Induced Connections on Submanifolds

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Abstract. The authors establish a relation of the theory of varieties with degenerate Gauss maps in projective spaces with the theory of congruences and pseudocongruences of subspaces and show how these two theories can be applied to the construction of induced connections on submanifolds of projective spaces and other spaces endowed with a projective structure.

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0 Introduction

The theory of congruences and pseudocongruences of subspaces of a projective space is closely related to the theory of varieties with degenerate Gauss maps.

In a three-dimensional space $\mathbb{P}^3$ as well as in three-dimensional spaces endowed with a projective structure (such as an affine, Euclidean, and non-Euclidean space), the theory of congruences was studied by many geometers. The extensive monographs on this subject were published (see, for example, the monograph [Fi 50]).

In this paper, we establish a relation of the theory of varieties with degenerate Gauss maps in projective spaces with the theory of congruences and pseudocongruences of subspaces and show how these two theories can be applied to the construction of induced connections on submanifolds of projective spaces and other spaces endowed with a projective structure.

1 Basic Equations of a Variety with a Degenerate Gauss Map.

A smooth n-dimensional variety X of a projective space $\mathbb{P}^N$ is called *tangentially degenerate* or *a variety with a degenerate Gauss map* if the rank of its Gauss map

$$\gamma : X \to \mathbb{G}(n, N)$$

is less than n, $0 \leq r = \text{rank } \gamma < n$. Here $x \in X$, $\gamma(x) = T_x(X)$, and $T_x(X)$ is the tangent subspace to X at x considered as an n-dimensional projective space $\mathbb{P}^n$. The number r is also called the *rank* of X, $r = \text{rank } X$. The case $r = 0$ is trivial one: it gives just an n-plane.

Let $X \subset \mathbb{P}^N$ be an almost everywhere smooth n-dimensional variety with a degenerate Gauss map. Suppose that $0 < \operatorname{rank} \gamma = r < n$. Denote by L a leaf of this map, $L = \gamma^{-1}(T_x) \subset X$; $\dim L = n - r = l$.

As was proved in [AG 04] (see Theorem 3.1 on p. 95), a variety with a degenerate Gauss map of rank r foliates into its plane leaves L of dimension l, along which the tangent subspace $T_x(X)$ is fixed.

The tangent subspace $T_x(X)$ is fixed when a point x moves along regular points of L. This is the reason that we denote it by T_L, $L \subset T_L$. A pair (L, T_L) on X depends on r parameters.

The foliation on X defined as indicated above is called the *Monge–Ampère foliation.*

The varieties of rank $r < n$ are multidimensional analogues of developable surfaces of a three-dimensional Euclidean space.

The main results on the geometry of varieties with degenerate Gauss maps and further references can be found in Chapter 4 of the book [AG 93] and in the recently published book [AG 04].

In this section, we find the basic equations of a variety X with a degenerate Gauss map of dimension n and rank r in a projective space $\mathbb{P}^N$.

In what follows, we will use the following ranges of indices:

$$a, b, c = 1, \ldots, l;\ p, q = l+1, \ldots, n;\ \alpha, \beta = n+1, \ldots, N.$$

A point $x \in X$ is said to be a *regular point* of the map γ and of the variety X if $\dim T_x X = \dim X = n$, and a point $x \in X$ is called *singular* if $\dim T_x X > \dim X = n$.

We associate a family of moving frames $\{A_u\}$, $u = 0, 1, \ldots, N$, with X in such a way that the point $A_0 = x$ is a regular point of X; the points A_a belong to the leaf L of the Monge–Ampère foliation passing through the point A_0; the points A_p together with the points A_0, A_a define the tangent subspace $T_L X$ to X; and the points A_α are located outside of the subspace $T_L X$.

The equations of infinitesimal displacement of the moving frame $\{A_u\}$ are

$$dA_u = \omega_u^v A_v, \quad u, v = 0, 1, \ldots, N, \tag{1}$$

where ω_u^v are 1-forms satisfying the structure equations of the projective space $\mathbb{P}^N$:

$$d\omega_u^v = \omega_u^w \wedge \omega_w^v, \quad u, v, w = 0, 1, \ldots, N. \tag{2}$$

As a result of the specialization of the moving frame mentioned above, we obtain the following equations of the variety X (see [AG 04], Section 3.1):

$$\omega_0^\alpha = 0, \tag{3}$$

$$\omega_a^\alpha = 0, \tag{4}$$

$$\omega_p^\alpha = b_{pq}^\alpha \omega^q, \ \ b_{pq}^\alpha = b_{qp}^\alpha, \tag{5}$$

$$\omega_a^p = c_{aq}^p \omega^q. \tag{6}$$

The 1-forms $\omega^q := \omega_0^q$ in these equations are basis forms of the Gauss image $\gamma(X)$ of the variety X, and the quantities b_{pq}^α form the second fundamental tensor of the variety X at the point $x = A_0$. The quantities b_{pq}^α and c_{aq}^p are related by the following equations:

$$b_{sq}^\alpha c_{ap}^s = b_{sp}^\alpha c_{aq}^s. \tag{7}$$

Equations (5) and (6) are called the *basic equations* of a variety X with a degenerate Gauss map (see [AG 04], Section 3.1).

Note that under transformations of the points A_p, the quantities c_{aq}^p are transformed as tensors with respect to the indices p and q. As to the index a, the quantities c_{aq}^p do not form a tensor with respect to this index. Nevertheless, under transformations of the points A_0 and A_a, the quantities c_{aq}^p along with the unit tensor δ_q^p are transformed as tensors. For this reason, the system of quantities c_{aq}^p is called a *quasitensor*.

Denote by B^α and C_a the $(r \times r)$-matrices of coefficients occurring in equations (5) and (6):

$$B^\alpha = (b_{pq}^\alpha), \;\; C_a = (c_{aq}^p).$$

Sometimes we will use the identity matrix $C_0 = (\delta_q^p)$ and the index $i = 0, 1, \ldots, l$, i.e., $\{i\} = \{0, a\}$. Then equations (5) and (7) can be combined and written as follows:

$$(B^\alpha C_i)^T = (B^\alpha C_i),$$

i.e., the matrices

$$H_i^\alpha = B^\alpha C_i = (b_{qs}^\alpha c_{ip}^s)$$

are symmetric.

The quadratic forms

$$\Phi^\alpha = b_{pq}^\alpha \omega^p \omega^q \tag{8}$$

are the *second fundamental forms* of the variety X at the point $x = A_0$, and the forms

$$\Phi^\alpha = b_{ps}^\alpha (\delta_q^s x^0 + c_{aq}^s x^a)\, \omega^p\, \omega^q$$

are the *second fundamental forms* of the variety X at the point $x = x^0 A_0 + x^a A_a \in L$.

Let $\{\alpha^u\}$ be the dual coframe (or tangential frame) in the space $(\mathbb{P}^N)^*$ to the frame $\{A_u\}$. Then the hyperplanes α^u of the frame $\{\alpha^u\}$ are connected with the points of the frame $\{A_u\}$ by the conditions

$$(\alpha^u, A_v) = \delta_v^u. \tag{9}$$

Conditions (9) mean that the hyperplane α^u contains all points A_v, $v \neq u$, and that the condition of normalization $(\alpha^u, A_u) = 1$ holds.

The equations

$$\xi_\beta \alpha^\beta = 0$$

defines the system of tangent hyperplanes passing through the tangent subspace $T_L X$, and this system of tangent hyperplanes defines the system of second fundamental forms

$$II = \xi_\beta b^\beta_{pq} \omega^p \omega^q \tag{10}$$

and the system of second fundamental tensors

$$\xi_\beta b^\beta_{pq}$$

of the variety X at the point $x = A_0$.

A variety X with a degenerate Gauss map is *dually nondegenerate* if the dimension of its dual variety X^* equals $N - l - 1$. By the generalized Griffiths–Harris theorem (see Theorem 3.2 and Corollary 3.3 in [AG 04], pp. 97–99), *a variety X with a degenerate Gauss map is dually nondegenerate if and only if at any smooth point $x \in X$ there is at least one nondegenerate second fundamental form in the system of second fundamental forms $\xi_\alpha b^\alpha_{pq} \omega^p \omega^q$ of X.*

In what follows, we will consider only dually nondegenerate varieties X with degenerate Gauss maps.

2 Focal Images of a Variety with a Degenerate Gauss Map.

Suppose that X is a variety with a degenerate Gauss map of dimension n and rank r in the space $\mathbb{CP}^N$. As we noted earlier, such a variety carries an r-parameter family of l-dimensional plane generators L of dimension $l = n - r$. Let $x = x^0 A_0 + x^a A_a$ be an arbitrary point of a generator L. For such a point, we have

$$dx = (dx^0 + x^0 \omega^0_0 + x^a \omega^0_a) A_0 + (dx^a + x^0 \omega^a + x^b \omega^a_b) A_a + (x^0 \omega^p + x^a \omega^p_a) A_p.$$

By (6), it follows that

$$dx \equiv (x^0 \delta^p_q + x^a c^p_{aq}) A_p \omega^q \pmod{L}. \tag{11}$$

The matrix $(J^p_q) = (x^0 \delta^p_q + x^a c^p_{aq})$ is the *Jacobi matrix* of the map $\gamma : X \rightarrow \mathbb{G}(n, N)$, and the determinant

$$J(x) = \det (J^p_q) = \det (x^0 \delta^p_q + x^a c^p_{aq})$$

of this matrix is the *Jacobian* of the map γ.

It is easy to see that $J(x) \neq 0$ at regular points and $J(x) = 0$ at singular points.

By (6) and (11), the set of singular points of a generator L of the variety X is determined by the equation

$$\det (\delta^p_q x^0 + c^p_{aq} x^a) = 0. \tag{12}$$

Hence this set is an *algebraic hypersurface of dimension $l-1$ and degree r in the generator L*. This hypersurface (in L) is called the *focus hypersurface* and is denoted F_L.

Because for $x^a = 0$ the left-hand side of equation (12) takes the form

$$\det(x^0 \delta^p_q) = (x^0)^r,$$

it follows that the point A_0 is a regular point of the generator L.

We call a tangent hyperplane $\xi = (\xi_\alpha)$ *singular* (or a *focus hyperplane*) if

$$\det(\xi_\alpha b^\alpha_{pq}) = 0, \tag{13}$$

i.e., if the rank of matrix $(\xi_\alpha b^\alpha_{pq})$ is reduced. Condition (13) is an equation of degree r with respect to the tangential coordinates ξ_α of the hyperplanes ξ containing the tangent subspace $T_L(X)$.

Because we consider only dually nondegenerate varieties with degenerate Gauss maps, *there exists at least one nondegenerate form in the system of second fundamental forms of X* (see the end of Section 1). Hence in the space $\mathbb{P}^N$, equation (13) defines an algebraic hypercone of degree r, whose vertex is the tangent subspace $T_L(X)$. This hypercone is called the *focus hypercone* and is denoted Φ_L (see [AG 93], p. 119).

Note that if a variety X is dually degenerate, then on such a variety, equations (13) are satisfied identically, and the variety X does not have focus hypercones.

The focus hypersurface $F_L \subset L$ and the focus hypercone Φ_L with vertex T_L are called the *focal images* of the variety X with a degenerate Gauss map.

3 Congruences and Pseudocongruences in a Projective Space

In a projective space $\mathbb{P}^n$, we consider a family Y of its l-dimensional subspaces L, $\dim L = l$, which depends on $r = n - l$ parameters. We assume that not more than a finite number of subspaces L passes through every point $x \in \mathbb{P}^n$. If we restrict ourselves by a small neighborhood of a subspace L, then we can assume that only one subspace $L \subset Y$ passes through a generic point $x \in L$. Such families of the space $\mathbb{P}^n$ are called the *congruences*.

The dual image for a congruence Y of l-dimensional subspaces in $\mathbb{P}^n$ is a *pseudocongruence* Y^* which is an r-parameter family of subspaces of dimension $r-1$. Every hyperplane $\xi \subset \mathbb{P}^n$ contains not more than a finite number of subspaces $L^* \subset Y^*$. However, if we consider an infinitesimally small neighborhood of the subspace L^* of the pseudocongruence Y^*, then there is only a single subspace L^* in the hyperplane ξ.

In what follows, we shall establish a relation of the theory of varieties with degenerate Gauss maps in projective spaces with the theory of congruences and pseudocongruences of subspaces and show how these two theories can be applied to the construction of induced connections on submanifolds of projective spaces and other spaces endowed with a projective structure.

So, consider in $\mathbb{P}^n$ a congruence Y of l-dimensional subspaces L. We associate with its element L a family of projective frames $\{A_0, A_1, \ldots, A_n\}$ chosen in such a way that the points $A_0, A_1, \ldots, A_l$ are located in L, and the points $A_{l+1}, \ldots, A_n$ are located outside of L. The equations of infinitesimal displacement of such frames have the form

$$\left\{\begin{array}{l} dA_i = \omega_i^j A_j + \omega_i^p A_p, \\ dA_p = \omega_p^i A_i + \omega_p^q A_q, \end{array}\right. \tag{14}$$

where $i, j = 0, 1, \ldots, l$; $p, q = l+1, \ldots, n$, and $L = A_0 \wedge A_1 \ldots \wedge A_l$ is a generator of the congruence Y in question. Because this generator depends on r parameters and is fixed, when $\omega_i^p = 0$, the forms ω_i^p are expressed linearly in terms of the differentials of these r parameters or in terms of linearly independent 1-forms θ^p—linear combinations of these differentials:

$$\omega_i^p = c_{iq}^p \theta^q. \tag{15}$$

Under admissible linear transformations of the basis forms θ^p, the matrices $C_i = (c_{iq}^p)$ are transformed according to the tensor law with respect to the indices p and q.

A point $F \in L \subset Y$ is called a *focus* of a generator L if $dF \in L$ under some condition on the basis forms θ^p. In order to find the foci, we represent them in the form $F = x^i A_i$. Then

$$dF \equiv x^i \omega_i^p A_p \pmod{L},$$

and as a result, the foci are determined by the system of equations

$$x^i \omega_i^p = 0.$$

By (15), this system takes the form

$$x^i c_{iq}^p \theta^q = 0. \tag{16}$$

This system has a nontrivial solution with respect to the forms θ^q if and only if

$$\det(x^i c_{iq}^p) = 0. \tag{17}$$

Equation (17) determines on L the focus hypersurface F_L which is an algebraic hypersurface of degree r.

Suppose that the point A_0 of our moving frame does not belong to the hypersurface F_L. Then the 1-forms ω_0^p are linearly independent, and we can take these forms as basis forms of the congruence Y. As a result, equations (15) become

$$\omega_a^p = c_{aq}^p \omega_0^q, \tag{18}$$

where $a = 1, \ldots, l$, and $c^p_{0q} = \delta^p_q$. Now equations (18) coincide with equations (6). As a result, equation (17) of the focus hypersurface F_L takes the form

$$\det(x^0 \delta^p_q + x^a c^p_{aq}) = 0. \tag{19}$$

Equation (19) coincides with equation (12) defining the foci on a plane generator L of a variety X with a degenerate Gauss map of rank r. However, unlike in Section 1, the quantities c^p_{aq} are not connected by any relations of type (7), because now there is no matrices $B^\alpha = (b^\alpha_{pq})$. Thus, the focus hypersurfaces F_L determined by equation (19) are arbitrary determinant varieties on generators L of the congruence Y in question.

In particular, if $l = 1$ and $n = r+1$, then Y becomes a rectilinear congruence. Equation (19) defining the focus hypersurfaces F_L of such a congruence becomes

$$\det(x^0 \delta^p_q + x^1 c^p_{1q}) = 0. \tag{20}$$

Hence, each of the focus hypersurfaces F_L of Y decomposes into r real or complex points if each is counted as many times as its multiplicity. Each of these points describes a *focal variety* in $\mathbb{P}^n$ tangent to the generators L of the congruence Y.

Next, we consider a pseudocongruence Y^* in the space $\mathbb{P}^n$. Its generator L^* is of dimension $r-1$ and depends on r parameters. We place the points A_p, $p = l+1, \ldots, n$, $l = n-r$, of our moving frame into the generator $L^* \subset Y^*$ and place the points A_i, $i = 0, 1, \ldots, l$, outside of L^*. The equations of infinitesimal displacement of such frames again have the form (14) but now the 1-forms ω^i_p are linear combinations of the basis forms θ^p defining a displacement of the generator $L^* = A_{l+1} \wedge \ldots \wedge A_n$. So now we have

$$\omega^i_p = b^i_{pq} \theta^q \tag{21}$$

and

$$dA_p = \omega^q_p A_q + b^i_{pq} \theta^q A_i. \tag{22}$$

Consider a hyperplane ξ passing through the generator $L^* \subset Y^*$. Relative to our moving frame, the equation of ξ is $\xi_i x^i = 0$, where ξ_i are tangential coordinates of the hyperplane ξ. The hyperplane ξ, which in addition to the generator L^* contains also a near generator ${}'L^*$ determined by the points A_p and dA_p, is called the *focus hyperplane*. By (22), the conditions defining the focus hyperplane are

$$\xi_i b^i_{pq} \theta^q = 0. \tag{23}$$

The system of equations (23) defines a displacement of the generator L^* if and only if this system has a nontrivial solution with respect to the forms θ^q. The necessary and sufficient condition for existence of such a nontrivial solution is the vanishing of the determinant of system (23):

$$\det(\xi_i b^i_{pq}) = 0. \tag{24}$$

Equation (24) defines the family of focus hyperplanes passing through the generator $L^* \subset Y^*$. This family is an algebraic hypercone of degree r whose vertex is the generator L^*. Note that equation (24) is similar to equation (13) of the focus hypercone Φ_L of a variety with a degenerate Gauss map.

4 Normalized Varieties in a Multidimensional Projective Space

1. Consider a smooth r-dimensional variety X in a projective space $\mathbb{P}^n$, $r < n$. The differential geometry on such a variety is rather poor. It is less rich than the differential geometry on varieties of the Euclidean space $\mathbb{E}^n$ or the spaces of constant curvature $\mathbb{S}^n$ and $\mathbb{H}^n$, where by $\mathbb{S}^n$ and $\mathbb{H}^n$ we denote, respectively, an n-dimensional elliptic and hyperbolic space. With a first-order neighborhood of a point $x \in X \subset \mathbb{P}^n$, only the tangent subspace $T_x(X)$ is associated. For example, in Section 1.4 of the book [AG 04], the authors showed that in order to enrich the differential geometry of a curve in the projective plane $\mathbb{P}^2$, it is necessary to use differential prolongations of rather higher orders of the curve equations.

However, we can enrich the differential geometry of $X \subset \mathbb{P}^n$ if we endow X with an additional construction consisting of a subspace $N_x(X)$ of dimension $n - r$ such that $T_x(X) \cap N_x(X) = x$, and an $(r-1)$-dimensional subspace $K_x(X)$, $K_x(X) \subset T_x(X)$, $x \notin K_x(X)$. We shall denote these subspaces simply by N_x and K_x and call the *normals of the first and second kind* (or simply the *first* and *second normals*) of the variety X, respectively (see [N 76], p. 198). The family of first normals forms a *congruence* N, and the family of second normals forms a *pseudocongruence* K in the space $\mathbb{P}^n$. If at any point of $x \in X$, there are assigned a single first normal N_x and a single second normal K_x, then the variety X is called *normalizated* (cf. [N 76], p. 198, and [AG 93], Chapter 6).

As we will see below, on varieties of the Euclidean space $\mathbb{E}^n$ and the non-Euclidean spaces $\mathbb{S}^n$ and $\mathbb{H}^n$, the first and second normals are determined by the geometry of these spaces while on varieties of the affine space $\mathbb{A}^n$ and the projective space $\mathbb{P}^n$, these normals should be assigned artificially, or to find them, one should use higher order neighborhoods of a point $x \in X$. In this section, we shall apply the first method. Note that the second method is connected with great computational difficulties. One can find more details on this method and a related bibliography in the books [AG 93], Chapters 6, 7, and [N 76], Chapter 5.

Thus, we now consider a normalized variety X of dimension r, $r = \dim X$, in the projective space $\mathbb{P}^n$. We associate with X a family of projective frames $\{A_0, A_1, \ldots, A_n\}$ in such a way that $A_0 = x; A_a \in N_x$, $a = 1, \ldots, l$, where $l = n - r$; and $A_p \in K_x$, $p = l+1, \ldots, n$. The equations of infinitesimal displacement of these frames have the form

$$\left\{\begin{array}{ll} dA_0 = \omega_0^0 A_0 & +\omega^p A_p, \\ dA_a = \omega_a^0 A_0 + \omega_a^b A_b & +\omega_a^p A_p, \\ dA_p = \omega_p^0 A_0 + \omega_p^a A_a & +\omega_p^q A_q, \end{array}\right. \tag{25}$$

Equations (25) show that for the family of moving frames in question, the system of differential equations

$$\omega^a = 0 \tag{26}$$

is satisfied, and the 1-forms ω^p are basis forms, because they determine a displacement of the point $A_0 = x$ along the variety X. Exterior differentiation of equations (26) and application of Cartan's lemma lead to the following equations:

$$\omega^a_p = b^a_{pq}\omega^q, \;\; b^a_{pq} = b^a_{qp} \tag{27}$$

The quantities b^a_{pq} form a tensor and are coefficients of the second fundamental forms of the variety X at the point x (see [AG 04], Section 2.1):

$$\Phi^a = b^a_{pq}\omega^p\omega^q. \tag{28}$$

2. The points A_p belong to the tangent subspace $T_x(X)$. We assume that these points belong to the second normal $K_x \subset T_x(X)$, $K_x = A_{l+1} \wedge \ldots \wedge A_n$. Then, for $\omega^p = 0$, the 1-forms ω^0_p must also vanish, and as a result, we have

$$\omega^0_p = l_{pq}\omega^q. \tag{29}$$

Next, we place the points A_a of our moving frame into the first normal N_x of X, $N_x = A_0 \wedge A_1 \wedge \ldots \wedge A_l$. Then, for $\omega^p = 0$, we obtain that $\omega^p_a = 0$, and hence

$$\omega^p_a = c^p_{aq}\omega^q. \tag{30}$$

Consider a point $y \in N_x$ on the first normal. For this point, we have $y = y^0 A_0 + y^a A_a$. Differentiating this point by means of (25), we find that

$$dy = (dy^0 + y^0\omega^0_0 + y^a\omega^0_a)A_0 + (y^0\omega^p + y^a\omega^p_a)A_p + (dy^a + y^b\omega^a_b)A_a. \tag{31}$$

A point y is a *focus* of the first normal N_x if $dy \in N_x$. By (31), this condition implies that

$$y^0\omega^p + y^a\omega^p_a = 0.$$

Applying relations (30), we find that

$$(y^0\delta^p_q + y^a c^p_{aq})\omega^q = 0.$$

This system has a nontrivial solution with respect to the forms ω^q if and only if

$$\det(y^0\delta^p_q + y^a c^p_{aq}) = 0. \tag{32}$$

Equation (32) differs from equation (19) only in notation, and it defines the focus hypersurface F_x in the generator N_x of the congruence of first normals associated with the variety X. It follows from equation (32) that the point $x \in X$, whose coordinates are $y^0 = 1$, $y^a = 0$, does not belong to the focus hypersurface F_x.

Let us find the focus hypercones Φ_x of the pseudocongruence K of second normals of X. The hypercones Φ_x are formed by the hyperplanes ξ of the space $\mathbb{P}^n$ containing the second normal $K_x = A_{l+1} \wedge \ldots \wedge A_n \subset T_x(X)$ and its neighboring normal $K_x + dK_x$, which contains not only the points A_p but also the points

$$dA_p \equiv \omega_p^0 A_0 + \omega_p^a A_a \pmod{N_x}.$$

As a result, tangential coordinates ξ_0 and ξ_a of such a hyperplane satisfy the equations

$$\xi_0 \omega_p^0 + \xi_a \omega_p^a = 0.$$

By (29) and (27), it follows from this equation that

$$(\xi_0 l_{pq} + \xi_a b_{pq}^a)\,\omega^q = 0.$$

This system has a nontrivial solution with respect to the forms ω^q if and only if its determinant vanishes,

$$\det(\xi_0 l_{pq} + \xi_a b_{pq}^a) = 0. \tag{33}$$

Equation (33) determines an algebraic hypercone of order r whose vertex is the generator K_x of the pseudocongruence K of the second normals. This hypercone is called the *focus hypercone* of the pseudocongruence K.

3. Next, we consider the tangent and normal bundles associated with a normalized variety X. The base of both bundles is the variety X itself, the fibers of the tangent bundle are the tangent subspaces T_x, and the fibers of the normal bundle are the second normals N_x,

Suppose that $'x = x + x^p A_p$ is an arbitrary point in the tangent subspace T_x, and $\boldsymbol{x} = {}'x - x = x^p A_p$ is a vector in the tangent bundle TX. The differential of this vector has the form

$$d\boldsymbol{x} = (dx^p + x^q \omega_q^p) A_p + x^p (l_{pq} A_0 + b_{pq}^a A_a)\omega^q. \tag{34}$$

The first term on the right-hand side of (34) belongs to the tangent subspace T_x, and the second term belongs to N_x. The 1-form

$$Dx^p = dx^p + x^q \omega_q^p$$

is called the *covariant differential* of the vector $\boldsymbol{x} = (x^p)$ in the connection γ_t, and the 1-forms ω_q^p are the components of the *connection form* $\omega = \{\omega_q^p\}$ of the affine connection γ_t induced on the variety X by a normalization of (N, K).

The vector field $\boldsymbol{x}$ is called *parallel* in the connection γ_t if its covariant differential DX^p vanishes, i.e, if

$$Dx^p = dx^p + x^q \omega_q^p = 0. \tag{35}$$

We find the exterior differentials of the components ω_q^p of the connection form ω. By (27), (29), and (30), these exterior differentials have the form

$$d\omega_q^p = \omega_q^s \wedge \omega_s^p + (l_{qs}\delta_t^p + b_{qs}^a c_{at}^p)\omega^s \wedge \omega^t. \tag{36}$$

The 2-form

$$\Omega_q^p = d\omega_q^p - \omega_q^s \wedge \omega_s^p$$

is said to be *the curvature form* of the affine connection γ_t induced on the variety X. From equation (36) it follows that

$$\Omega_q^p = \frac{1}{2} R_{qst}^p \omega^s \wedge \omega^t, \tag{37}$$

where

$$R_{qst}^p = l_{qs}\delta_t^p + b_{qs}^a c_{at}^p - l_{qt}\delta_s^p - b_{qt}^a c_{as}^p \tag{38}$$

is the *curvature tensor* of the affine connection γ_t on X. Equations (38) allow us to compute the curvature tensor for different normalizations of the variety X.

If $R_{qst}^p = 0$ on the variety X, then the affine connection γ_t on X is *flat*, and a parallel translation of a vector $\boldsymbol{x}$ does not depend on the path of integration (see, for example, [N 76], p. 118, or [KN 76], p. 70).

4. Further, we consider a vector field $\boldsymbol{y}$ in the normal bundle $N(X)$. This vector is determined by the point x and a point $y = y^0 A_0 + y^a A_a$ of the fiber $N_x \subset N(X)$. The differential of the point y is defined by equation (31).

The 1-form

$$Dy^a = dy^a + y^b \omega_b^a \tag{39}$$

is called the *covariant differential* of the vector field $\boldsymbol{y}$ in the normal bundle $N(X)$, and the forms ω_a^b are the components of the *connection form of the normal connection* γ_n on a normalized variety X (see, for example, [Ca 01], p. 242; see more on the normal connection in [AG 95] and Section 6.3 of the book [AG 93]). The 2-form

$$\Omega_b^a = d\omega_b^a - \omega_b^c \wedge \omega_c^a$$

is called the *curvature form* of the normal connection γ_n. Note that Cartan in [Ca 01] called this form the *Gaussian torsion* of an embedded variety X.

Differentiating the forms ω_b^a and applying formulas (27) and (30), we find the expression of the curvature form Ω_b^a:

$$\Omega_b^a = \frac{1}{2} R_{bst}^a \omega^s \wedge \omega^t, \tag{40}$$

where

$$R_{bst}^a = c_{bs}^p b_{pt}^a - c_{bt}^p b_{ps}^a. \tag{41}$$

The tensor R_{bst}^a is called the *tensor of normal curvature* of the variety X.

The second normals K_x associated with the variety X allow us to find a distribution Δ_y of r-dimensional subspaces associated with X. The elements of the distribution Δ_y are linear spans of the points $y \in N_x$ and the second normals K_x, $\Delta_y = y \wedge K_x$. By (31), the distribution Δ_y is determined by the system of equations

$$dy^a + y^b \omega_b^a = 0. \tag{42}$$

In the general case, the system of equations (42) is not completely integrable, and when a point x moves along a closed contour $l \subset X$, the corresponding point y does not describe a closed contour.

But the point y describes a closed contour l' if system (42) is completely integrable. The condition of complete integrability of (42) is the vanishing of the tensor of normal curvature (41) of the variety X. In this case, the distribution Δ_y defined by system (42) is completely integrable, and the closed contours l' lie on integral varieties of this distribution. These integral varieties form an $(n-r)$-parameter family of r-dimensional subvarieties $X(y)$ which are "parallel" to the variety X in the sense that the subspaces $T_x(X)$ and $T_x(X(y))$ pass through the same second normal K_x.

5. A normalization of a variety $X \subset \mathbb{P}^n$ is called *central* if all its first normals N_x form a bundle with an $(l-1)$-dimensional vertex S.

The following theorem gives necessary and sufficient conditions for a normalization of a variety X to be central.

Theorem 1. *A normalization of a normalized variety $X \subset \mathbb{P}^n$ is central if and only if the quantities l_{pq} and c^p_{aq} in equations* (29) *and* (30) *vanish:*

$$l_{pq} = 0, \;\; c^p_{aq} = 0.$$

Proof. Necessity: Suppose that a normalization of a variety $X \subset \mathbb{P}^n$ is central with an $(l-1)$-dimensional vertex S. If we place the points A_a into this vertex S, then we get

$$dA_a = \omega^b_a A_b.$$

By (25), this implies that

$$\omega^0_a = 0, \;\; \omega^p_a = 0.$$

By (29) and (30), this means that

$$l_{pq} = 0, \;\; c^p_{aq} = 0.$$

Sufficiency: If $l_{pq} = 0$, $c^p_{aq} = 0$, then $\omega^0_a = 0$, $\omega^p_a = 0$, and $dA_a = \omega^b_a A_b$, (i.e., the subspace $S = A_{l+1} \wedge \cdots \wedge A_n$ is fixed), then all first l-dimensional first normals N_x pass through S, and a normalization of X is central with the $(l-1)$-dimensional vertex S. □

Corollary 2. *The induced affine connection γ_t and the normal connection γ_n of a centrally normalized variety $X \subset \mathbb{P}^n$ are flat.*

Proof. Because for a centrally normalized variety X, we have $l_{pq} = 0$, $c^p_{aq} = 0$, and the curvature tensor of the induced affine connection γ_t has the form (38), it follows that

$$R^p_{qst} = 0,$$

i.e., the curvature tensor of the induced affine connection γ_t of a centrally normalized variety vanishes.

In the same way, it follows from (41) that the tensor of the normal connection γ_n of a centrally normalized variety also vanishes.

Note that both results also follow from the fact that a centrally normalized variety $X \subset \mathbb{P}^n$ can be bijectively projected onto an r-dimensional subspace T

that is complementary to the vertex S of the bundle of first normals N_x, and the geometry on the variety X induced by this central normalization is equivalent to the plane geometry in the subspace T. □

Atanasyan [A 52] found necessary and sufficient conditions for normalization of a variety X in an affine space $\mathbb{A}^N$ to be central and trivial. The trivial normalization of X in $\mathbb{A}^N$ is a normalization for which all first normals N_x of X are parallel to some constant l-plane (i.e., they form a bundle of parallel l-planes).

In our notations, his conditions for a normalization to be central are

$$c^p_{aq} = \delta^p_q c_a,$$

where δ^p_q is the Kronecker delta, and c_a are $(1,0)$-tensors, and the conditions for a normalization to be trivial are

$$c^p_{aq} = 0.$$

But for an affine space (and in particular, for a Euclidean space) we always have $l_{pq} = 0$. In addition, in the projective setting (as well as in the affine setting), a trivial normalization is a central normalization whose vertex is an $(l-1)$-plane at infinity. Therefore, Atanasyan's results follow from Theorem 1.

6. Consider the normalization dual to a central normalization. For such a normalization all second normals K_x belong to a fixed hyperplane α. We will call such a normalization *affine*.

Theorem 3. *A normalization of a variety $X \subset \mathbb{P}^n$ is affine if and only if the* 1*-forms ω^0_p and ω^0_a occurring in equations* (25) *vanish,*

$$\omega^0_p = 0, \;\; \omega^0_a = 0.$$

If a normalization of variety $X \subset \mathbb{P}^n$ is affine, then the space $\mathbb{P}^n$ carries an affine structure, i.e., $\mathbb{P}^n$ is an affine space $\mathbb{A}^n$

Proof. We place the points $A_1, \dots, A_n$ of our moving frame into the fixed hyperplane α. Since for an affine normalization $K_x \subset \alpha$, and hence $dA_p \subset \alpha$, $p = l+1, \dots, n$, it follows that

$$\omega^0_p = 0.$$

Moreover, the points A_a, $a = 1, \dots, l$, of the first normal N_x can be also placed into the hyperplane α. Then $dA_a \subset \alpha$, and as a result, we have

$$\omega^0_a = 0.$$

Conversely, if $\omega^0_p = 0, \omega^0_a = 0$, then

$$dA_p \subset \alpha, \;\; dA_a \subset \alpha,$$

where $\alpha = A_1 \wedge \dots \wedge A_n$. Hence the plane α is fixed, and the normalization of X is affine. This proves the first part of Theorem 3.

To prove the second part of Theorem 3, note that we can take the hyperplane α as the hyperplane at infinity $\mathbb{P}^{n-1}_\infty$ of the space $\mathbb{P}^n$. Hence, this hyperplane defines an affine structure of the space $\mathbb{P}^n$. Thus, the space $\mathbb{P}^n$ become an affine space $\mathbb{A}^n$. $\square$

7. Now suppose that a normalized variety $X \subset \mathbb{P}^n$ has a flat normal connection γ_n, i.e., $R^a_{bst} = 0$. By (41), these conditions lead to the relation

$$b^a_{pt} c^p_{bs} = b^a_{ps} c^p_{bt}. \tag{43}$$

Relations (43) differ from relations (7) only in notation. If we introduce the matrix notations

$$B^a = (b^a_{pq}), \;\; C_b = (c^p_{bq}),$$

then relations (43) take the form

$$(B^a C_b) = (B^a C_b)^T. \tag{44}$$

We proved in Chapters 3 and 4 of [AG 04] that these relations imply that the matrices B^a and C_b can be simultaneously reduced to a diagonal form or a block diagonal form. Thus, we have proved the following result.

Theorem 4. *The focus hypersurfaces $F_x \subset N_x$ of a normalized variety X with a flat normal connection decompose into the plane generators of different dimensions.*

This property of the varieties X with a flat normal connection γ_n allows us to construct a classification of such varieties in the same way as this was done for the varieties with degenerate Gauss maps in a projective space. For varieties in an affine space and a Euclidean space, such a classification was outlined in the papers [ACh 75, 76, 01].

5 Normalization of Varieties in Affine and Euclidean Spaces

1. An affine space $\mathbb{A}^n$ differs from a projective space $\mathbb{P}^n$ by the fact that in $\mathbb{A}^n$ a hyperplane at infinity $\mathbb{P}_\infty$ is fixed. If we place the points A_i, $i = 1, \ldots, n$, of our moving projective frame into this hyperplane, then the equations of infinitesimal displacement of the moving frame take the following form (see equations (1.81) in AG 04]):

$$\left\{ \begin{array}{ll} dA_0 = \omega_0^0 A_0 + & \omega_0^i A_i, \\ dA_i = & \omega_i^j A_j, \;\; i, j = 1, \ldots, n, \end{array} \right. \tag{45}$$

and the structure equations of the affine space $\mathbb{A}^n$ take the form

$$d\omega_0^0 = 0, \;\; d\omega_0^i = \omega_0^j \wedge \omega_j^i, \;\; d\omega_j^i = \omega_j^k \wedge \omega_k^i. \tag{46}$$

Consider a variety X of dimension r in the affine space $\mathbb{A}^n$. The tangent space $T_x(X)$ intersects the hyperplane at infinity $\mathbb{P}_\infty$ in a subspace K_x of dimension $r-1$, $K_x = T_x \cap \mathbb{P}_\infty$. Thus, for a normalization of X, it is sufficient to assign only a family of first normals N_x. If we place the points A_a, $a = 1, \ldots, l$, of our moving frame into the subspace $N_x \cap \mathbb{P}_\infty$, and the points A_p, $p = l+1, \ldots, n$, into the subspace K_x, then equations (45) take the form

$$\begin{cases} dA_0 = \omega_0^0 A_0 + & & +\omega_0^p A_p, \\ dA_a = & \omega_a^b A_b & +\omega_a^p A_p, \\ dA_p = & \omega_p^a A_a & +\omega_p^q A_q \end{cases} \tag{47}$$

(cf. equations (25)).

As was in the projective space, we have equations (27),

$$\omega_p^a = b_{pq}^a \omega^q, \;\; b_{pq}^a = b_{qp}^a, \tag{48}$$

where b_{pq}^a is the second fundamental tensor of the variety X. Equations (30) also preserve their form:

$$\omega_a^p = c_{aq}^p \omega^q, \tag{49}$$

but equations (29) become

$$\omega_p^0 = 0. \tag{50}$$

As a result, because $l_{pq} = 0$, the equation of the focus hypersurface $F_x \subset N_x$ preserves its form (32):

$$\det(y^0 \delta_q^p + y^a c_{aq}^p) = 0. \tag{51}$$

As to equation (33) of the focus hypercone Φ_x, by (50), this equation takes the form

$$\det(\xi_a b_{pq}^a) = 0. \tag{52}$$

Expressions (38) for the components of the curvature tensor of the affine connection γ_t induced on the normalized variety $X \subset \mathbb{A}^n$ take now the form

$$R_{qst}^p = b_{qs}^a c_{at}^p - b_{qt}^a c_{as}^p, \tag{53}$$

and the expression (41) for the components of the tensor of normal curvature of the variety X preserves its form:

$$R_{bst}^a = b_{pt}^a c_{bs}^p - b_{ps}^a c_{bt}^p. \tag{54}$$

Consider the tensor R_{st} obtained from the curvature tensor R_{qst}^p of the affine connection γ_t with respect to the indices p and q. This tensor is called the *Ricci tensor of the connection* γ_t. It follows from (53) that

$$R_{st} = b_{ps}^a c_{at}^p - b_{pt}^a c_{as}^p.$$

Similarly we can define the *Ricci tensor of the normal connection* γ_n. We denote it by $\widetilde{R}_{st}$. It follows from (54) that

$$\widetilde{R}_{st} = b_{pt}^a c_{as}^p - b_{ps}^a c_{at}^p.$$

Comparing the last two equations, we see that

$$R_{st} = -\widetilde{R}_{st}.$$

Hence, the following theorem is valid.

Theorem 5. *On a variety $X \subset \mathbb{A}^n$ endowed with an affine normalization, the Ricci tensors of the connections γ_t and γ_n are equal in absolute value but opposite in sign.*

Note that if a normalized variety X is a hypersurface in the space $\mathbb{A}^n$, then the following theorem is valid.

Theorem 6. *If on a normalized variety $X \subset \mathbb{A}^n$ the induced affine connection γ_t is flat, then the normal connection γ_n is also flat.*

Proof. In fact, for a hypersurface X we have the following ranges of the indices:

$$a, b = 1; \;\; p, q, s, t = 2, \ldots, n.$$

Hence the curvature tensor of the normal connection γ_n has the components R^1_{1st}. Now it follows from Theorem 5 that

$$R^1_{1st} = -R^p_{pst}.$$

But if the connection γ_t is flat, then $R^p_{qst} = 0$ and hence $R^p_{pst} = 0$. As a result, we have $R^1_{1st} = 0$, and the connection γ_n is also flat. $\square$

As was the case in the projective space, the vanishing of tensor of normal curvature R^a_{bst} is equivalent to the complete integrability of the system defining the distribution $\Delta_y = y \wedge K_x$, where $y \in N_x$. But in the affine space, the elements Δ_y of this distribution are parallel to the subspace $T_x(X)$.

Thus, we have proved the following result.

Theorem 7. *A variety $X \subset \mathbb{A}^n$ has a flat normal connection γ_n if and only if this variety admits an l-parameter family of parallel varieties $X(y)$, where $y \in N_x$.*

2. Another relation of the theory of varieties with degenerate Gauss maps and the theory of normalized varieties was established in Theorem 4 of the paper [Cha 78] by Chakmazyan (see also p. 39 of his book [Cha 90]).

We will now present this theorem.

Suppose that at an arbitrary point x of a normalized variety X, an s-dimensional direction $\nu^s(x)$ (i.e., an a s-dimensional plane passing through x) belonging to the virst normal $N_x(X)$ is given. This means we have a smooth field of s-dimensional directions $\nu^s(x)$ on X, where $s \leq l = n - r$. This field determines a normal subbundle $\nu^s(X)$, whose s-dimensional fibres are s-dimensional centroprojective spaces.

The plane $N^s(x)$ of this field corresponding to an arbitrary point $x \in X$, can be defined by the point x and points B_f given by

$$B_f = \xi_f^a A_a \in N_x, \tag{55}$$

where $f, g, h = 1, \ldots, s$.

In addition, the plane $N^s(x)$ must be invariant with respect to admissible transformations of the moving frame in $N_x(X)$. The necessary and sufficient conditions for this invariance are

$$dB_f = \theta_f^g B_g + \theta_f^0 A_0 \pmod{\omega^p}, \tag{56}$$

where θ_f^g and θ_f^0 are linearly independent 1-forms whose structure can be obtained by taking exterior derivatives of (56). Since we do not need these conditions, we will not derive them.

A field ν^s is called *parallel* with respect to the normal connection γ_n if under any infinitesimal displacement of an arbitrary point $x \in X$, the displacement of the s-dimensional direction $\nu^s(x)$ remains in the $(r+s)$-dimensional plane passing through the tangent subspace $T_x(X)$, $x \in X$, and the direction $\nu^s(x)$.

Let us find analytic conditions for a field ν^s to be parallel. Any direction belonging to an s-dimensional element of the field ν^s is determined by the point $A_0 = x$ and

$$A = A_0 + \xi^f B_f, \tag{57}$$

where B_f are defined by (55).

Taking exterior derivative of (57) and applying (25), we find that

$$dA = (\omega_0^0 + \xi^f \xi_f^a \omega_a^0) A_0 + d\xi^f B_f + (\omega^p + \xi^f \xi_f^a \omega_a^p) A_p + \xi^f (d\xi_f^a + \xi_f^b \omega_b^a) A_a. \tag{58}$$

The field ν^s is parallel with respect to the normal connection γ_n if and only if

$$(d\xi_f^a + \xi_f^b \omega_b^a) A_a = \theta_f^g B_g + \theta_f^0 A_0.$$

This and (55) implies that

$$d\xi_f^a + \xi_f^b \omega_b^a = \theta_f^g \xi_g^a. \tag{59}$$

Theorem 8. *A field ν^s of s-dimensional normal directions $\nu^s(x)$ on a normalized variety $X \subset \mathbb{P}^n$ is parallel with respect to the normal connection γ_n if and only if all the planes $N^s(x)$, $x \in X$, form a variety V_r^{r+s} with a degenerate Gauss map of rank r with s-dimensional plane generators in the space $\mathbb{P}^n$.*

Proof. Necessity: Suppose that a normalized variety $X \subset \mathbb{A}^n$ admits a parallel field ν^s of s-dimensional normal directions $\nu^s(x)$. Consider the variety V_r^{r+s} formed by the planes $N^s(x)$ of this variety. An arbitrary point of a plane $N^s(x)$ of this variety can be written as

$$A = A_0 + \xi^f B_f.$$

By (59), we find that

$$dA = (\omega_0^0 + \xi^f \xi_f^a \omega_a^0) A_0 + (\omega^p + \xi^f \xi_f^a \omega_a^p) A_p + (d\xi^f + \xi^g \theta_g^f) B_f. \tag{60}$$

Equation (60) shows that the tangent subspace to the variety V_r^{r+s} is spanned by the points A_0, A_p, and B_f and does not depend on a location of the point A in $N^s(x)$. Thus, V_r^{r+s} is a variety with a degenerate Gauss map of rank r with s-dimensional plane generators in the space $\mathbb{P}^n$.

Sufficiency: Suppose that the planes $N^s(x)$ of a field ν^s of s-dimensional normal directions form a variety V_r^{r+s} with a degenerate Gauss map of rank $n-l$ with s-dimensional plane generators in the space $\mathbb{P}^n$. Then the tangent $(r+s)$-dimensional subspaces to V_r^{r+s} formed by points of the planes $N^s(x)$ are the same at all points $A = A_0 + \xi^f B_f$ of an arbitrary plane $N^s(x)$. Since the differential of the point A has the form (58), it follows that

$$(d\xi_f^a + \xi_f^b \omega_b^a) A_a = \theta_f^g B_g + \theta_f^0 A_0.$$

This means that the field ν^s is parallel. $\square$

Theorem 8 indicates a method of construction of a variety V_r^{r+s} with a degenerate Gauss map departing from a normalized variety X of some special kind.

3. Further consider a variety X of dimension r in the Euclidean space $\mathbb{E}^n$. On X, both the second normal $K_x = T_x \cap \mathbb{P}_\infty$ and the first normal N_x orthogonal to the tangent subspace $T_x(X)$ are naturally defined.

In the Euclidean space $\mathbb{E}^n$, there is defined a scalar product of vectors, and a scalar product of points in the hyperplane at infinity $\mathbb{P}_\infty$ is induced by the scalar product in $\mathbb{E}^n$. Because in our moving frame, we have $A_a \in N_x \cap \mathbb{P}_\infty$; $A_p \in T_x \cap \mathbb{P}_\infty = K_x$, $a = 1, \ldots, l$; $p = l+1, \ldots, n$; and $T_x \perp N_x$, we find that

$$(A_a, A_p) = 0, \tag{61}$$

where, as usually, the parentheses denote the scalar product of points in the hyperplane at infinity $\mathbb{P}_\infty$. In addition, we set

$$(A_a, A_b) = g_{ab}, \;\; (A_p, A_q) = g_{pq}, \tag{62}$$

where g_{ab} and g_{pq} are nondegenerate symmetric tensors.

Differentiating equations (61) and using formulas (47), (61), and (62), we find that

$$g_{ab}\, \omega_p^b + g_{pq}\, \omega_a^q = 0.$$

It follows that

$$\omega_a^p = -g^{pq}\, g_{ab}\, \omega_q^b. \tag{63}$$

Equations (57) and (48) imply that

$$\omega_a^p = -g^{pq}\, g_{ac}\, b_{qs}^c \omega^s. \tag{64}$$

Comparing (64) and (49), we obtain

$$c^p_{as} = -g^{pq}\, g_{ac}\, b^c_{qs}. \tag{65}$$

Now we find the equation of the focus hypersurface F_x of the variety $X \in \mathbb{E}^n$. By (51) and (65), we have the following equation for F_x:

$$\det(y^0 \delta^p_q - y^a g^{ps} g_{ac} b^c_{sq}) = 0.$$

The last equation is equivalent to the equation

$$\det(y^0 g_{pq} - y_a b^a_{pq}) = 0, \tag{66}$$

where $y_a = g_{ab} y^b$.

In our moving frame, the hyperplane at infinity $\mathbb{P}_\infty$ is determined by the equation $y^0 = 0$. Hence by (60), the intersection $F_x \cap \mathbb{P}_\infty$ of the focus hypersurface F_x with the hyperplane at infinity $\mathbb{P}_\infty$ is defined by the equation

$$\det(y_a b^a_{pq}) = 0. \tag{67}$$

But this equation differs only in notation from equation (52) of the focus hypercone Φ_x of the variety X. Equations (52) and (67) coincide if $\xi_a = y_a = g_{ab} y^b$. Thus, we have proved the following result.

Theorem 9. *The focus hypercone Φ_x of the variety $X \subset \mathbb{E}^n$ is formed by the hyperplanes ξ containing the tangent subspace T_x and orthogonal at the points $\widetilde{y}$ of the hyperplane at infinity $\mathbb{P}_\infty$ lying in the intersection $F_x \cap \mathbb{P}_\infty$.*

This result clarifies the geometric meaning of the focus hypercone Φ_x for the variety $X \subset \mathbb{E}^n$ and its relation with the focus hypersurface F_x of X.

We also find the curvature tensor of the affine connection induced on the variety $X \subset \mathbb{E}^n$. Substituting the values of c^p_{aq} from (65) into formula (53), we find that

$$R^p_{qst} = g^{pu} g_{ac} (b^a_{qt} b^c_{us} - b^a_{qs} b^c_{ut}). \tag{68}$$

Contracting equation (68) with the tensor g_{pv} and changing the summation indices (if necessary), we find that

$$R_{pqst} = g_{ac} (b^a_{ps} b^c_{qt} - b^a_{pt} b^c_{qs}), \tag{69}$$

where $R_{pqst} = g_{pu} R^u_{qst}$. Formulas (68) and (69) give the usual expressions for the curvature tensor of the affine connection γ_t induced on a normalized variety $X \subset \mathbb{E}^n$.

But in addition to the curvature tensor of the affine connection induced on a normalized variety $X \subset \mathbb{E}^n$, we considered also the tensor R^a_{bst} of normal curvature defined by equation (54). Substituting the values of c^p_{aq} from (65) into formula (54), we find that

$$R^a_{bst} = g^{pq} g_{bc} (b^c_{qt} b^a_{ps} - b^c_{qs} b^a_{pt}). \tag{70}$$

As we noted earlier, in the book [C 01], the exterior 2-form

$$\Omega^a_b = d\omega^a_b - \omega^c_b \wedge \omega^a_c = \frac{1}{2} R^a_{bst} \omega^s \wedge \omega^t$$

is called the *Gaussian torsion* of a variety $X \subset \mathbb{E}^n$.

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