On the Structure of Varieties with Degenerate Gauss Mappings

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ON THE STRUCTURE OF VARIETIES WITH DEGENERATE GAUSS MAPPINGS

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Let $V = \mathbb{C}^{N+1}$ and let $X^n \subset \mathbb{P}V$ be a variety. Let $x \in X$ be a smooth point, and let $\tilde{T}_x X \subset \mathbb{P}V$ denote the embedded tangent projective space to X at x. Let

$$\begin{aligned} \gamma : X &\dashrightarrow \mathbb{G}(n, \mathbb{P}V) \\ x &\mapsto \tilde{T}_x X \end{aligned}$$

denote the *Gauss map* of X, where $\mathbb{G}(n, \mathbb{P}V)$ denotes the Grassmanian of $\mathbb{P}^n$'s in $\mathbb{P}V$.

In [GH], Griffiths and Harris present a structure theorem for varieties with *degenerate* Gauss mappings, that is X such that $\dim \gamma(X) < \dim X$. Namely, such varieties are "built up from cones and developpable varieties" [GH, p392]. In this note we illustrate their result with examples and indicate invariants that might be used to obtain a refined structure theorem.

Fixing $X^n \subset \mathbb{P}V$, let r denote the rank of γ and set $f = n - r$, the dimension of a general fiber. If $x \in X$ is a smooth point, we let $F = \gamma^{-1}\gamma(x)$ denote the fiber of γ (which is a $\mathbb{P}^f$). Let $Z_F = F \cap X_{sing}$ (the *focus* of F). Z_F is a codimension one subset of F of degree $n - f$. The number of components of Z_F and the dimension of the varieties each of these components sweeps out as one varies F furnish invariants of X.

Here are some examples of varieties with degenerate Gauss mappings (which are not mutually exclusive):

I. Joins.

Form the join of k varieties $Y_1, \dots, Y_k \subset \mathbb{P}V$,

$$X = S(Y_1, \dots, Y_k) = \overline{\cup_{y_j \in Y_j} \mathbb{P}_{y_1, \dots, y_k}}$$

where $\mathbb{P}_{y_1,\dots,y_k}$ denotes the projective space spanned by $y_1, \dots, y_k$ (generically a $\mathbb{P}^{k-1}$). Note that $\dim X \leq \Sigma_j \dim Y_j + (p-1)$ with equality expected.

Joins have degenerate Gauss maps with at least $(k-1)$-dimensional fibers because Terracini's lemma (see [Z, II.1.10]) implies that the tangent space to $S(Y_1, \dots, Y_k)$ is constant along each $\mathbb{P}^{k-1}_{y_1,\dots,y_k}$.

Two special cases of this construction:

1. Let L be a linear space. Then $S(Y, L)$ is a cone over Y with vertex L.
2. $Y_j = Y$ for all j. Then X is the union of the secant $\mathbb{P}^{k-1}$'s to Y.

Joins are built out of cones in the sense that one can use e.g. the family of cones over Y_2 with vertices the points of Y_1 to sweep out X.

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II. Varieties built from tangent lines.

IIA. Tangential varieties. Let $Y \subset \mathbb{P}V$ be a variety and let $\tau(Y) \subset \mathbb{P}V$ denote the union of tangent stars to Y. (If Y is smooth, $\tau(Y)$ is the union of embedded tangent lines to Y.) $\tau(Y)$ has a degenerate Gauss map with at least one dimensional fibers. (see [L] for definitions). Examples IIB and IIC below generalize $\tau(Y)$.

IIB. Bands. Let $Y \subset \mathbb{P}V$ be a smooth variety of dimension m and fix $1 \leq k \leq N-m-1$. For each $y \in Y$, let $L_y \subset \mathbb{G}(m+k, \mathbb{P}V)$ be such that $\tilde{T}_y Y \subset L_y$ and let $X = \cup_{y \in Y} L_y$. Then $\dim X \leq m + k$ (with equality occurring generically) and X will have degenerate Gauss map with at least one-dimensional fibers.

Note that the higher osculating varieties of Y are special cases of this construction.

Bands are unions of cones with vertices the points of Y.

One could seek to generalize tangential varieties in a different way, namely by taking a subspace of the tangent lines through each point of Y. If $x \in \mathbb{P}V$ and $v \in T_x\mathbb{P}V$, we let $\mathbb{P}^1_{x,v}$ denote the line passing through x with tangent space spanned by v. Let $\Delta \subset TY$ be a distribution. One could consider the variety $X = \cup_{y \in Y, v \in \Delta_y} \mathbb{P}^1_{y,v}$ consisting of the union of tangent lines tangent to Δ. In general X will *not* have a degenerate Gauss map, but it will in some special cases. The case where $Y \subset Z$ and one takes $X = \cup_{y \in Y} \tilde{T}_y Z$ is one special case (This case is covered by example IIB). Here is another construction:

IIC. Unions of conjugate spaces.

Let $II = II_{Y,y} \in S^2T^*_yY \otimes N_yY$ denote the projective second fundamental form of Y at y (see [AG], [GH] or [L] for a definition).

Let $Y^{n-1} \subset \mathbb{P}^{n+1}$ be a variety such that at general points there exist $n-1$ simultaneous eigen-directions for the quadrics in its second fundamental form. This condition holds for generic varieties of codimension two. (To make the notion of eigen-direction precise, choose a nondegenerate quadric in II to identify T with T^* and consider the quadrics as endomorphisms of T. The result is independent of the choices.) Let $X^n \subset \mathbb{P}^{n+1}$ be the union of one of these families of embedded tangent lines.

The directions indicated above are called *conjugate directions* on Y^{n-1}.

In higher dimensions it is still possible to have a conjugate direction or conjugate space, but in this case Y must satisfy a certain exterior differential system. As is shown in [AG, p 85] local solutions to this system exist and depend on $n(n-1)$ arbitrary functions of two variables.

In this case X is the union of the tangential varieties of the integral curves for the distribution defined by the conjugate directions to Y.

III. Varieties with $f = 1$.

IIIA. Generic varieties with $f = 1$. We say a variety $X \subset \mathbb{P}V$ with $f = 1$ is *generic among varieties with* $f = 1$ if Z_F consists of $n-1$ distinct points and the variety each point sweeps out is $(n-1)$-dimensional. The following theorem follows from results in [AG]:

Theorem. *The varieties $X^n \subset \mathbb{P}^{n+a}$ generic among varieties with $f = 1$ are the union of conjugate lines to some variety Y^{n-1}, with a finite number of lines tangent to a general point of Y.*

IIIB. Classification of $X^3 \subset \mathbb{P}^4$ with $f = 1$. Here F is a $\mathbb{P}^1$ and the focus Z_F is of degree two. There are two classes:

Class 1: Z_F consists of two distinct points, z_1, z_2.

1a. (Generic case) Each z_j traces out a surface S_j. Here X is the dual variety of a II-generic surface in $\mathbb{P}^{4*}$ (its Gauss image). Locally X may be described as the union of a family of lines tangent to conjugate directions on either surface (one must take the family that corresponds to conjugate directions on the other surface). It may be the case that globally $S_1 = S_2$ and there is a unique construction.

1b. z_1 traces out a surface S and z_2 traces out a curve C. Here X may be described as the union of a family of conjugate lines to S, where the conjugate lines intersect along a curve.

1c. Both z_j's trace out curves, C_j. In this case $X = S(C_1, C_2)$.

Class 2: Z_F is a single point z of multiplicity two.

2a. z traces out a surface S. In this case S will have a family of asymptotic lines and X is the union of the asymptotic lines to S.

2b. z traces out a curve C. An example of X in this case is the union of a family of planes that are tangent to C. We conjecture that this is the only example.

2c. z is fixed, then X is a cone over z.

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