Role of weakly singular integral equations in surface water wave scattering

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Abstract

A class of boundary value problems arising in the study of scattering of surface water waves by barriers, under the assumption of the linearized theory, is reduced to singular integral equations of the first kind, involving only "weakly singular" kernels. The unified treatment presented here is observed to be most suitable to handle the class of scattering problems under consideration and it is shown that reduction of these scattering problems to either the Cauchy type "strongly singular" integral equation or the so called "hyper singular" integral equations can be avoided altogether.

Key words: Integral equations, weakly singular kernels, Green's function, water waves.
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1 Introduction

Weakly singular integral equations(I.E.'s) of the first kind, are given by the general form (see Mikhlin [1])

$$\int_B f(t)K(t,u)dt = g(u), \quad \text{for } u \in B, \tag{1.1}$$

in which B is a piecewise connected arc in complex t-plane, f and g are differentiable functions and the kernel K is a weakly singular function of the two variables t and u, which are either of the following two types:

$$\begin{aligned} &\textbf{(i)} \qquad K(t,u) = k_0^{(1)}(t,u)\ \log(t-u) + k_1(t,u) \\ &\textbf{(ii)} \qquad K(t,u) = k_0^{(2)}(t,u)\ (t-u)^{-\alpha} + k_2(t,u), \quad (0 < \alpha < 1) \end{aligned} \tag{1.2}$$

where k_1 and k_2 are regular square-integrable functions.

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A large class of problems involving scattering of surface water waves by obstacles, under the assumption of the linearized theory (see Stoker [2]), has been handled in the literature (see Ursell [3], Porter [4], Tuck [5], Chakrabarti [6] etc.) by way of reducing the problems to those of solving certain weakly singular integral equations of the first kind, which are of the form of the equation (1.1), with the kernels being of the type **(i)**, mentioned above.

In recent times, more complicated class of scattering problems in the theory of water waves, than the ones handled previously, has been solved (see Martin [7], [8], Parsons and Martin [9], [10] etc.) by reducing these problems to "hypersingular" integral equations of the first kind, which are of the form (1.1), but with their kernels K being given by

$$K(t,u) = k_0^{(3)}(t,u)\ (t-u)^{-\beta} + k_3(t,u), \quad (\beta > 1) \tag{1.3}$$

where $k_0^{(3)}$ and k_3 are regular and square integrable functions.

The other type of singular integral equations (1.1), which can be called as "strongly singular" singular integral equations, in which the kernels K have a Cauchy type singularity (see Gakhov [11], Muskhelishvili [12]), with

$$K(t,u) = k_0^{(4)}(t,u)\ (t-u)^{-1} + k_4(t,u), \tag{1.4}$$

where $k_0^{(4)}$ and k_4 are regular and square integrable, has also been used as the media to solve water wave scattering problems (see Ursell [3]).

In the present review article, we explain various aspects of weakly singular integral equations and their solutions with applications to scattering of surface water waves by obstacles present in deep water. That integral equations with stronger singularity like the Cauchy-type and a hyper singular kernels also arise in studying the same problems, have been shown in a straight forward manner. Our emphasis here is on the treatment of these scattering problems only by the aid of weakly singular integral equations and we find that complete solutions of the problems at hand can be derived via such weakly singular integral equations only.

2 Study of weakly singular I.E.'s and their solutions

Weakly singular integral equations which arise in many areas of application, viz., mechanics, electromagnetic theory, theory of water waves etc., are studied extensively by many workers (see [4], [5], [6], [13] etc.) Many problems involving scattering of surface water waves by thin vertical barriers can be converted into weakly singular integral equations with its kernel being either of the types (1.2) mentioned earlier. Several methods of handling these problems will be explained in the next section. In this section, we give a brief account of the various types of weakly singular integral equations and their solutions.

2.1 Abel type integral equation

One of the classical types of weakly singular integral equation is Abel integral equation. Many boundary value problems which occur in the applications, for example, water wave scattering by thin vertical barriers etc., can be converted at one stage or the other into Abel type integral equation to be solved.
The two general Abel integral equations are given by

$$\begin{aligned} &\textbf{(i)} \quad \int_a^x \frac{f(t)}{\{h(x)-h(t)\}^{\alpha}}dt = g(x), \quad \text{for} \;\; a \le x \le b, \quad 0 \le \alpha \le 1, \\ &\textbf{(ii)} \quad \int_x^b \frac{f(t)}{\{h(t)-h(x)\}^{\alpha}}dt = g(x), \quad \text{for} \;\; a \le x \le b, \quad 0 \le \alpha \le 1, \end{aligned} \tag{2.1}$$

where $h(x)$ is monotonically increasing function.

The above integral equations (2.1) are solved by many methods in the literature and the solution is given by

$$\begin{aligned} &\textbf{(i)} \quad f(x) = \frac{\sin \alpha\pi}{\pi}\frac{d}{dx}\int_a^x \frac{h'(t)g(t)}{\{h(x)-h(t)\}^{1-\alpha}}dt, \quad \text{for} \;\; a \le x \le b, \quad 0 \le \alpha \le 1, \\ &\textbf{(ii)} \quad f(x) = -\frac{\sin \alpha\pi}{\pi}\frac{d}{dx}\int_x^b \frac{h'(t)g(t)}{\{h(t)-h(x)\}^{1-\alpha}}dt, \quad \text{for} \;\; a \le x \le b, \quad 0 \le \alpha \le 1. \end{aligned} \tag{2.2}$$

2.2 I.E.'s with logarithmic singular kernels

We consider here the integral equations of the first kind, involving the kernels of the form $\log|t-u|$, $\log|t^2-u^2|$, $\log\left|\frac{t+u}{t-u}\right|$ etc., in the interval (a,b), where a, b are real.
The integral equation of the first kind

$$\int_a^b f(t)\log|t-u|dt = g(u), \quad \text{for} \;\; a < u < b, \tag{2.3}$$

where f and g are assumed to be differentiable functions, can be solved (see Estrada Kanwal [14]), by utilizing the solution of a Cauchy singular integral equation, and the solution is given by

$$f(t) = \frac{1}{\pi^2}\Big\{(t-a)(b-t)\Big\}^{-\frac{1}{2}}\Big[\frac{1}{\log(\frac{b-a}{4})}\int_a^b \frac{g(u)}{\{(u-a)(b-u)\}^{\frac{1}{2}}}du + \int_a^b \frac{\{(u-a)(b-u)\}^{\frac{1}{2}}g'(u)}{(t-u)}du\Big],$$
$$\text{for} \;\; a < t < b \;\; \text{and} \;\; b-a \neq 4. \tag{2.4}$$

In the special case, when $b-a=4$, the solution of the equation (2.3) is same as the corresponding solution of the Cauchy singular integral equation involving an arbitrary constant and is given by

$$f(t)=\frac{1}{\pi^2}\Big\{(t-a)(b-t)\Big\}^{-\frac{1}{2}}\Big[C\pi+\int_a^b\frac{\{(u-a)(b-u)\}^{\frac{1}{2}}g'(u)}{(t-u)}du\Big],\ \text{for}\ \ a<t<b, \tag{2.5}$$

where C is an arbitrary constant, provided that

$$\int_a^b\frac{g(u)}{\{(u-a)(b-u)\}^{\frac{1}{2}}}du=0. \tag{2.6}$$

The solution of the logarithmic singular integral equation

$$\int_a^b f(t)\log|t^2-u^2|dt=g(u),\quad \text{for}\ \ a<u<b, \tag{2.7}$$

where f and g are assumed to be differentiable functions, can be derived by reducing it to an integral equation of the type (2.3) through the transformations

$$t^2=\tau,\ \ u^2=\xi,\ \ a^2=\alpha,\ \ b^2=\beta.$$

we next consider the logarithmic singular integral equation

$$\int_a^b f(t)\log|\frac{t+u}{t-u}|dt=g(u),\quad \text{for}\ \ a<u<b, \tag{2.8}$$

where f and g are assumed to be differentiable functions.

The solution of the above integral equation (2.8) can be obtained by solving the associated Cauchy-type singular integral equation which involves an arbitrary constant, and then, determining the arbitrary constant by the aid of the original integral equation under consideration. We also give various forms of solutions depending on the behaviour of the unknown function at the endpoints (see Chakrabarti and Manam [15]).

The general solution of the equation (2.8) is found to be unique and is given by

$$f(t)=\frac{1}{\pi^2\Big\{(t^2-a^2)(b^2-t^2)\Big\}^{-\frac{1}{2}}}\Big[C_2+2\int_a^b\frac{\{(u^2-a^2)(b^2-u^2)\}^{\frac{1}{2}}ug'(u)}{(t^2-u^2)}du\Big],\ (a<t<b), \tag{2.9}$$

where C_2 is given by

$$C_2=2\Big(\frac{a\pi-I_1}{I_2}\Big)\int_a^b\frac{g(x)}{\Big\{(x^2-a^2)(b^2-x^2)\Big\}^{\frac{1}{2}}}dx+2\int_a^b\bigg(\frac{x^2-a^2}{b^2-x^2}\bigg)^{\frac{1}{2}}g(x)dx, \tag{2.10}$$

with

$$I_1=\int_a^b\left(\frac{x^2-a^2}{b^2-x^2}\right)^{\frac{1}{2}}\log\left|\frac{a+x}{a-x}\right|dx,$$

and

$$I_2=\int_a^b\frac{\log\left|\frac{u+x}{u-x}\right|}{\left\{(x^2-a^2)(b^2-x^2)\right\}^{\frac{1}{2}}}dx=\int_a^b\frac{\log\left|\frac{a+x}{a-x}\right|}{\left\{(x^2-a^2)(b^2-x^2)\right\}^{\frac{1}{2}}}dx.$$

Different forms of solutions of this integral equation with the unknown function depending on the endpoints are given as:

(**I**) $f(t)$ **bounded at both the endpoints** $t=a$ **and** $t=b$**:**

$$f(t)=\frac{2}{\pi^2}\left\{(t^2-a^2)(b^2-t^2)\right\}^{\frac{1}{2}}\int_a^b\frac{ug'(u)}{\left\{(u^2-a^2)(b^2-u^2)\right\}^{\frac{1}{2}}(t^2-u^2)}du,\ (a<t<b),\quad (2.11)$$

provided that

$$\left.\begin{aligned}&(i)\ \int_a^b\frac{tg'(t)}{\left\{(t^2-a^2)(b^2-t^2)\right\}^{\frac{1}{2}}}dt=0\\ \text{and}\qquad&(ii)\ C_2+2\int_a^b\left(\frac{t^2-a^2}{b^2-t^2}\right)^{\frac{1}{2}}tg'(t)dt=0.\end{aligned}\right\}\quad (2.12)$$

(**II**) $f(t)$ **bounded at the end** $t=a$ **but unbounded at** $t=b$**:**

$$f(t)\ =\ \frac{2}{\pi^2}\left(\frac{t^2-a^2}{b^2-t^2}\right)^{\frac{1}{2}}\int_a^b\left(\frac{b^2-u^2}{u^2-a^2}\right)^{\frac{1}{2}}\frac{ug'(u)}{t^2-u^2}du,\ (a<t<b),\quad (2.13)$$

provided that

$$C_2-2\int_a^b\left(\frac{b^2-t^2}{t^2-a^2}\right)^{\frac{1}{2}}tg'(t)dt=0.\quad (2.14)$$

(**III**) $f(t)$ **unbounded at the end** $t=a$ **but bounded at** $t=b$**:**

$$f(t)\ =\ \frac{2}{\pi^2}\left(\frac{b^2-t^2}{t^2-a^2}\right)^{\frac{1}{2}}\int_a^b\left(\frac{u^2-a^2}{b^2-u^2}\right)^{\frac{1}{2}}\frac{ug'(u)}{(t^2-u^2)}du,\ (a<t<b),\quad (2.15)$$

provided that

$$C_2+2\int_a^b\left(\frac{t^2-a^2}{b^2-t^2}\right)^{\frac{1}{2}}tg'(t)dt=0,\quad (2.16)$$

where C_2 is given by the relation (2.10).

3 Occurrence in water wave scattering by obstacles

We present a short review of the various methods available to tackle a class of surface water wave scattering problem in the presence of an obstacle. We consider here only the case of infinite depth, in which the irrotational motion assumed to be that of an incompressible inviscid fluid under the action of gravity. We use a rectangular cartesian coordinate system in which the y-axis is taken vertically downwards so that $y > 0$, $x \in I\!R$ is the region occupied by the fluid. The motion is also assumed to be two-dimensional and time-harmonic and is described by a velocity potential $\Phi(x, y, t)$ which is the real part of $\phi(x, y)e^{-i\omega t}$, $\omega(> 0)$ denoting angular frequency and t denoting the non-dimensional time. The time-dependent factor $e^{-i\omega t}$ is suppressed throughout the analysis. Then $\phi(x, y)$ satisfies

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, x \in I\!R, y > 0. \tag{3.1}$$

On the surface of the fluid occupied by $y = 0, x \in I\!R$, ϕ satisfies the boundary condition

$$\frac{\partial \phi}{\partial y} + K\phi = 0, \tag{3.2}$$

where $K = \frac{\omega^2}{g}$, with g being acceleration due to gravity.
On the piecewise curve B, denoting the boundary of the obstacle, ϕ satisfies the Neumann boundary condition

$$\frac{\partial \phi}{\partial x} = 0. \tag{3.3}$$

and

$$\phi, \ \frac{\partial \phi}{\partial x}, \ \frac{\partial \phi}{\partial y} \to 0 \text{ as } y \to \infty. \tag{3.4}$$

The behaviours of $\phi(x, y)$ at infinity, are given by

$$\phi(x, y) \to \begin{cases} e^{iKx - Ky} + Re^{-iKx - Ky}, & \text{as } \ x \to -\infty \\ Te^{iKx - Ky}, & \text{as } \ x \to \infty, \end{cases} \tag{3.5}$$

where R and T are two unknown complex constants, called the reflection and transmission coefficients respectively, to be determined.
In the case of sharp cornered obstacles, for example, vertical thin barriers, we have to mention the singular behaviour of the velocity by the edge conditions which are given by

$$\frac{\partial \phi}{\partial x}(0, y) \sim O(|y - t|^{-\frac{1}{2}}) \text{ as } y \to t, \tag{3.6}$$

where t, denoting the sharp edge of the obstacle under consideration.

3.1 Vertical barrier problem

Many methods of solution are available in the literature when the obstacle is in the form of a single thin vertical barrier, viz., a completely submerged barrier, a surface piercing, a barrier with a gap, a completely submerged plate etc. The submerged and surface piercing barrier problems have been solved first by Ursell [3] in 1947. Later on, in 1966, the same problem has been attacked by Williams [16] by utilizing a reduction method.

These problems can be viewed as those of solving pairs of dual integral equations (see Chakrabarti [6]) and then the dual integral equations can be converted into a logarithmic singular integral equation to be solved. Also, these dual integral equations can be transformed into a hyper singular integral equation and then be solved numerically. Another method (see Porter [4]) of solving these problems involves application of the Green's theorem to an appropriate Green's function of the problem and the velocity potential and obtaining an integral equation with its kernel being the tangential derivatives of the Green's function and finally solving the integral equation numerically. This integral equation is found to be equivalent to the logarithmic singular integral equation discussed in section-2.

We present various approaches of solving the thin vertical barrier problem very concisely. The purely analytical method to solve the boundary value problems posed for the thin vertical barriers is to formulate them into logarithmic singular integral equations of the type whose analytical solutions are known. One of the ways to achieve this is through dual integral equations. There are many semi-analytical methods, namely, Green's function technique, hyper singular integral equation technique etc., available and we discuss them one by one.

In case of a thin vertical barrier, the velocity potential $\phi(x,y)$ satisfies the continuity condition

$$\phi(0^-,y)=\phi(0^+,y),\ \text{ for all }\ y\in L, \tag{3.7}$$

with L representing $(0,\infty)\setminus B$.
We represent the unknown velocity potential $\phi(x,y)$, in the two regions $x<0$ and $x>0$, as given by,

$$\phi(x,y)=\begin{cases} e^{iKx-Ky}+Re^{-iKx-Ky}+\displaystyle\int_0^\infty A(\xi)[\xi\cos\xi y-\sin\xi y]e^{\xi x}\,d\xi,\ x<0\\ Te^{iKx-Ky}+\displaystyle\int_0^\infty B(\xi)[\xi\cos\xi y-\sin\xi y]e^{-\xi x}\,d\xi,\ x>0, \end{cases} \tag{3.8}$$

where $A(\xi)$ is an unknown function to be determined ,along with the unknown constants R and T,which are the reflection and the transmission coefficients of the incoming wave e^{iKx-Ky}.

Since the horizontal velocity component is continuous across the positive y-axis, we get, by Havelock's expansion theorem that

$$T = 1 - R; \ A(\xi) = -B(\xi). \tag{3.9}$$

It can be easily seen that the conditions (3.3), (3.7) and (3.8) along with the relations (3.9) give rise to a pair of dual integral equations as given by

$$\int_0^\infty \xi A(\xi)(\xi \cos \xi y - K \sin \xi y) d\xi = iK(1-R)e^{-Ky}, \text{ on } y \in B, \tag{3.10}$$

$$\int_0^\infty A(\xi)(\xi \cos \xi y - K \sin \xi y) d\xi = Re^{-Ky}, \text{ on } y \in L. \tag{3.11}$$

and these can be rewritten in an alternative form, as given by

$$\left(\frac{d}{dy} - K\right) \int_0^\infty \xi A(\xi) \sin \xi y d\xi = iK(1-R)e^{-Ky}, \text{ on } y \in B, \tag{3.12}$$

$$\left(\frac{d}{dy} - K\right) \int_0^\infty A(\xi) \sin \xi y d\xi = Re^{-Ky}, \text{ on } y \in L. \tag{3.13}$$

(I) Formulation of logarithmic singular integral equation

(a) Through dual integral equations

In the case, when $B = (0, a) \cup (b, \infty)$, the above ordinary differential equations (3.12) and (3.13) can be easily solved to give the following new dual integral equations:

$$\int_0^\infty \xi A(\xi) \sin \xi y d\xi = \begin{cases} D_1 e^{Ky} + i(1-R)\sinh Ky, \text{ for } y \in (0, a) \\ D_2 e^{Ky} - \frac{i}{2}(1-R)e^{-Ky}, \text{ for } y \in (b, \infty) \end{cases} \tag{3.14}$$

and

$$\int_0^\infty A(\xi) \sin \xi y d\xi = E_1 e^{Ky} - \frac{R}{2K} e^{-Ky}, \text{ for } y \in (a, b), \tag{3.15}$$

where D_1, D_2 and E_1 are arbitrary constants.

In order to accommodate the origin as well as the point at infinity along the y-axis, the arbitrary constants D_1 and D_2 in (3.14) are taken as zero.

Then the dual integral equations (3.14)-(3.15) can be rewritten as

$$\int_0^\infty \xi A(\xi) \sin \xi y d\xi = \begin{cases} i(1-R)\sinh Ky, \text{ for } y \in (0, a) \\ -\frac{i}{2}(1-R)e^{-Ky}, \text{ for } y \in (b, \infty) \end{cases} \tag{3.16}$$

and

$$\int_0^\infty A(\xi) \sin \xi y d\xi = E_1 e^{Ky} - \frac{R}{2K} e^{-Ky}, \text{ for } y \in (a, b). \tag{3.17}$$

Now we define

$$\int_0^\infty \xi A(\xi) \sin \xi y d\xi = f(y), \text{ for } y \in (a, b), \tag{3.18}$$

where $f(y)$ is an unknown function to be determined.
Utilizing the relations (3.16) and (3.18), we obtain, by using Fourier sine transform

$$A(\xi) = \frac{2}{\pi\xi}\int_0^\infty P(y)\,\sin\xi y dy, \tag{3.19}$$

where

$$P(y) = \begin{cases} i(1-R)\sinh Ky, \text{ for } y \in (0,a) \\ -\frac{i}{2}(1-R)e^{-Ky}, \text{ for } y \in (b,\infty) \\ f(y), \text{ for } y \in (a,b). \end{cases}$$

By putting $A(\xi)$ into the equation (3.17) and after utilizing the result

$$\int_0^\infty \frac{\sin\xi y \sin\xi t}{\xi} d\xi = -\frac{1}{2}\log\left|\frac{y-t}{y+t}\right|, \text{ for } y,t \in (0,\infty),$$

we obtain the following special logarithmic singular integral equation, to be solved for the unknown function $f(y)$, as given by

$$\frac{1}{\pi}\int_a^b f(u)\log\left|\frac{u+x}{u-x}\right| du = g(x), \text{ for } x \in (a,b), \tag{3.20}$$

where

$$\begin{aligned} g(x) \;&=\; -\frac{i(1-R)}{\pi}\int_0^a \sinh Kt\log\left|\frac{x+t}{x-t}\right|dt + \frac{i(1-R)}{2\pi}\int_b^\infty e^{-Kt}\log\left|\frac{x+t}{x-t}\right|dt \\ &\quad + E_1 e^{Kx} - \frac{R}{2K}e^{-Kx}, \end{aligned}$$

with E_1 and R as two unknowns occurring in the forcing function.

It can be easily seen that (see Chakrabarti and Manam [15]) that the unknown function $f(x)$ in the integral equation (3.20) is bounded at both the endpoints a and b. Consequently, we utilize the bounded solution (2.14) at both the endpoints along with the conditions (2.15) to determine the unknown constants E_1 and R, the reflection coefficient of the water wave problem under consideration.

In the case when $a = 0$, the barrier B is completely submerged and since $E_1 = \frac{R}{2K}$ for the consistency of the equation (3.17), the above logarithmic integral equation becomes

$$\frac{1}{\pi}\int_0^b f(u)\log\left|\frac{u+x}{u-x}\right| du = g(x), \text{ for } x \in (0,b), \tag{3.21}$$

where

$$g(x) \;=\; -\frac{i(1-R)}{2}\int_b^\infty e^{-Kt}\log\left|\frac{x+t}{x-t}\right|dt - \frac{R}{2K}\sinh kx,$$

with R as an unknown occurring in the forcing function.

We note here that the two conditions (2.15) to be satisfied by the forcing function $g(x)$ are turned out to be same when $a = 0$.

In the case when $b = \infty$, the barrier B is surface piercing and in this case the unknown constant $E_1 = 0$ for the consistency of the equation (3.17), and the logarithmic integral equation (3.20) becomes

$$\frac{1}{\pi}\int_a^\infty f(u)\log\Big|\frac{u+x}{u-x}\Big|du = g(x), \text{ for } x \in (a,\infty), \tag{3.22}$$

where

$$g(x) = \frac{i(1-R)}{\pi}\int_0^a \sinh Kt\ \log\Big|\frac{x+t}{x-t}\Big|dt - \frac{R}{2K}e^{-Kx},$$

with R as an unknown occurring in the forcing function.

It is not difficult to see that the integral equation (3.22) can be converted into an integral equation of the type (3.21) to determine the solution as well as the unknown reflection coefficient R.

The above two specific problems of scattering of surface water waves by fully submerged and surface piercing barriers can be handled directly by converting them into a logarithmic singular integral equations, of the type as obtained by Ursell [3], and is given by

$$\frac{1}{\pi}\int_0^b \tilde{f}(u)\log\Big|\frac{u+x}{u-x}\Big|du = Ae^{-Ky} - 2\int_0^b \tilde{f}(u)e^{-K(y+u)}I(y+u)du, \text{ for } x \in (0,b), \tag{3.23}$$

and

$$\frac{1}{\pi}\int_a^\infty \tilde{f}(u)\log\Big|\frac{u+x}{u-x}\Big|du = Ae^{-Ky} - 2\int_a^\infty \tilde{f}(u)e^{-K(y+u)}I(y+u)du, \text{ for } x \in (a,\infty), \tag{3.24}$$

where $\tilde{f}$ representing the value of $\frac{\partial\phi}{\partial x}$ along the gap outside the barrier and $I(t) = \int_{-\infty}^{Kt}\frac{e^v}{v}dv$, whose solution has been obtained directly using the reduction method by Chakrabarti [17].

We note that the logarithmic singular integral equation with domain as a pair of intervals arises in the case of thin barrier in the form of a plate, whose solution can be obtained analytically, in a similar manner as done in the case of the single general interval (a,b).

(b) An alternative approach (Equivalence with Green's function technique)

We define

$$\widehat{f}(y) = \phi(0^+,y) - \phi(0^-,y), \text{ for } y \in B, \tag{3.25}$$

where $\widehat{f}(y)$ is an unknown function to be determined in B and $\widehat{f}(y) = 0$ in L.

Utilizing the relations (3.8) and (3.9) in the relation (3.25), we have

$$2\int_0^\infty A(\xi)(\xi\cos\xi y - K\sin\xi y)d\xi = \widehat{f}(y) + 2Re^{-Ky}, \quad y\in B \tag{3.26}$$

giving

$$A(\xi) = -\frac{1}{(\xi^2+K^2)}\int_B \widehat{f}(y)(\xi\cos\xi y - K\sin\xi y)dy \tag{3.27}$$

and

$$R = -K\int_B \widehat{f}(y)e^{-Ky}dy = -\int_B \frac{d\widehat{f}}{dy}e^{-Ky}dy. \tag{3.28}$$

Using the condition (3.3), we get from the relation (3.8) that

$$\frac{d}{dy}\int_0^\infty A(\xi)(K\cos\xi y + \xi\sin\xi y)d\xi = -i\frac{d}{dy}[(1-R)e^{-Ky}] \tag{3.29}$$

By integrating the above equation (3.29) and using the relations (3.27) and (3.28), we get

$$\int_B \widehat{f}(t)\frac{\partial M(t,y)}{\partial t}dt = -ie^{-Ky} + C_0, \tag{3.30}$$

where

$$M(t,y) = \frac{1}{\pi}\int_0^\infty \frac{(K\cos\xi t + \xi\sin\xi t)(K\cos\xi y + \xi\sin\xi y)}{\xi(\xi^2+K^2)}d\xi - ie^{-K(y+t)}$$

and C_0 is an arbitrary constant.
By integrating by parts, the integral equation (3.30) becomes

$$\int_B \frac{d\widehat{f}}{dt}M(t,y)dt = ie^{-Ky} - C_0, \tag{3.31}$$

which is equivalent to the logarithmic singular integral equation obtained in Porter [4] using the Green's function technique and can be solved numerically.

Note that the reflection coefficient R can be obtained by utilizing the solution of the equation (3.31) into the relation (3.28) and by determining the arbitrary constant C_0 by utilizing an appropriate consistency condition for the equation (3.31). We have not taken up this study here.

(II) Formulation of hyper singular integral equation

We rewrite the relation (3.8), using the relation (3.9), for the velocity potential $\phi(x,y)$, as given by

$$\phi(x,y) = \begin{cases} e^{iKx-Ky} + Re^{-iKx-Ky} + \frac{\partial^2}{\partial y^2}\int_0^\infty \frac{A(\xi)}{\xi^2}[\xi\cos\xi y - \sin\xi y]e^{\xi x}\,d\xi,\ x<0 \\ Te^{iKx-Ky} - \frac{d^2}{dy^2}\int_0^\infty \frac{A(\xi)}{\xi^2}[\xi\cos\xi y - \sin\xi y]e^{-\xi x}\,d\xi,\ x>0. \end{cases} \tag{3.32}$$

Utilizing the relation (3.32) into the relation (3.25), we get

$$\frac{4}{\pi}\frac{d}{dy^2}\int_0^\infty \frac{A(\xi)}{\xi^2}(\xi\cos\xi y - K\sin\xi y)d\xi = -2Re^{-Ky} - \widehat{f}(y), \tag{3.33}$$

giving

$$A(\xi) = -\frac{1}{2(\xi^2+K^2)}\int_B \widehat{f}(y)(\xi\cos\xi y - K\sin\xi y)dy \tag{3.34}$$

and

$$R = -K\int_B \widehat{f}(y)e^{-Ky}dy \tag{3.35}$$

Utilizing the conditions (3.3), (3.34) and the relation, derived in Ursell [3],

$$\int_0^\infty \frac{(\xi\cos\xi u - K\sin\xi u)(\xi\cos\xi y - K\sin\xi y)}{\xi(\xi^2+K^2)}d\xi = \log\left|\frac{y+u}{y-u}\right| - 2e^{-K(y+u)}\int_{-\infty}^{K(y+u)}\frac{e^v}{v}dv,$$

we get a hyper singular integral equation for $\widehat{f}(y)$ as

$$\int_B \widehat{f}(u)\left[\frac{1}{(y-u)^2} + P(y,u)\right]du = 2i\pi Ke^{-Ky}, \tag{3.36}$$

where

$$P(y,u) = \frac{[1+2K(y+u)]}{(y+u)^2} + 2i\pi K^2e^{-K(y+u)} - 2K^2e^{-K(y+u)}\int_{-\infty}^{K(y+u)}\frac{e^v}{v}dv,$$

which can be solved numerically (see [7], [8], [9], [10] etc.). Consequently, we see that the reflection coefficient R can be determined from the relation (3.35).

3.2 Non-vertical barrier problem

Since the Havelock's type of expansion for the velocity potential is not possible in the case of the non-vertical barrier, Green's function technique has been used to solve the problem under consideration semi-analytically (see Porter [4]). Utilizing the Cauchy-Riemann type conditions and changing the normal derivatives into tangential derivatives, a weakly singular integral equation can be obtained and henceforth can be solved numerically.
In order to solve the non-vertical barrier problem we use the Green's identity as given by

$$\int\int_D(\phi\nabla^2G - G\nabla^2\phi)dxdy = \int_B(\phi\frac{\partial G}{\partial n} - G\frac{\partial\phi}{\partial n})db, \tag{3.37}$$

where B denotes the boundary of the region D, b measures the arc-length on B and $\frac{\partial}{\partial n}$ is the normal derivative on B outward from D.

With the parameterization $(x, y) = (X(\theta), Y(\theta))$, $0 \leq \theta < 2\pi$, for the boundary of the obstacle B, in the positive sense, we define

$$\left.\begin{aligned} \frac{\partial}{\partial n} &= \frac{1}{\sigma(\theta)}(Y'(\theta)\frac{\partial}{\partial x} - X'(\theta)\frac{\partial}{\partial y}, \\ \frac{\partial}{\partial b} &= \frac{1}{\sigma(\theta)}(X'(\theta)\frac{\partial}{\partial x} + Y'(\theta)\frac{\partial}{\partial y}, \end{aligned}\right\} \quad 0 \leq \theta < 2\pi, \tag{3.38}$$

where $\sigma(\theta) = \sqrt{(X'(\theta))^2 + (Y'(\theta))^2}$, $\quad 0 \leq \theta < 2\pi$ is a scaling factor and $\frac{\partial}{\partial n}$ is the inward normal to the obstacle and $\frac{\partial}{\partial b}$ is the tangential derivative.
The appropriate Green's function $G(x, y|x_0, y_0)$ for the problem satisfying the partial differential equation

$$\nabla^2 G = \delta(x - x_0)\delta(y - y_0) \;\text{ in }\; -\infty < x < \infty,\; 0 < y < \infty, \tag{3.39}$$

where $-\infty < x_0 < \infty$, $0 < y < \infty$, the boundary conditions (3.2) and (3.4), and the radiation condition

$$G(x, y|x_0, y_0) \sim -ie^{-K(y+y_0)+iK|x-x_0|}, \;\text{ as }\; |x - x_0| \to \infty, \tag{3.40}$$

is given by

$$\begin{aligned} G(x, y|x_0, y_0) &= \frac{1}{2\pi}\log\left(\frac{r_0}{r_1}\right) - \frac{1}{\pi}\int_0^\infty \frac{\xi\cos\xi(y + y_0) - K\sin\xi(y + y_0)}{\xi^2 + K^2} e^{-\xi|x-x_0|}d\xi \\ &\quad -ie^{-K(y+y_0)+iK|x-x_0|}, \end{aligned} \tag{3.41}$$

where

$$r_0^2 = (x - x_0)^2 + (y - y_0)^2, \quad r_1^2 = (x - x_0)^2 + (y + y_0)^2 \tag{3.42}$$

Application of the Green's identity (3.37) to the functions ϕ and G in the region $D = \{-X < x < X,\; 0 < y < \infty\}$, where X is large enough to accomodate the barrier and the singular point (x_0, y_0), gives the integral representation for the function ϕ as

$$\phi(x_0, y_0) = e^{-Ky_0+iKx_0} + \int_B \phi(x, y)\frac{\partial}{\partial n}G(x, y|x_0, y_0)db \;\; (x_0, y_0) \in D, \tag{3.43}$$

with

$$R = -i\int_B \phi(x, y)\frac{\partial}{\partial n}(e^{-Ky+iKx})db \tag{3.44}$$

and

$$T = 1 - i\int_B \phi(x, y)\frac{\partial}{\partial n}(e^{-Ky-iKx})db, \tag{3.45}$$

where n is the inward normal to the boundary of the obstacle B.

Moving the point (x_0, y_0) on to the barrier B, we get an integral equation of the second kind to solve for the velocity potential ϕ as given by

$$\frac{1}{2}\phi(x_0, y_0) - \int_B \phi(x, y)\frac{\partial}{\partial n}G(x, y|x_0, y_0)db = e^{-Ky_0+iKx_0} \quad (x_0, y_0) \in B, \tag{3.46}$$

Applying normal derivative to both sides of the relation (3.43), and utilizing the Cauchy-Riemann conditions as given by

$$\left.\begin{array}{ll} \textbf{(i)} & \frac{\partial \phi}{\partial n} = -\frac{\partial \psi}{\partial b},\ \psi \text{ being the stream function,} \\ \textbf{(ii)} & \frac{\partial}{\partial b}(e^{-Ky\pm iKx}) = \mp\frac{\partial}{\partial n}(-ie^{-Ky\pm iKx}), \end{array}\right\} \tag{3.47}$$

and

$$\frac{\partial^2 G}{\partial n_0 \partial n} = -\frac{\partial^2 H}{\partial b_0 \partial b}, \tag{3.48}$$

where

$$\begin{aligned} H(x, y|x_0, y_0) &= \frac{1}{2\pi}\log(r_0 r_1) + \frac{1}{\pi}\int_0^\infty \frac{\xi\cos\xi(y+y_0) - K\sin\xi(y+y_0)}{\xi^2+K^2}e^{-\xi|x-x_0|}d\xi \\ &\quad +ie^{-K(y+y_0)+iK|x-x_0|}, \end{aligned} \tag{3.49}$$

we get the following integral relation:

$$-\frac{\partial}{\partial b_0}\psi(x_0, y_0) = \frac{\partial}{\partial b_0}(-ie^{-Ky+iKx}) - \frac{\partial}{\partial b_0}\int_B \phi(x, y)\frac{\partial}{\partial b}H(x, y|x_0, y_0)db, \quad (x_0, y_0) \in D. \tag{3.50}$$

Integrating the above equation (3.50) with respect to b_0 and using integration by parts to transfer the tangential derivative from H to ϕ gives the following integral equation, after moving the point (x_0, y_0) on to the obstacle B:

$$\int_B H(x, y|x_0, y_0)\frac{\partial}{\partial b}\phi(x, y)db = -ie^{-Ky+iKx} + \psi_B, \tag{3.51}$$

where ψ_B is the value of the stream function at (x_0, y_0) which is considered to be an arbitrary constant.

The equation (3.51), is a weakly singular integral equation which can be solved numerically for specific barrier configurations. Porter [4] has solved this weakly singular integral equation numerically when the obstacle boundary B is fully immersed closed curve by using Galerkin method.

In order to calculate the reflection and transmission coefficients R and T, one utilizes the relation **(ii)** of (3.47) into the relations (3.44)-(3.45) and uses integration by parts to transfer the tangential derivative onto the function ϕ, giving

$$R = \int_B \frac{\partial}{\partial b}\phi(x, y)e^{-Ky+iKx}db \tag{3.52}$$

and

$$T = 1 - \int_B \frac{\partial}{\partial b}\phi(x, y)e^{-Ky-iKx}db. \tag{3.53}$$

4 Conclusions

Varieties of types of integral equations have been shown above to occur in the study of the scattering of surface water waves by obstacles present in deep water, under the assumption of the linearized theory. The present observations lead to conclude that only weakly singular integral equations (Cf. equations (2.1), (2.3), (2.7), (3.20)-(3.24), (3.31), (3.51)) need to be handled for complete solutions of the problems at hand and thus, singularities of higher order (including hypersingular integrals) can be totally avoided for these problems.

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