Nonlinear Toys

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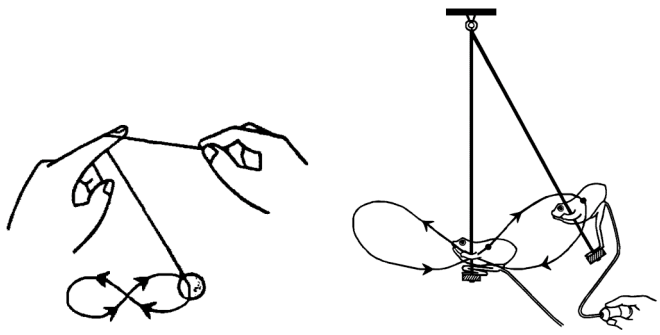
NL2717 **Nonlinear toys**

Toys are ubiquitous and some are unique to particular cultures. Many action toys are intriguing because they perform in unexpected ways as a result of underlying nonlinear principles, yielding both amusement and surprises for the unwary. Here we give several examples of toys and devices that operate on linear and nonlinear physical principles.



(a) (b)

Figure 1: Parametric excitation demonstrated by (a) a pendulum over a finger and (b) a frog on a swing.

Frog on a swing. Several toys are based on the mechanism of *parametric excitation*. Although primarily a linear mechanism, an increase in amplitude in parametric excitation leads to the nonlinear range of amplitude dependent phenomena. One can demonstrate parametric excitation by taking a string with a weight attached to its end and make it into a pendulum by hanging it over your forefinger (see Figure 1(a)). Now, force the weight into a small amplitude oscillation, and then move the weight up and down with the same phase as the oscillation by pulling on the string periodically. The result is an increase in the amplitude of the oscillation of the pendulum. The phase of the oscillation will vary slowly as the amplitude increases and the inherently nonlinear nature of the pendulum becomes important. The basic linearized equation for parametric excitation of the nonlinear pendulum is the Mathieu equation

$$\frac{d^2\theta}{dt^2} + \left(\omega_0^2 - \alpha \sin 2\omega_0 t\right)\theta = 0\,,$$

where θ is the angle of the pendulum from the vertical, ω_0 is the fundamental small amplitude frequency, and α is the amplitude of the periodically changing pendulum length.

This principle of parametric excitation is active in the toy called the "frog on a swing" (see Figure 1(b)). The frog in this toy is made of rubber and sits on a swing (pendulum). It can be inflated periodically with air using a tube and bulb. This causes the frog to stand up and sit down repeatedly and, hence, moves the center of mass of the frog up and down, as illustrated above. With proper timing in inflating the bulb at the correct phase during each swing, the amplitude of the swing with the oscillating frog on it gradually increases. The same principle applies to the playground swing where children can increase the amplitude of their swing by "pumping" the swing in a standing position (Wirkus et al., 1998).

HANDSTAND PENDULUM. A related, but completely different phenomenon occurs in the "handstand pendulum." The normal pendulum hangs stably downwards from its fulcrum as a result of gravity. However, applying a high frequency forced oscillation to the fulcrum can cause the pendulum to be inverted, so that it now "hangs" stably pointing upwards (Figure 2). Again the governing linearized equation is the Mathieu equation, but with the signs changed:

$$\frac{d^2\theta}{dt^2} + \left(-\frac{g}{\ell} + 2\alpha \sin 2\omega t\right)\theta = 0$$

where g is the acceleration due to gravity and ℓ is the length of the pendulum arm.

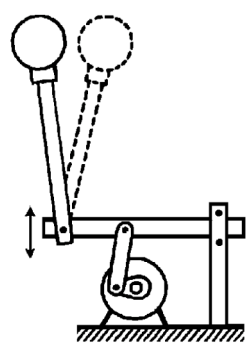


Figure 2: The handstand pendulum.

SHISHI-ODOSHI. In some Japanese gardens, there is a device that was designed originally to scare deer and other animals away from rice fields. Now, its purpose is either to scare away the deer that may intrude into a garden to eat the plants or to serve as an aesthetic fixture in the garden for its novelty and silence breaking "clack". This device, while not strictly a toy, is based on nonlinear principles and is called a shishi-odoshi (deer-scarer) or shika-oi. The shishi-odoshi is made from a thick bamboo culm with a length of three internal cells, i.e., four nodes (the raised rings at regular intervals along the bamboo). At one end, the last node has been cut off and the end shaped to receive water (see Figure 3). From this open end, the internal membrane of the first node also has been removed so there is a cylindrical cup that is two cells long. A pivot rod is inserted into the bamboo between the two inner nodes so that when the tube is empty, the closed end of the bamboo rests on a stone. However, the position of the pivot is chosen so that as a steady stream of water flows into the open end of the tube, the center of mass of the bamboo and water shifts across the pivot and the open end of the tube will dip down, thus pouring out the water. The empty tube now returns quickly to its stable position on the stone with a loud "clack," which is the noise that is supposed to scare the deer.

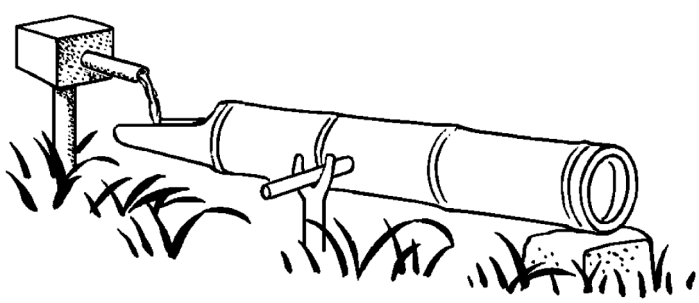
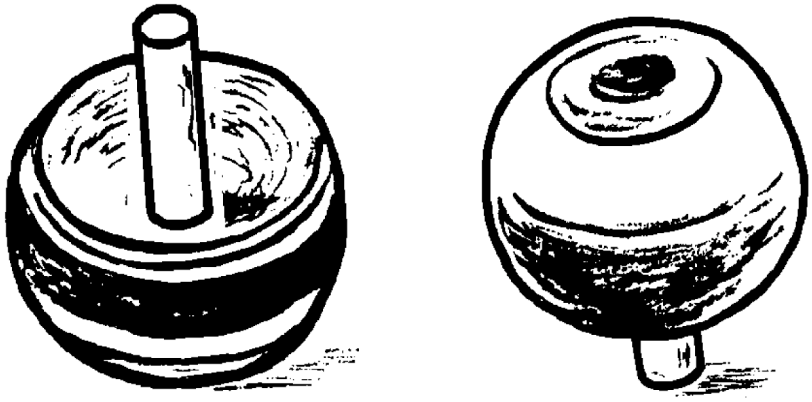


Figure 3: Bamboo deer alarm.

The oscillatory motion of the bamboo tube caused by the steady flow of water is called a *relaxation oscillation* and can be regarded as an example of a self-induced oscillation. There are many toys that use this principle to change their orientation periodically, namely, water filling up a reservoir, and then the water being discharged as a result of the change of orientation of the toy.

Tippe top. A fascinating toy is the tippe top, which doesn't act like a conventional top. The tippe top consists of a little more than a hemisphere with a short cylindrical stem (see Figure 4(a)). It has a low center of gravity, so when placed on a surface, it will simply sit like an ordinary top on its hemispheric side. However, if the stem is held between a finger and the thumb and given a spin on the hemispheric end, then unlike a conventional top, the tippe top will readily turn over and continue to spin on the stem (see Figure 4(b)). As it turns over, the behavior of the tippe top is unusual in two respects: the center of gravity is raised and the spinning direction with respect to fixed body coordinates is reversed. When the top is spun, the low center of gravity is centrifugally moved away from the vertical spin axis. Before and after the top turns over, the angular momentum of the top about the vertical axis is the dominant momentum component. During the inversion process, the center of gravity is raised and the increase in potential energy reduces the rotational kinetic energy. The torque needed to execute this inversion comes from the sliding frictional forces between the top's hemispherical surface and the surface on which it is rotating. The mechanics of this top are described by nonlinear equations of motion that are solvable only in special cases (Cohen, 1977; Or, 1994; Gray & Nickel, 2001).



(a) (b)

Figure 4: A tippe top in its stable resting position and stable rotating position.

Pecking woodpecker. A cute toy providing entertainment for young children is the pecking woodpecker going down a pole. To make this toy, coil a soft wire loosely around a smooth pole to make a soft spring. Now leave several turns of the coil on the pole and the remaining coils extended outwards from the pole (see Figure 5a). The weight of the protruding part of the spring will tilt the spring coiled around the pole so that the static friction is sufficient to keep it fixed on the pole. If one pushes the extended coil downwards and releases it, the extended coil vibrates up and down and the spring around the pole descends in a staccato fashion. The reason for the stuttering movement down the pole is that the coil around the pole oscillates periodically between being stopped by static friction and essentially free falling down the pole when the coil is aligned with the pole. Now attach a small wooden bird to the end of the extended wire, and the bird will peck the pole with its beak as it descends the pole (see Figure 5b).

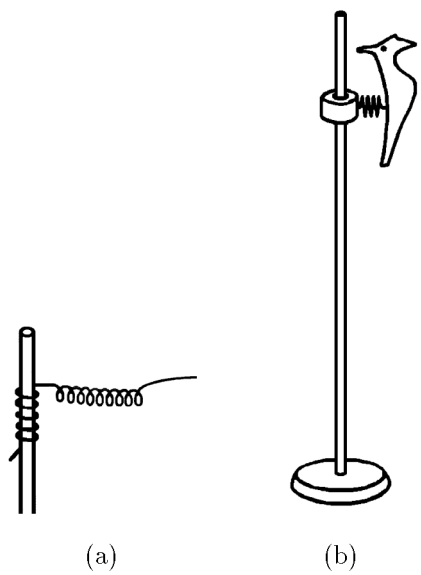


Figure 5: The pecking woodpecker with a detail of the spring on the left.

There are many toys that move by the principle of self-induced excitation (see Figure 6). Basic equation for such oscillations are those of Duffing and van der Pol.

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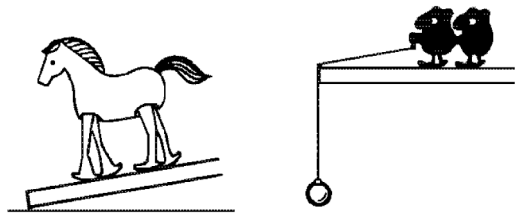

Figure 6: Two other toys that use self-induced oscillations.

See also Duffing equation; Equations, nonlinear; Laboratory modles of nonlinear waves; Parametric amplification; Pendulum; Relaxation oscillators; Stick-slip friction; van der Pol equation

Further Reading

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