A note on surface water waves for finite depth in the presence of an ice-cover

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Abstract

A class of boundary value problems involving propagation of two-dimensional surface water waves,associated with water of uniform finite depth,against a plane vertical wave maker is investigated under the assumption that the surface is covered by a thin sheet of ice. It is assumed that the ice-cover behaves like a thin isotropic elastic plate.Then the problems under consideration lead to those of solving the two-dimensional Laplace equation in a semi-infinite strip,under Neumann boundary conditions on the vertical boundary as well as on one of the horizontal boundaries, representing the bottom of the fluid region, and a condition involving upto fifth order derivatives of the unknown function on the top horizontal ice-covered boundary,along with two appropriate edge-conditions ,at the ice-covered corner,ensuring the uniqueness of the solutions. The mixed boundary value problems are solved completely, by exploiting the regularity property of the Fourier cosine transform.

1 Introduction

The problem concerning the propagation of two-dimensional time-harmonic surface waves in the case of deep water against a vertical cliff was considered long ago by Stoker[11] under the assumption that surface tension was negligible. In order to pose the problem well enough for its mathematical treatment for solution, a source/sink like behavior of the associated irrotational fluid motion was assumed and the resulting boundary value problem for the Laplace's equation was solved completely (see Chakrabarti [1], Mandal and Kundu [8], Chakrabarti and Sahoo [2]).The effect of surface tension in this type of surface wave phenomena was investigated by Packham [9] who showed that the assumption on the source/sink like behavior of the flow was not necessary to specify the mathematical boundary value problem completely. Packham also assumed that there was no reflection by the wall, for an incoming

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plane wave-train. Rhodes-Robinson [10] re-investigated the problem of Packham and had shown that there should exist reflected waves against a vertical wall, under the influence of surface tension and that the associated boundary value problem for the quarter-plane could be solved only within an unknown constant which was related to the contact angle of the surface elevation at the vertical wall.

In the present paper we have investigated a class of such surface water wave problems,in the case when the depth of the water is finite, under the assumption that there exists a thin ice-cover on the top surface. Such two-dimensional problems, within the linearised theory of surface water waves, in the presence of an ice-cover, can be formulated (Gol'dshtein and Marchenko [5] and Fox and Squire [4]) in terms of the two-dimensional Laplace's equation for the velocity potential $\mathrm{Re}\{\phi(x,y)e^{-it}\}$ (x, y represent non-dimensional cartesian coordinates and t represents non-dimensional time, with $i^2 = -1$). The boundary condition on the ice-cover, i.e, on $y = 0$, is given by the relation: $D\dfrac{\partial^5\phi}{\partial y\partial x^4}+\dfrac{\partial\phi}{\partial y}+\phi = 0$, in which the small positive constant D is known in terms of the constant density of the water under consideration,the acceleration due to gravity, the wave-length of the propagating surface water waves as well as the Young's modulus and the Poisson's ratio of the ice-sheet, considered as an isotropic homogeneous thin elastic plate (see [5] for details). On the plane vertical boundary $x = 0$, we assume a general condition $\left.\dfrac{\partial\phi}{\partial x}\right|_{x=0} = u(y)$, $(0 \le y \le H)$ to hold good, where $u(y)$ is a known differentiable function of y, giving rise to different wave problems for different choices of u(y). The boundary at the bottom, $y = H$,is assumed to be rigid, on which holds the Neumann condition $\dfrac{\partial\phi}{\partial y} = 0$. We also require two edge-conditions at the ice-covered edge $x \to 0^+, y = 0$, for the purpose of obtaining the unique solutions of the boundary value problems under our consideration.

We have described in the next three sections, the mathematical formulation of the boundary value problems and the method of solution.

We have employed a Fourier cosine transform technique to solve the boundary value problems under consideration and have determined the complete solutions of the problems by exploiting the regularity property of Fourier transform, as in the work of Mandal and Bandyopadhyay[7].

Because of the complexity of the boundary condition on the boundary $y = 0$ of the quarter-plane, the algebraic calculations are a bit involved and we have finally determined the reflection coefficients under special choices of the function $u(y)$,numerically, for various values of the parameter D and the depth H of the fluid. The numerical results are presented in a tabular forms.

2 The boundary value problems

The class of boundary value problems under consideration,is formulated, by using the notations of [5], as described below.

We consider the irrotational motion of an incompressible inviscid fluid due to a harmonically oscillating vertical plane wavemaker under the action of gravity. We use a rectangular cartesian coordinate system in which the y-axis is taken vertically downwards so that $y = 0$, $x > 0$ is the undisturbed ice-covered surface and $x = 0$ is the wavemaker, and in the undisturbed state the fluid occupies the region $x > 0$ and $0 \leq y \leq H$. The motion is assumed to be two-dimensional and time-harmonic and is described by a velocity potential $\Phi(x, y, t)$ which is the real part of $\phi(x, y)e^{-it}$, where t denotes the non-dimensional time. The time-dependent factor e^{-it} is suppressed throughout the analysis.

The function $\phi(x, y)$ satisfies the p.d.e.

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \text{ in } x > 0, 0 < y < H, \tag{2.1}$$

with

$$D\frac{\partial^5\phi}{\partial y\partial x^4} + \frac{\partial\phi}{\partial y} + \phi = 0, \text{ on } y = 0, x > 0, \tag{2.2}$$

$$\frac{\partial\phi}{\partial x} = u(y), \text{ on } x = 0, y > 0, \tag{2.3}$$

$$\frac{\partial\phi}{\partial y} = 0, \text{ on } x > 0, y = H. \tag{2.4}$$

where $D(> 0)$, is a known constant and $u(y)$ is a known function.

Also, $\phi(x, y)$ behaves like $\cosh[\lambda(H - y)]\ e^{i\lambda x}$, as $x \to \infty$, where λ is the positive root of the transcendental equation

$$\alpha(1 + D\alpha^4)\tanh\lambda H - 1 = 0, \tag{2.5}$$

which ensures the wave-like behavior for ϕ as $x \to \infty$.

Finally, for the uniqueness of the solution of the above boundary value problem, we must impose two appropriate edge-conditions,at the corner (edge) point $(x \to 0^+, y \to 0)$. The relevant edge-conditions are given by

$$(i)\ \frac{\partial^4\phi}{\partial y\partial x^3} \to \mu_1 \text{ (a known constant), as } x \to 0^+, y \to 0, \tag{2.6}$$

and

$$(ii)\ \frac{\partial^3\phi}{\partial y\partial x^2} \to \mu_2 \text{ (a known constant), as } x \to 0^+, y \to 0. \tag{2.7}$$

which are related to the concentrated force and concentrated moment at the ice-edge (see [5]), respectively.

The equation (2.5) has a similar structure as the one appearing in the work of Balmforth and Craster while modelling an ice-cover(see [3]) and it has been shown in the Appendix that the equation does not have any other root except one positive real root λ, four complex conjugate roots $\lambda_1, \overline{\lambda}_1, \lambda_2, \overline{\lambda}_2$ with $Re(\lambda_1) > 0, Re(\lambda_2) < 0$ and $Im(\lambda_1) > 0, Im(\lambda_2) > 0$ and an infinite number of imaginary roots, $k = ik_n$ (k_n real, n= 1, 2, 3,...).

3 The method of solution

By seperating out all possible wave-like solutions, we write the general solution of the problem (2.1)–(2.4) as,

$$\begin{aligned}\phi(x,y) \;=\; & A_0\frac{\cosh[\lambda(H-y)]}{\cosh[\lambda H]}\,e^{-i\lambda x} + R\frac{\cosh[\lambda(H-y)]}{\cosh[\lambda H]}\,e^{i\lambda x} + A_1\frac{\cosh[\lambda_1(H-y)]}{\cosh[\lambda_1 H]}\,e^{-i\lambda_1 x}\\ & + A_2\frac{\cosh[\overline{\lambda}_1(H-y)]}{\cosh[\overline{\lambda}_1 H]}\,e^{-i\bar{\lambda}_1 x} + \Psi(x,y),\end{aligned} \tag{3.1}$$

where R, A_1 and A_2 are unknowns to be determined whilst the constant A_0 is assumed to be known and $\Psi(x, y)$, is an unknown function, which satisfies the following equation and conditions.

$$\nabla^2\Psi(x,y) = 0, \text{ in } x > 0, 0 < y < H, \tag{3.2}$$

$$D\frac{\partial^5\Psi}{\partial y\partial x^4} + \frac{\partial\Psi}{\partial y} + \Psi = 0, \text{ on } y = 0, x > 0, \tag{3.3}$$

$$\frac{\partial\Psi}{\partial x} = v(y), \text{ on } x = 0, y > 0, \tag{3.4}$$

$$\frac{\partial\Psi}{\partial y} = 0, \text{ on } x > 0, y = H, \tag{3.5}$$

where

$$v(y) = u(y) - i(R - A_0)\lambda\frac{\cosh[\lambda(H-y)]}{\cosh\lambda H} - iA_1\lambda_1\frac{\cosh[\lambda_1(H-y)]}{\cosh\lambda_1 H} + iA_2\overline{\lambda}_1\frac{\cosh[\overline{\lambda}_1(H-y)]}{\cosh\overline{\lambda}_1 H}.$$

The edge conditions (2.6) and (2.7), in terms of $\Psi(x,y)$, become

$$\begin{aligned}
(i)\ \frac{\partial^4\Psi}{\partial y\partial x^3} \to \quad & \mu_1 - i(R-A_0)\lambda^4\tanh\lambda H - iA_1\lambda_1^4\tanh\lambda_1 H \\
& + iA_2\overline{\lambda}_1^4\tanh\overline{\lambda}_1 H, \text{as } x\to 0^+\ , y\to 0, \\
(ii)\ \frac{\partial^3\Psi}{\partial y\partial x^2} \to \quad & \mu_2 - (R+A_0)\lambda^3\tanh\lambda H - A_1\lambda_1^3\tanh\lambda_1 H \\
& - A_2\overline{\lambda}_1^3\tanh\overline{\lambda}_1 H, \text{as } x\to 0^+\ , y\to 0. \qquad (3.6)
\end{aligned}$$

Now we utilize the Fourier cosine transform of Ψ to reduce the boundary value problem (3.2)-(3.5) into a new boundary value problem, as explained below.

We set

$$\chi(y,\xi) = \int_0^\infty \Psi(x,y)\cos\xi x dx.$$

Then the problem is to determine for $\chi(y,\xi)$ satisfying

$$\begin{aligned}
\frac{\partial^2\chi}{\partial y^2} - \xi^2\chi &= v(y), 0\le y\le H, \qquad (3.7)\\
\chi + (1+D\xi^4)\frac{\partial\chi}{\partial y} &= D[(\mu_1-\mu_3\xi^2) - i(R-A_0)\lambda^2(\xi^2+\lambda^2)\tanh\lambda H \\
&\quad - iA_1\lambda_1^2(\xi^2+\lambda_1^2)\tanh\lambda_1 H + iA_2\overline{\lambda}_1^2(\xi^2+\overline{\lambda}_1^2)\tanh\overline{\lambda}_1 H], \\
&\qquad \text{on } \ y=0, \quad (3.8)\\
\frac{\partial\chi}{\partial y} &= 0, \quad \text{on} \quad y=H. \qquad (3.9)
\end{aligned}$$

where $\frac{\partial^2\phi}{\partial y\partial x}(x,y) \to \mu_3$, (an unknown constant) , as $x\to 0^+$, $y\to 0$, giving

$$\frac{\partial^2\Psi}{\partial y\partial x}(x,y) \to \mu_3 + i(R-A_0)\lambda^2\tanh\lambda H + iA_1\lambda_1^2\tanh\lambda_1 H - iA_2\overline{\lambda}_1^2\tanh\overline{\lambda}_1 H,$$

as $x\to 0^+, y\to 0$,

We will finally determine μ_3 by using the edge condition (3.6).

Again, setting

$$\eta(y,\xi) = \chi(y,\xi) - f(\xi),$$

where

$$\begin{aligned}
f(\xi) = \quad & D[(\mu_1-\mu_3\xi^2) - i(R-A_0)\lambda^2(\xi^2+\lambda^2)\tanh\lambda H - iA_1\lambda_1^2(\xi^2+\lambda_1^2)\tanh\lambda_1 H \\
& + iA_2\overline{\lambda}_1^2(\xi^2+\overline{\lambda}_1^2)\tanh\overline{\lambda}_1 H].
\end{aligned}$$

the problem (3.7)–(3.9) becomes

$$\frac{\partial^2\eta}{\partial y^2} - \xi^2\eta = g(y,\xi), \tag{3.10}$$

$$\eta + (1 + D\xi^4)\frac{\partial\eta}{\partial y} = 0, \text{ on } y = 0, \tag{3.11}$$

$$\frac{\partial\eta}{\partial y} = 0, \text{ on } y = H. \tag{3.12}$$

where

$$g(y,\xi) = v(y) + \xi^2 f(\xi).$$

Now, the Green's function for the problem (3.10)–(3.12) is

$$G(y,t) = -\frac{\cosh[\xi(t-H)]P(y,\xi)}{\xi\Delta(\xi)}, \text{ for } 0 < y < t, \tag{3.13}$$

(For $0 \le t \le y$, y and t are to be interchanged in (3.13))

where

$$\begin{aligned} P(y,\xi) &= \{\xi(1 + D\xi^4)\cosh\xi y - \sinh\xi y\}, \\ \Delta(\xi) &= \{\xi(1 + D\xi^4)\cosh\xi H - \sinh\xi H\}. \end{aligned}$$

Using the above Green's function the solution for the problem (3.10)–(3.12) can be written as

$$\eta(y,\xi) = -\int_0^H \frac{P(y,\xi)}{\xi\Delta(\xi)}\cosh[\xi(t-H)]g(t,\xi)dt - \frac{1}{\xi}\int_0^y \sinh[\xi(t-y)]g(t,\xi)dt.$$

Therefore, the solution of the problem (3.7)–(3.9) can be expressed as

$$\chi(y,\xi) = (R-1)L(y,\xi,\lambda) + A_1 L(y,\xi,\lambda_1) - A_2 L(y,\xi,\overline{\lambda}_1) - M(y,\xi), \tag{3.14}$$

where

$$\begin{aligned} L(y,\xi,x) &= \frac{ixP(y,\xi)}{2\xi\Delta(\xi)\cosh xH}\left[\frac{\sinh[(\xi+x)H]}{\xi+x} + \frac{\sinh[(\xi-x)H]}{\xi-x}\right] \\ &\quad + iDx^2(\xi^2+x^2)\tanh xH\left[\frac{P(y,\xi)}{\Delta(\xi)}\sinh\xi H - \cosh\xi y\right] \\ &\quad + \frac{ix}{2\xi\cosh xH}\left[\frac{2\xi\cosh[x(H-y)]}{\xi^2-x^2} - \frac{\cosh[(xH+\xi y)]}{\xi+x} - \frac{\cosh[(xH-\xi y)]}{\xi-x}\right], \\ M(y,\xi) &= \frac{P(y,\xi)}{\xi\Delta(\xi)}\int_0^H u(t)\cosh[\xi(t-H)]dt + \frac{1}{\xi}\int_0^y u(t)\sinh[\xi(t-y)]dt \\ &\quad + D(\mu_1 - \mu_3\xi^2)\left[\frac{P(y,\xi)}{\Delta(\xi)}\sinh\xi H - \cosh\xi y\right]. \end{aligned}$$

Then the Fourier cosine inversion formula gives $\Psi(x,y)$ as

$$\Psi(x,y)=\frac{2}{\pi}\int_0^\infty \chi(y,\xi)\cos\xi x\;d\xi, \tag{3.15}$$

where $\chi(y,\xi)$ is given in (3.14).

The contour in the relation (3.15) can be extended to the whole real axis with 2 $\cos\xi x$ replaced by $e^{i\xi x}$. This may then be evaluated by the method of residues at the poles $k=ik_n$, for n = 1, 2,... . We note here that the poles at $\xi=\lambda,\lambda_1$ and $\overline{\lambda}_1$ can be removed by chosing the constants R, A_1 and A_2 suitably (This will be explained little later). Then we obtain

$$\Psi(x,y)=2i\sum_{j=1}^{\infty}e^{-k_jx}\left[(R-A_0)L^{'}(y,k_j,\lambda)+A_1L^{'}(y,k_j,\lambda_1)-A_2L^{'}(y,k_j,\overline{\lambda}_1)-M^{'}(y,k_j)\right], \tag{3.16}$$

where

$$\begin{aligned}
L^{'}(y,k_j,x) &= \frac{Q(y,k_j)}{2k_j\cosh xH}\left[x\Big\{\frac{\sin[(k_j-ix)H]}{k_j-ix}+\frac{\sin[(k_j+ix)H]}{k_j+ix}\Big\}\right.\\
&\left.-2Dk_jx^2(x^2-k_j^2)\sinh xH\sin k_jH\right]\\
&-\frac{ix}{k_j(x^2+K_j^2)}\left[k_j(\cosh xy-\tanh xH\sinh xy-\cos k_jy)+x\tanh xH\sin k_jy\right]\\
&-iDx^2(x^2-k_j^2)\cos k_jy\tanh xH, \qquad (3.17)\\
M^{'}(y,k_j) &= \frac{Q(y,k_j)}{k_j}\left[-i\int_0^H u(t)\cos[k_j(t-H)]dt+iDk_j(\mu_1+\mu_3k_j^2)\sin k_jH\right]\\
&+\frac{1}{k_j}\int_0^y u(t)\sin[k_j(t-y)]dt-D(\mu_1+\mu_3k_j^2)\cos k_jy.
\end{aligned}$$

with

$$Q(y,k_j)=\frac{k_j(1+Dk_j^4)\cos k_jy-\sin k_jy}{(1+5Dk_j^4)\sin k_jH+H\{k_j(1+Dk_j^4)\cos k_jH-\sin k_jH\}}.$$

By some simple algebraic manipulations ,utilizing the second edge condition (3.6), we get the unknown constant μ_3 as:

$$\mu_3=\frac{1}{I_5}[(R-A_0)P_1+A_1P_2-A_2P_3-P_4], \tag{3.18}$$

where

$$P_1 = 2i\sum_{j=1}^{\infty}k_j^2L^{''}(\lambda,k_j)+\lambda^3\tanh\lambda H,$$

$$
\begin{aligned}
P_2 &= 2i\sum_{j=1}^{\infty} k_j^2 L''(\lambda_1, k_j) + \lambda_1^3 \tanh \lambda_1 H, \\
P_3 &= 2i\sum_{j=1}^{\infty} k_j^2 L''(\overline{\lambda}_1, k_j) - \overline{\lambda}_1^3 \tanh \overline{\lambda}_1 H, \\
P_4 &= \mu_2 - 2\lambda^3 \tanh \lambda H + \mu_1 I_3 - 2\sum_{j=1}^{\infty} \frac{k_j^2 \int_0^H u(t)\cos[k_j(t-H)]dt}{\sin k_j H\left[-H\{k_j^2(1+Dk_j^4)^2+1\}+(1+5Dk_j^4)\right]},
\end{aligned}
$$

with

$$
\begin{aligned}
L''(x, k_j) &= \frac{-x\left(\dfrac{\sin[(k_j-ix)H]}{k_j-ix} + \dfrac{\sin[(k_j+ix)H]}{k_j+ix}\right) + 2Dk_j x^2(x^2-k_j^2)\sinh xH \sin k_j H}{2\cosh xH \sin k_j H\left[-H\{k_j^2(1+Dk_j^4)^2+1\}+(1+5Dk_j^4)\right]}, \\
I_3 &= 2D\sum_{j=1}^{\infty} \frac{k_j^3}{\left[-H\{k_j^2(1+Dk_j^4)^2+1\}+(1+5Dk_j^4)\right]}, \\
I_5 &= 2D\sum_{j=1}^{\infty} \frac{k_j^5}{\left[-H\{k_j^2(1+Dk_j^4)^2+1\}+(1+5Dk_j^4)\right]}.
\end{aligned}
$$

At this stage, we can obtain the unknown function $\Psi(x, y)$ in terms of the unknown constants R, A_1 and A_2 which are determined as described below.

Since $\chi(y, \xi)$ is the Fourier cosine transform of the function $\Psi(x, y)$, then, treated as a function of the complex variable ξ, $\chi(y, \xi)$ cannot have singularities in the half plane $Re(\xi) > 0$. This forces us to choose the unknown constants $(R - A_0), A_1$ and A_2 in such a way that the function $\chi(y, \xi)$, as given by the relation (3.14), is analytic in the half plane $Re(\xi) > 0$. In other words, the constants $(R - A_0), A_1$ and A_2 must be so chosen as to meet the regularity requirements of the function $\chi(y, \xi)$ at the points λ, λ_1 and $\overline{\lambda}_1$.

The above regularity considerations give rise to the following system of linear equations for the determination of the unknown constants $(R - A_0), A_1$ and A_2:

$$
\left.
\begin{aligned}
r_1(R-A_0) + a_1A_1 + b_1A_2 &= s_1 \\
r_2(R-A_0) + a_2A_1 + b_2A_2 &= s_2 \\
r_3(R-A_0) + a_3A_1 + b_3A_2 &= s_3
\end{aligned}
\right\} \qquad (3.19)
$$

where

$$
a_1 = \frac{i\lambda_1}{2\cosh \lambda_1 H}\Big[\frac{\sinh[(\lambda+\lambda_1)H]}{\lambda+\lambda_1} + \frac{\sinh[(\lambda-\lambda_1)H]}{\lambda-\lambda_1}\Big] - \frac{iD\lambda_1\lambda^3 \sinh \lambda H}{I_5 \cosh \lambda_1 H} P(\lambda_1)
$$

$$- \left[2iD^2\lambda^3\lambda_1^2 - iD\lambda\lambda_1^2(\lambda^2+\lambda_1^2) - \frac{\{D\lambda^3\lambda_1^3 + 2iD^2\lambda^3\lambda_1^4 I_3\}}{I_5} \right] \sinh\lambda H \tanh\lambda_1 H,$$

$$a_2 = \frac{i}{4\cosh\lambda_1 H}\Big[\sinh[2\lambda_1 H] + 2\lambda_1 H\Big] - \frac{iD\lambda_1^4\tanh\lambda_1 H}{I_5}P(\lambda_1)$$
$$- \left[2iD(D-1)\lambda_1^5 - \frac{\{D\lambda_1^6 + 2iD^2\lambda_1^7 I_3\}}{I_5} \right] \sinh\lambda_1 H \tanh\lambda_1 H,$$

$$a_3 = \frac{i\lambda_1}{2\cosh\lambda_1 H}\Big[\frac{\sinh[(\overline{\lambda}_1+\lambda_1)H]}{\overline{\lambda}_1+\lambda_1} + \frac{\sinh[(\overline{\lambda}_1-\lambda_1)H]}{\overline{\lambda}_1-\lambda_1}\Big] - \frac{iD\lambda_1\overline{\lambda}_1^3\sinh\overline{\lambda}_1 H}{I_5\cosh\lambda_1 H}P(\lambda_1)$$
$$- \left[2iD^2\overline{\lambda}_1^3\lambda_1^2 - iD\overline{\lambda}_1\lambda_1^2(\overline{\lambda}_1^2+\lambda_1^2) - \frac{\{D\overline{\lambda}_1^3\lambda_1^3 + 2iD^2\overline{\lambda}_1^3\lambda_1^4 I_3\}}{I_5} \right] \sinh\overline{\lambda}_1 H \tanh\lambda_1 H,$$

$$b_1 = -\frac{i\overline{\lambda}_1}{2\cosh\overline{\lambda}_1 H}\Big[\frac{\sinh[(\lambda+\overline{\lambda}_1)H]}{\lambda+\overline{\lambda}_1} + \frac{\sinh[(\lambda-\overline{\lambda}_1)H]}{\lambda-\overline{\lambda}_1}\Big] + \frac{iD\overline{\lambda}_1\lambda^3\sinh\lambda H}{I_5\cosh\overline{\lambda}_1 H}P(\overline{\lambda}_1)$$
$$+ \left[2iD^2\lambda^3\overline{\lambda}_1^2 - iD\lambda\overline{\lambda}_1^2(\lambda^2+\overline{\lambda}_1^2) + \frac{\{D\lambda^3\overline{\lambda}_1^3 - 2iD^2\lambda^3\overline{\lambda}_1^4 I_3\}}{I_5} \right] \sinh\lambda H \tanh\overline{\lambda}_1 H,$$

$$b_2 = -\frac{i\overline{\lambda}_1}{2\cosh\overline{\lambda}_1 H}\Big[\frac{\sinh[(\lambda_1+\overline{\lambda}_1)H]}{\lambda_1+\overline{\lambda}_1} + \frac{\sinh[(\lambda_1-\overline{\lambda}_1)H]}{\lambda_1-\overline{\lambda}_1}\Big] + \frac{iD\overline{\lambda}_1\lambda_1^3\sinh\lambda_1 H}{I_5\cosh\overline{\lambda}_1 H}P(\overline{\lambda}_1)$$
$$+ \left[2iD^2\lambda_1^3\overline{\lambda}_1^2 - iD\lambda_1\overline{\lambda}_1^2(\lambda_1^2+\overline{\lambda}_1^2) + \frac{\{D\lambda_1^3\overline{\lambda}_1^3 - 2iD^2\lambda_1^3\overline{\lambda}_1^4 I_3\}}{I_5} \right] \sinh\lambda_1 H \tanh\overline{\lambda}_1 H,$$

$$b_3 = -\frac{i}{4\cosh\overline{\lambda}_1 H}\Big[\sinh[2\overline{\lambda}_1 H] + 2\overline{\lambda}_1 H\Big] + \frac{iD\overline{\lambda}_1^4\tanh\overline{\lambda}_1 H}{I_5}P(\overline{\lambda}_1)$$
$$+ \left[2iD(D-1)\overline{\lambda}_1^5 + \frac{\{D\overline{\lambda}_1^6 - 2iD^2\overline{\lambda}_1^7 I_3\}}{I_5} \right] \sinh\overline{\lambda}_1 H \tanh\overline{\lambda}_1 H,$$

$$r_1 = \frac{i}{4\cosh\lambda H}\Big[\sinh[2\lambda H] + 2\lambda H\Big] - \frac{iD\lambda^4\tanh\lambda H}{I_5}P(\lambda)$$
$$- \left[2iD(D-1)\lambda^5 - \frac{\{D\lambda^6 + 2iD^2\lambda^7 I_3\}}{I_5} \right] \sinh\lambda H \tanh\lambda H,$$

$$r_2 = \frac{i\lambda}{2\cosh\lambda H}\Big[\frac{\sinh[(\lambda_1+\lambda)H]}{\lambda_1+\lambda} + \frac{\sinh[(\lambda_1-\lambda)H]}{\lambda_1-\lambda}\Big] - \frac{iD\lambda\lambda_1^3\sinh\lambda_1 H}{I_5\cosh\lambda H}P(\lambda)$$
$$- \left[2iD^2\lambda_1^3\lambda^2 - iD\lambda_1\lambda^2(\lambda_1^2+\lambda^2) - \frac{\{D\lambda_1^3\lambda^3 + 2iD^2\lambda_1^3\lambda^4 I_3\}}{I_5} \right] \sinh\lambda_1 H \tanh\lambda H,$$

$$
\begin{aligned}
r_3 &= \frac{i\lambda}{2\cosh\lambda H}\Big[\frac{\sinh[(\overline{\lambda}_1+\lambda)H]}{\overline{\lambda}_1+\lambda}+\frac{\sinh[(\overline{\lambda}_1-\lambda)H]}{\overline{\lambda}_1-\lambda}\Big]-\frac{iD\lambda\overline{\lambda}_1^3\sinh\overline{\lambda}_1 H}{I_5\cosh\lambda H}P(\lambda) \\
&\quad -\left[2iD^2\overline{\lambda}_1^3\lambda^2-iD\overline{\lambda}_1\lambda^2(\overline{\lambda}_1^2+\lambda^2)-\frac{\{D\overline{\lambda}_1^3\lambda^3+2iD^2\overline{\lambda}_1^3\lambda^4 I_3\}}{I_5}\right]\sinh\overline{\lambda}_1 H\tanh\lambda H, \\
s_1 &= \int_0^H u(t)\cosh[\lambda(t-H)]dt+\mu_1 D\lambda\Big[1+\lambda^2\frac{I_3}{I_5}\Big]\sinh\lambda H \\
&\quad -\frac{D\lambda^3\sinh\lambda H}{I_5}\Big[2\lambda^3\tanh\lambda H-\mu_2+2\sum_{j=1}^{\infty}\frac{k_j^2\int_0^H u(t)\cos[k_j(t-H)]dt}{\sin k_j H\left[-H\{k_j^2(1+Dk_j^4)^2+1\}+(1+5Dk_j^4)\right]}\Big], \\
s_2 &= \int_0^H u(t)\cosh[\lambda_1(t-H)]dt+\mu_1 D\lambda_1\Big[1+\lambda_1^2\frac{I_3}{I_5}\Big]\sinh\lambda_1 H \\
&\quad -\frac{D\lambda_1^3\sinh\lambda_1 H}{I_5}\Big[2\lambda^3\tanh\lambda H-\mu_2+2\sum_{j=1}^{\infty}\frac{k_j^2\int_0^H u(t)\cos[k_j(t-H)]dt}{\sin k_j H\left[-H\{k_j^2(1+Dk_j^4)^2+1\}+(1+5Dk_j^4)\right]}\Big], \\
s_3 &= \int_0^H u(t)\cosh[\overline{\lambda}_1(t-H)]dt+\mu_1 D\overline{\lambda}_1\Big[1+\overline{\lambda}_1^2\frac{I_3}{I_5}\Big]\sinh\overline{\lambda}_1 H \\
&\quad -\frac{D\overline{\lambda}_1^3\sinh\overline{\lambda}_1 H}{I_5}\Big[2\lambda^3\tanh\lambda H-\mu_2+2\sum_{j=1}^{\infty}\frac{k_j^2\int_0^H u(t)\cos[k_j(t-H)]dt}{\sin k_j H\left[-H\{k_j^2(1+Dk_j^4)^2+1\}+(1+5Dk_j^4)\right]}\Big],
\end{aligned}
$$

with

$$
P(x) = 2\sum_{j=1}^{\infty}\frac{k_j^3\{\cosh xH-x(1+Dk_j^4)\sinh xH\}}{(k_j^2+x^2)\left[-H\{k_j^2(1+Dk_j^4)^2+1\}+(1+5Dk_j^4)\right]}.
$$

We find that the constants $R, A_1 \text{and} A_2$, satisfying the system (3.19), are given by the following formulae:

$$
\left.
\begin{aligned}
R &= A_0+\frac{[s_3(a_1b_2-a_2b_1)-a_3(b_2s_1-b_1s_2)-b_3(a_1s_2-a_2s_1)]}{[r_3(a_1b_2-a_2b_1)-a_3(b_2r_1-b_1r_2)-b_3(a_1r_2-a_2r_1)]}, \\
A_1 &= \frac{[(b_2s_1-b_1s_2)+(R-A_0)(b_1r_2-b_2r_1)]}{[a_1b_2-a_2b_1]}, \\
A_2 &= \frac{[(a_1s_2-a_2s_1)+(R-A_0)(a_2r_1-a_1r_2)]}{[a_1b_2-a_2b_1]}.
\end{aligned}
\right\} \tag{3.20}
$$

The final form of the solution $\phi(x,y)$ can be obtained by using the relations (3.20) in the relations (3.16), (3.18) and (3.1).

4 Numerical results for a special case

As a special case of the general problems considered in the present work, we take up the problem of reflection of incoming surface water waves against a vertical cliff, under an ice-cover. In such circumstances, we must choose $A_0 = 1$ and $u(y) = 0$. We have selected various combinations of the values of the edge constants μ_1 and μ_2 for the choices of the depths $H = 10$ and $H = 50$ and have determined the reflection coefficient R numerically , for particular choices of the ice-cover parameter D, in the case of the problem of an incoming surface water wave against a vertical cliff. The absolute values of the reflection coefficient are found to be almost equal to unity in all the cases considered, as is expected because of energy considerations since we have not assumed any source/sink like behaviour of the fluid motion, at the corner point $x = 0, y = 0$. The tables provided below give the values of R and its absolute value $|R|$, for different values of D, obtained by the method explained above.

D	R	$\|R\|$
0.01	0.9998121960445526 + i 0.019379696606730636	0.9999999
0.02	0.9990229484796528 + i 0.04419443868883628	1
0.03	0.9975596108733402 + i 0.06981993092398515	1
0.04	0.9954739736306375 + i 0.09503456120816377	0.9999999
0.05	0.9928543287579676 + i 0.11933265213916151	1

Table1 : Numerical values of R , $|R|$ for $\mu_1 = 0$, $\mu_2 = 0$ and $H = 10$.

D	R	$\|R\|$
0.01	0.999902192372107 + i 0.01009287663764384	0.9999531
0.02	0.9994730732258614 + i 0.02383419147118705	0.9997572
0.03	0.9986755885068032 + i 0.037891628822523844	0.9993941
0.04	0.9975509421349259 + i 0.05142372593236471	0.9988755
0.05	0.9961612199939124 + i 0.06410759515644035	0.9982218

Table2 : Numerical values of R , $|R|$ for $\mu_1 = 0.2$, $\mu_2 = 0.5$ and $H = 10$.

D	R	$\|R\|$
0.01	0.9999825247692546 + i 0.0018032882703128267	0.9999841
0.02	0.9998715714242288 + i 0.005809139742992588	0.9998884
0.03	0.9996594052376961 + i 0.009744471698114134	0.9997068
0.04	0.9993770975752817 + i 0.013079300423291319	0.9994626
0.05	0.9990634794928692 + i 0.01563988491959552	0.9991858

Table3 : Numerical values of R , $|R|$ for $\mu_1 = 0.4$, $\mu_2 = 0.9$ and $H = 10$.

D	R	$\|R\|$
0.01	0.9998072463903708 + i 0.019633391589439633	0.99999
0.02	0.9989930890674694 + i 0.04486432876390823	1
0.03	0.9974791633747385 + i 0.07095997909547949	0.99999
0.04	0.9953177815442427 + i 0.0966566797580392	1
0.05	0.9926004755974633 + i 0.12142609210416275	0.99999

Table4 : Numerical values of R , $|R|$ for $\mu_1 = 0$, $\mu_2 = 0$ and $H = 50$.

D	R	$\|R\|$
0.01	0.9998990663673147 + i 0.010280842671947631	0.9999519
0.02	0.999454094020879 + i 0.024323606517907086	0.9997500
0.03	0.9986246898420348 + i 0.038714123351378096	0.9993748
0.04	0.9974529074587096 + i 0.052580525749461225	0.9988378
0.05	0.9960033813631122 + i 0.06558445601473398	0.9981603

Table5 : Numerical values of R , $|R|$ for $\mu_1 = 0.2$, $\mu_2 = 0.5$ and $H = 50$.

D	R	$\|R\|$
0.01	0.9999809677145101 + i 0.0019385801105487235	0.9999828
0.02	0.9998618946804774 + i 0.006153476200252212	0.9998808
0.03	0.9996336987117664 + i 0.010311152851099582	0.9996868
0.04	0.9993285647764047 + i 0.013860673097277626	0.9994246
0.05	0.9989872810617524 + i 0.01661870363305906	0.9991255

Table6 : Numerical values of R , $|R|$ for $\mu_1 = 0.4$, $\mu_2 = 0.9$ and $H = 50$.

It is observed that the values of $|R|$ are almost equal to unity and this fact can be attributed to the special choice of the edge constants in our numerical work. We emphasize that the values of the reflection coefficient have been computed by using the approximate values of the roots of the transcendental equation (2.5).

Conclusions

A special class of mixed boundary value problems, involving the two-dimensional Laplace's equation in a semi-infinite strip, have been considered for their solutions, this class being the appropriate one in connection with the propagation of linearised surface water waves in the case of finite depth of fluid, in the presence of an ice-cover . A method of solution has been demonstrated which utilizes the well-known Fourier analysis. Numerical results have been determined for the reflection coefficients of waves against a vertical cliff, for special choices of the various parameters involved.

Appendix

It is easy to demonstrate, numerically, that the transcendental equation

$$f(z) \equiv z(1+Dz^4)\sinh zH - \cosh zH = 0, \quad D > 0, H > 0. \qquad (A1)$$

has two real roots $\pm\lambda$, with $\lambda > 0$, four complex conjugate roots $\lambda_1, \overline{\lambda}_1, \lambda_2, \overline{\lambda}_2$ with $Re(\lambda_1) > 0, Re(\lambda_2) < 0$ and $Im(\lambda_1) > 0, Im(\lambda_2) > 0$ and an infinite number of imaginary roots, $\pm ik_n$ with $k_n > 0$, n= 1, 2, 3,... and it is found that $0 < k_1H < \pi < k_2H < 2\pi <$ ... and $k_nH \to (n-1)\pi$ as $n \to \infty$.

We prove, by Rouche's theorem, that the transcendental equation (A1) does not have any other roots. We choose, for comparison, the function

$$g(z) = z(1 + Dz^4)\sinh zH,$$

whose roots are given by $z = 0$, four complex conjugate roots $\beta_1, \overline{\beta}_1, \beta_2, \overline{\beta}_2$ and $zH = 0, \pm i\pi, \pm i2\pi, ...$ on the imaginary axis. We see from the above that the functions $f(z)$ and $g(z)$ have $2m + 4$ number of zero's in the square with vertices $zH = (2m-1)\frac{\pi}{2}(\pm 1 \pm i)$, for sufficiently large $m > 0$. Note that the square is assumed to be chosen in a such a way that it does not pass through any of the zero's of the functions f and g.
Since

$$\left|\frac{f(z)}{f(z) - g(z)}\right| = \left|z(1 + Dz^4)\tanh zH - 1\right| > 1,$$

uniformly on the square, we conclude, by the Rouche's theorem, that there are no other roots for the transcendental equation (A1).

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