1 and t_2 be the failure times of a system. Then the reliability $R(t_1, t_2)$ of the system is defined as $1 - F(t_1, t_2)$, where $F(t_1, t_2)$, is the joint cumulative distribution function of t_1 and t_2. The reliability $R(t_1, t_2)$ is the probability that the system is still operating at time t_1 and t_2. Let $(\sum y - \sum x) = \gamma_1$, $(\sum x + \sum' y) = \gamma_2$, and $(\sum' x - \sum' y) = \gamma_3$, then the bivariate predictive reliability density obtained from model (2) is $R_p(t_1, t_2)$

$$
= \begin{cases}
\frac{1}{\gamma_1\gamma_2}\Bigg(r^2\int_{z=t_2}^{+\infty}\Bigg(\frac{1}{r(\gamma_2+\gamma_1+z)}\Bigg(\gamma_1\gamma_2\Big(1+\frac{z}{\gamma_2}\Big)^{-n}\Big(1-\frac{\gamma_1}{\gamma_2+\gamma_1+z}\Big)^{n} \\
\times\, {}_2F_1\Big[-r,\ 1+n,\ 1-r,\ \frac{\gamma_1}{\gamma_2+\gamma_1+z}\Big]\Bigg) - \frac{1}{r(\gamma_2+\gamma_1+z)}\Bigg(\gamma_1\gamma_2\Big(1+\frac{t_1}{\gamma_2}\Big)^{-n} \\
\times\Big(1+\frac{-t_1+z}{\gamma_1}\Big)^{-r}\Big(1-\frac{\gamma_1-t_1+z}{\gamma_2+\gamma_1+z}\Big)^{n}\, {}_2F_1\Big[-r,\ 1+n,\ 1-r,\ \frac{\gamma_1-t_1+z}{\gamma_2+\gamma_1+z}\Big]\Bigg)\Bigg)dz\Bigg) \\
\text{for } 0<t_1<t_2, \\
\frac{1}{\gamma_2\gamma_3}\Bigg((n-r)^2\int_{w=t_1}^{+\infty}\Bigg(\frac{1}{n(\gamma_3+\gamma_2+w)}\Bigg(\gamma_2\gamma_3(1+\frac{t_2}{\gamma_3})^{-n}\Big(1-\frac{\gamma_2+t_2}{\gamma_3+\gamma_2+w}\Big)^{n-r} \\
\times\Big(1+\frac{-t_2+w}{\gamma_3}\Big)^{-n+r}\, {}_2F_1\Big[-n,\ 1+n-r,\ 1-n,\ \frac{\gamma_2+t_2}{\gamma_3+\gamma_2+w}\Big]\Bigg) - \frac{1}{n(\gamma_3+\gamma_2+w)}\Bigg(\gamma_2\gamma_3 \\
\times\Big(1+\frac{w}{\gamma_2}\Big)^{-n}\Big(1-\frac{\gamma_2+w}{\gamma_3+\gamma_2+w}\Big)^{n-r}\, {}_2F_1\Big[-n,\ 1+n-r,\ 1-n,\ \frac{\gamma_2+w}{\gamma_3+\gamma_2+w}\Big]\Bigg)\Bigg)dw\Bigg) \\
\text{for } 0<t_2<t_1,
\end{cases} \tag{3}
$$

where the hypergeometric function ${}_2F_1(\eta,\ \xi;\ \zeta;\ z) = \sum_{k=0}^{\infty} z^k\Big((\eta)_k(\xi)_k/(\zeta)_k\Big)/k!$ and $(\eta)_k = \Gamma(\eta+k)/\Gamma(\eta)$.
This reliability function can be readily evaluated for a given set of data.

Predictive moment generating function

The moment generating function for the future responses w and z is given by

$$
M_p(t_1,t_2) = \int_{z=0}^{+\infty}\int_{w=0}^{+\infty} \exp\{(wt_1+zt_2)\}\, p(w,z|\mathbf{x},\mathbf{y})dwdz\ .
$$

Let I_1 and I_2 be the moment generating function given in terms of the representation of the joint predictive density (2) corresponding respectively to the first and second regions.
For $0<w<z$,

$$
\begin{aligned}
I_1 &= \frac{r^2}{\gamma_1\gamma_2}\int_{w=0}^{+\infty}\int_{z=w}^{+\infty}\exp\left\{(wt_1+zt_2)\right\}p(w,z|\mathbf{x},\mathbf{y})dzdw \\
&= \frac{r^2}{\gamma_1\gamma_2}\int_{w=0}^{+\infty}\exp\left\{wt_1\right\}\left[1+\frac{w}{\gamma_2}\right]^{-(n+1)}\int_{z=w}^{+\infty}\exp\left\{zt_2\right\}\left[1+\frac{z-w}{\gamma_1}\right]^{-(r+1)}dzdw,
\end{aligned}
$$

and for $0<z<w$,

$$
\begin{aligned}
I_2 &= \frac{(n-r)^2}{\gamma_2\gamma_3}\int_{z=0}^{+\infty}\int_{w=z}^{+\infty}\exp\left\{(wt_1+zt_2)\right\}p(w,z|\mathbf{x},\mathbf{y})dwdz \\
&= \frac{(n-r)^2}{\gamma_2\gamma_3}\int_{z=0}^{+\infty}\exp\left\{zt_2\right\}\left[1+\frac{z}{\gamma_2}\right]^{-(n+1)}\int_{w=z}^{+\infty}\exp\left\{wt_1\right\}\left[1+\frac{w-z}{\gamma_3}\right]^{-(n-r+1)}dwdz.
\end{aligned}
$$

For a given set of data, the mean and the second moment of the future failure times, w and z, are obtained by differentiating I_1 and I_2 once and twice, and evaluating the derivatives at $t_1=t_2=0$.

Predictive hazard rate function

In this case, the predictive hazard rate function is defined as the ratio of the probability of the failure rate at times t_1 and t_2 of a system denoted by $f_p(t_1,t_2)$, where $f_p(.\,,\,.)$ is defined in (2) to the probability that the system has not yet failed at times t_1 and t_2, and is denoted by $R_p(t_1,t_2)$ as given in (3), that is,

$$
H_p(t_1,t_2)=f_p(t_1,t_2)/R_p(t_1,t_2).
$$

After some simplifications, the predictive hazard rate function is obtained as $H_p(t_1,t_2)$

$$= \begin{cases} \left\{\frac{r^2}{\gamma_1\gamma_2}\left[1+\frac{t_1}{\gamma_2}\right]^{-(n+1)}\left[1+\frac{t_2-t_1}{\gamma_1}\right]^{-(r+1)}\right\} \Big/ \left\{\frac{1}{\gamma_1\gamma_2}\left(r^2\int_{z=t_2}^{+\infty}\left(\frac{1}{r(\gamma_2+\gamma_1+z)}\left(\gamma_1\gamma_2\left(1+\frac{z}{\gamma_2}\right)^{-n}\right.\right.\right.\right. \\ \times\left(1-\frac{\gamma_1}{\gamma_2+\gamma_1+z}\right)^{n} {}_2F_1\left[-r,\ 1+n,\ 1-r,\ \frac{\gamma_1}{\gamma_2+\gamma_1+z}\right]\Bigg) - \frac{1}{r(\gamma_2+\gamma_1+z)}\left(\gamma_1\gamma_2\left(1+\frac{t_1}{\gamma_2}\right)^{-n}\right. \\ \left.\left.\left.\left.\times\left(1+\frac{-t_1+z}{\gamma_1}\right)^{-r}\left(1-\frac{\gamma_1-t_1+z}{\gamma_2+\gamma_1+z}\right)^{n} {}_2F_1\left[-r,\ 1+n,\ 1-r,\ \frac{\gamma_1-t_1+z}{\gamma_2+\gamma_1+z}\right]\right)\right)dz\right)\right\} \\ \text{for } 0<t_1<t_2, \\ \left\{\frac{(n-r)^2}{\gamma_2\gamma_3}\left[1+\frac{t_2}{\gamma_2}\right]^{-(n+1)}\left[1+\frac{t_1-t_2}{\gamma_3}\right]^{-(n-r+1)}\right\} \Big/ \left\{\frac{1}{\gamma_2\gamma_3}\left((n-r)^2\int_{w=t_1}^{+\infty}\left(\frac{1}{n(\gamma_3+\gamma_2+w)}\left(\gamma_2\gamma_3\right.\right.\right.\right. \\ \times(1+\frac{t_2}{\gamma_3})^{-n}\left(1-\frac{\gamma_2+t_2}{\gamma_3+\gamma_2+w}\right)^{n-r}\left(1+\frac{-t_2+w}{\gamma_3}\right)^{-n+r} {}_2F_1\left[-n,\ 1+n-r,\ 1-n,\ \frac{\gamma_2+t_2}{\gamma_3+\gamma_2+w}\right]\Bigg) \\ -\frac{1}{n(\gamma_3+\gamma_2+w)}\left(\gamma_2\gamma_3\left(1+\frac{w}{\gamma_2}\right)^{-n}\left(1-\frac{\gamma_2+w}{\gamma_3+\gamma_2+w}\right)^{n-r}\right. \\ \left.\left.\left.\left.\times\ {}_2F_1\left[-n,\ 1+n-r,\ 1-n,\ \frac{\gamma_2+w}{\gamma_3+\gamma_2+w}\right]\right)\right)dw\right)\right\} \\ \text{for } 0<t_2<t_1. \end{cases}$$

There is no closed form representation of the hazard rate function; however, for a given set of data it can be evaluated.

2.3 Numerical example

For this example which makes use of the representation of the predictive density $p(w,z|\mathbf{x},\mathbf{y})$ derived in Section 2.1, we generated a bivariate sample of size n=10 from the marginal densities whose distribution are mixtures of exponential random variables for some specific values of the parameters, namely $\lambda = 7,\ \ \lambda' = 4,\ \ \eta = 5$, and $\eta' = 3.5$. Given this sample, it was found that $\gamma_1 = 0.217732$, $\gamma_2 = 2.006680$ and $\gamma_3 = 0.399440$. Figure 1 shows the graph of the joint predictive density of w and z. The graph in Figure 2 represents the system's reliability with respect to the failure times t_1 and t_2.

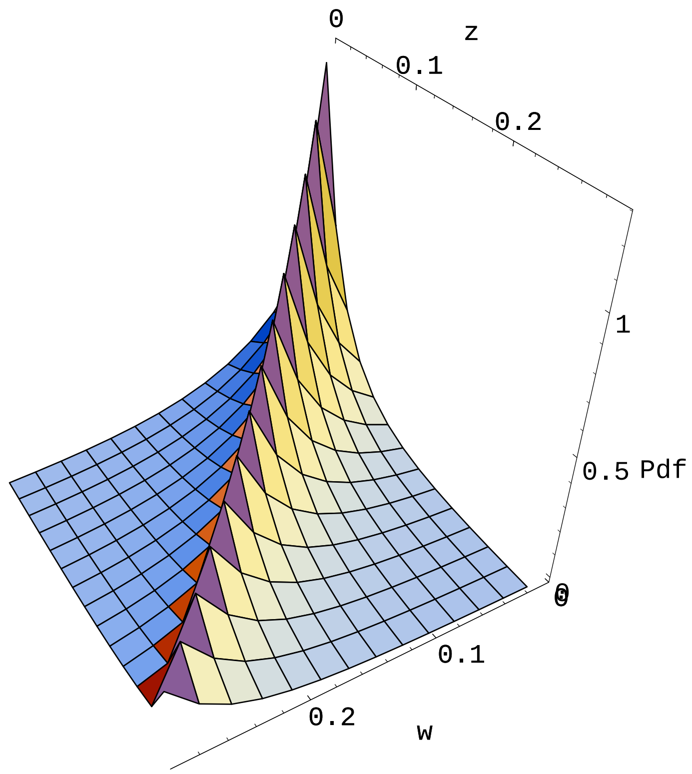

Figure 1: Predictive density of two future responses for dependent components connected in parallel

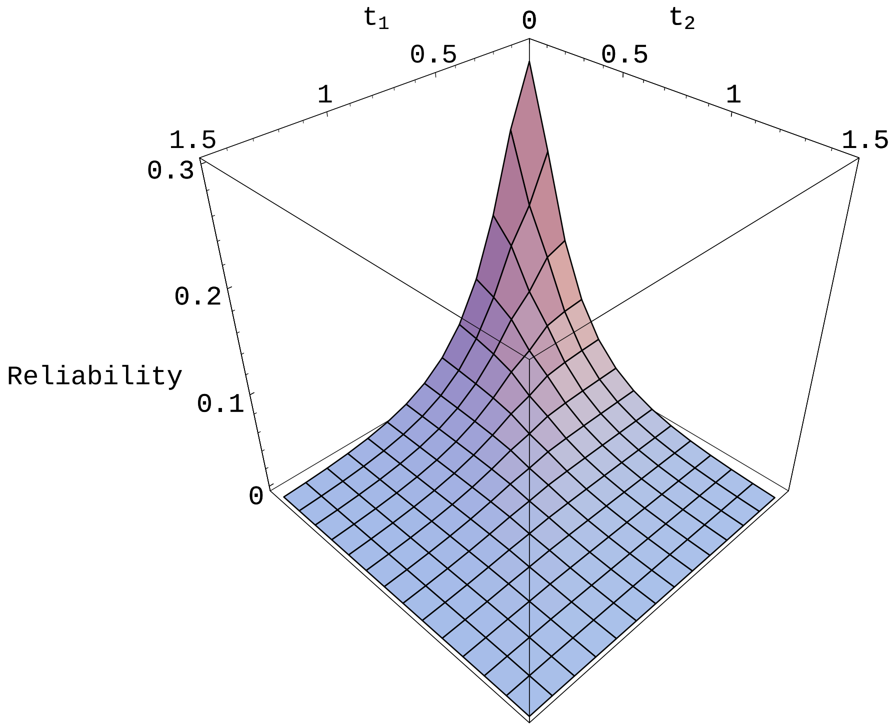

Figure 2: Predictive reliability function associated with two future responses for dependent components connected in parallel

3. COMPONENTS CONNECTED IN SERIES

Consider a system consisting of two components connected in series so that the system fails if either one of the components fails. It is assumed that the components are independently and identically distributed. Let the two failure times, x for component C_1 and y for component C_2, be independently and exponentially distributed random variables with parameters $\lambda_i,\ i = 1,\ 2$ and let $z = \min(x,\ y)$. Then

$$\Pr(z > t) = \Pr(x > t,\ y > t) \quad = \quad \exp\left\{-(\lambda_1 + \lambda_2)t\right\}.$$

Thus, z is exponentially distributed with parameter $(\lambda_1 + \lambda_2)$. Similarly, let the observed failure times for the component C_1 be $x_1 < x_2 < \cdots < x_n$, and $y_1 < y_2 < \cdots < y_n$ be the failure times for the component C_2. Furthermore, let

$$z_i = \min(x_i,\ y_i);\quad i = 1, 2, \ldots, n,$$

then $z_1, z_2, \ldots, z_n$ are independently and exponentially distributed with parameter $(\lambda_1 + \lambda_2)$.

3.1 Joint predictive density function

The likelihood function of λ_1 and λ_2 for a given data set $\mathbf{z} = z_1, z_2, \ldots, z_n$ is

$$\mathrm{L}(\lambda_1, \lambda_2|\mathbf{z}) \quad = \quad (\lambda_1 + \lambda_2)^n \exp\left\{-(\lambda_1 + \lambda_2)\sum_{i=1}^{n} z_i\right\}.$$

Following Jeffreys (1961), let a non-informative prior for the parameters λ_1 and λ_2 be

$$p(\lambda_1, \lambda_2) \propto 1/(\lambda_1\ \lambda_2)\ .$$

Then the posterior density of λ_1 and λ_2 given $\mathbf{z}$ is

$$\begin{aligned} p(\lambda_1, \lambda_2|\mathbf{z}) &= \psi(\mathbf{z})\frac{(\lambda_1 + \lambda_2)^n}{\lambda_1\lambda_2}\exp\left\{-(\lambda_1 + \lambda_2)\sum_{i=1}^{n} z_i\right\} \\ &= \psi(\mathbf{z})\sum_{j=0}^{n}\binom{n}{j}\lambda_1^{j-1}\lambda_2^{n-j-1}\exp\left\{-\lambda_1\sum_{i=1}^{n} z_i\right\}\exp\left\{-\lambda_2\sum_{i=1}^{n} z_i\right\}, \end{aligned}$$

where the normalizing constant is

$$\begin{aligned}\psi(\mathbf{z})^{-1} &= \int_{\lambda_1=0}^{+\infty}\int_{\lambda_2=0}^{+\infty}\sum_{j=0}^{n}\binom{n}{j}\lambda_1^{j-1}\lambda_2^{n-j-1}\exp\left\{-\lambda_1\sum_{i=1}^{n}z_i\right\}\exp\left\{-\lambda_2\sum_{i=1}^{n}z_i\right\}d\lambda_2 d\lambda_1\\ &= \sum_{j=0}^{n}\binom{n}{j}\frac{\Gamma(j)\Gamma(n-j)}{\left(\sum_{i=1}^{n}z_i\right)^n}\ .\end{aligned}$$

Let x_f and y_f be two future responses from the components C_1 and C_2, which follow the exponential model with parameters λ_1 and λ_2 respectively; then the predictive density of $x_f,\ y_f$ given $\mathbf{z}$ is

$$\begin{aligned}p(x_f, y_f|\mathbf{z}) &= \psi(\mathbf{z})\int_{\lambda_1=0}^{+\infty}\int_{\lambda_2=0}^{+\infty}\lambda_1\exp\{-\lambda_1 x_f\}\,\lambda_2\exp\{-\lambda_2 y_f\}\\ &\quad\times\sum_{j=0}^{n}\binom{n}{j}\lambda_1^{j-1}\lambda_2^{n-j-1}\exp\left\{-\lambda_1\sum_{i=1}^{n}z_i\right\}\exp\left\{-\lambda_2\sum_{i=1}^{n}z_i\right\}d\lambda_2 d\lambda_1\\ &= \psi(\mathbf{z})\sum_{j=0}^{n}\binom{n}{j}\frac{\Gamma(j+1)\Gamma(n-j+1)}{\left(x_f+\sum_{i=1}^{n}z_i\right)^{j+1}\left(y_f+\sum_{i=1}^{n}z_i\right)^{n-j+1}},\ \ x_f,\ y_f>0.\end{aligned}$$

For instance, n=2 we have

$$p(x_f, y_f|\mathbf{z}) = \frac{\left(\sum_{i=1}^{2}z_i\right)^2}{4}\sum_{j=0}^{2}\binom{2}{j}\frac{\Gamma(j+1)\Gamma(3-j)}{\left(x_f+\sum_{i=1}^{2}z_i\right)^{j+1}\left(y_f+\sum_{i=1}^{2}z_i\right)^{3-j}},\ \ x_f,\ y_f>0.$$

3.2 Numerical example

For this case, we generated two samples of size 10 from an exponential distribution with parameters $\lambda_1 = 15$ and $\lambda_2 = 25$ respectively, from which a sample of the $z_i\ 's$ $(= \min(\mathrm{x_i}, \mathrm{y_i}))$ was then obtained. Figure 3 shows the predictive density for bivariate future responses in a system whose components are connected in series.

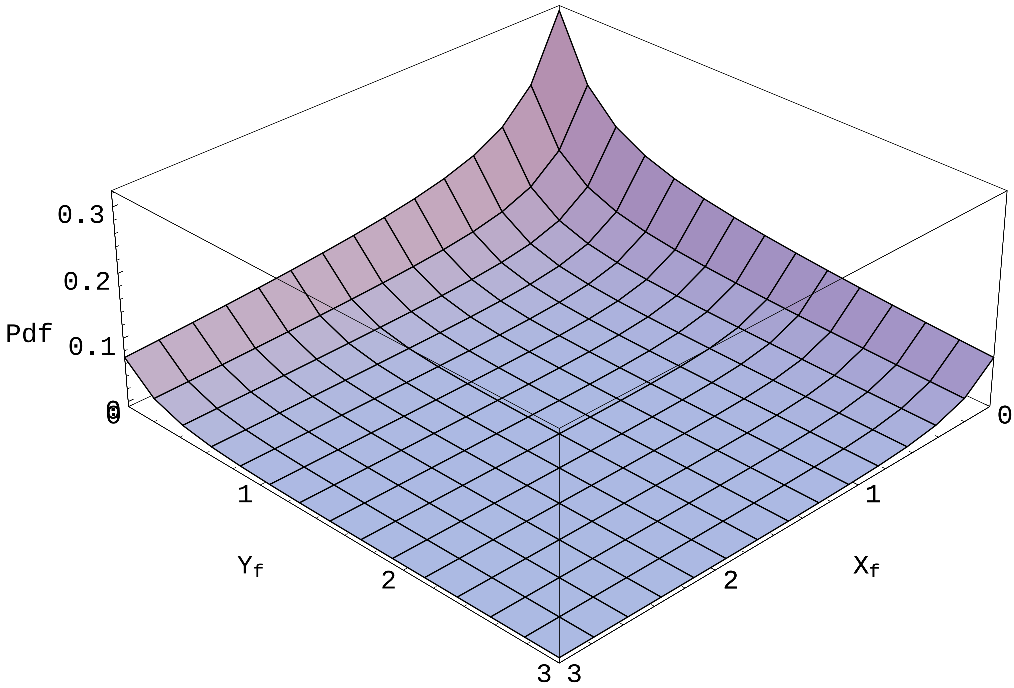

Figure 3: Predictive density of two future responses for components connected in series

4. DISCUSSION

The graph in Figure 1 for the case of components connected in parallel exhibits a diagonal ridge from which the joint density decreases rapidly. Figure 2 appears to be in conformity with the idea that a system's reliability decreases with time as the components are more likely to fail. It is seen from Figure 3 that the density function decreases exponentially from the origin when the two components are connected in series. The graph exhibits a flatter pattern which reflects the fact that the components are connected in series, in which case the failure of one response will result in the failure of the system. The calculations for each of the numerical example were carried out with symbolic computational package *Mathematica*.

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