Implementation of the Barrier Method to the Variational Inequality (VI) Parking Spatial Price Equilibrium Problem

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Abstract

The game played between parking facility owners and travelers who want to park in a specific geographic area such as the Central Business District (CBD) is formulated as a Variational Inequality (VI) spatial price equilibrium problem. The parking facilities owners place a price for parking (supply price) and the users set their own price that they are willing to pay (demand price). After a certain period of trade-offs, usually the system reaches equilibrium, where the supply price for each parking facility equals the demand price of the user groups that have decided to accept the price. This means that there is a flow from each of these user groups to the specific parking facilities, otherwise there is no flow. The user groups whose demand prices are lower than the supply prices will not park in this area and either use another parking facility further away, park on the street, visit the area through another transportation means (e.g. ride share, transit, bicycle, etc.), or avoid the area and go somewhere else to conduct their business and personal functions. This VI mathematical formulation is solved using the barrier method that forces the solution to stay in the feasible region. Two small problems are solved that demonstrate the implementation of the algorithms to this application.

Keywords: variational inequality, spatial price equilibrium, parking pricing, barrier method, nonlinear programming, game theory.

Implementation of the Barrier Method to the Variational Inequality (VI) Parking Equilibrium Problem

1 Introduction

The problem addressed in this study is the game played in a geographical area (e.g. the Central Business District (CBD)) between the parking facilities owners and the users who want to visit this area to carry out their personal and/or business functions. The parking facility owners place a price for parking (supply price) and the users set their own price that they are willing to pay (demand price). After a certain period of trade offs, usually the system reaches equilibrium, where the supply price for each parking facility equals the demand price of the user groups that have decided to accept this price. The user groups whose demand prices are lower than the supply prices will not park in this area and either use another parking facility further away of the designated geographical area, park on the street, visit the area through another transportation means (e.g. ride share, transit, bicycle, etc.), or avoid the area and go somewhere else to conduct their business and personal functions.

This problem falls within the category of spatial price equilibrium mathematical formulation that is well addressed in [Nagurney, 1993]. In general the problem involves a set of suppliers (parking lot owners) where each has its own supply price function, a set of users (parking lot customer groups) that are characterized by their specific demand price function, and the transaction (transportation, parking payment, vehicle arrival and departure to/from the parking lot) cost functions from each supply (parking lot) to each potential user group destination (office, restaurant, theater, retail shop, public building, etc.). Spatial price equilibrium problems are usually formulated as variational inequalities (VI).

The solution algorithms are usually problem specific and dictated by the size of the problem as well as the functional form of the supply, demand, and transportation cost functions (e.g. linear or nonlinear). In the parking equilibrium VI formulation we assumed that the demand and supply price functions are linear. The feasible region is defined naturally by the positiveness of the variables (flow of users from each demand group to each of the candidate parking lots). The unique characteristics of this problem let us to consider the use of the barrier method to find feasible solutions. The barrier method provided a more transparent way of deploying a numerical method by providing at the same time an analytical justification for its use. Although it has been applied to other nonlinear constrained problems this is the first implementation to solve VI problems.

This paper is organized as follows: First, we present the parking spatial price equilibrium problem. Second, we provide a background on the VI mathematical formulations for spatial price equilibrium problems. Third we present the implementation of the barrier function to solve the VI parking spatial price equilibrium problem, which we demonstrate on two case studies. Finally we present the main conclusions and future work.

1.1 Parking Equilibrium Problem Definition

Assume that a geographical area has a set of *m* parking facilities (supply), *n* user groups who want to visit this area, the parking facility price functions, the user group price functions and the transaction cost functions. The problem is to find the supply (number of parking spaces) for each parking facility, the demand (number of people who park at each parking facility) for each user group, the flow from each user group to each parking facility and the associated parking facility prices, user group prices, and transaction cost prices given that the system operates in equilibrium.

Parking facility price functions. The parking facility price is a function of the number of parking spaces available at each facility, the number of spaces at other competing facilities, and some fixed cost, a_0, that captures the operating, maintenance cost and the capital cost. The operating cost includes the human recources necessary to operate the facility such as parking guards, energy costs (lighting, heating, other) and the parking guidance or information system (PGS/PIS). The maintenance costs include: pavement maintenance (repaving, line striping, cleaning, etc.), structures (painting, resurfacing, other), regular maintenance of the various fixtures (lights, bicycle stools, signs, parking gates, detectors (e.g. loop detectors), and the PGS/PIS). The capital cost includes the cost to build the facility, and the added elements such as a gate, detectors, cost to install a parking guidance system, cost to implement a parking information system, other.

In this paper we assume that the parking lot price functions are linear and there is an influence to this price from other competing parking lots. The functional form of the parking lot price functions should be determined based on market studies for the specific area of interest. The functional form chosen for the parking lot price functions is,

$\pi_i(s) = a_1 s_1 + a_2 s_2 + a_3 s_3 + ...a_i s_i..... + a_m s_m + a_0$, for $i = 1, 2, 3, ..., m$

$\pi_i(s)$: The parking space supply price function for parking lot *i* as a function of the parking space supply of each competing parking lot,

a_o: The fixed term, a_0, represents the operating, maintenance and other fixed cost of each parking lot,

a_i: This coefficient represents the impact on the supply price of parking lot *m* of the number of parking spaces available at each parking lots,

s_i: The parking space supply for parking lot *i.*

For the parking space supply price of parking lot *i*, the coefficient a_i is higher than all the other coefficients that correspond to the contribution of the number of parking spaces available from the other competing parking lots.

It is also noted that the contribution of each parking lot's parking space supply to the other parking lots parking space supply price is different. For example this interaction for parking lots *i* and *j* is asymmetric:

$\frac{\partial \pi_i(s)}{\partial s_j} \neq \frac{\partial \pi_j(s)}{\partial s_i}$, (asymmetric interaction).

We note here that this asymmetric interaction is also assumed for the user group demand price function as well as the associated transaction cost functions.

User group demand price functions. Users are grouped together based on similar sociological characteristics that are represented in the functional form of each demand price function. The user group demand price functions are assumed to have a linear form that is a function of the various user groups that compete for parking in the same geographical area. The negative signs in the demand price function coefficients indicate that the higher the demand is, the lower the demand price is. The user group demand price functions in this implementation are assumed to have the following functional form:

$$\rho_j(d) = -b_1 d_1 - b_2 d_2 - b_3 d_3 - ..... - b_j d_j ..... - b_n d_n + b_0 .$$

$\rho_j(d)$: The demand price function for group *j* as a function of each user group's demand.

b_o: The fixed term, b_0, represents the maximum price that each user group is willing to pay for parking.

b_j: This coefficient represents the impact on the demand price of each user group.

d_j: The demand for user group *j*.

Transaction cost functions. The transaction cost functions for the parking problem may include the travel time (walking) from the parking lot to the user group's destination plus the parking lot cost, which is represented by the fixed term in the equation. The walking travel time usually is converted into cost based on the value of time and it may include other factors such as safety (the path from one destination to a parking lot may be classified as safe or non-safe and this can be captured into the corresponding transaction cost function). The coefficients here are positive implying that the higher the number of users, the higher the corresponding transaction costs will be. This is usually attributed to the time required to park the car (e.g. arriving at the parking lot and trying to park or in case of valet parking the time required to leave the keys to the attendant that may some delay depending on the number of people waiting on line), and the time required to pay for parking (leaving the parking lot). It is also noted that these functions are assumed linear, where in real situations they are may be nonlinear. The corresponding transaction functions are:

$$c_{ij}(Q) = e_{i1}Q_{i1} + e_{i2}Q_{i2} + e_{i3}Q_{i3} + ... + e_{in}Q_{in} + e_{i0}, \; i = 1, 2, 3, ..., m$$

$c_{ij}(Q)$: The transaction cost to park at parking lot *i* and go to destination *j* as a function of the flow to each potential destination from parking lot *i*.

Q_{ij}: The number of people that park at parking lot *i* and go to destination *j*. Here the destination *j* is synonymous to a user demand group. Therefore, two different demand groups may have the same physical destination but the demand subscript will be different.

e_{i0}: The fixed cost e_{i0} represents the walking time from parking lot *i* to the user's destination plus the fixed parking cost at parking lot *i*.

e_{ij}: The coefficient, e_{ij}, represents the "congestion" cost at the time of arrival and departure from the parking lot.

This implies that the price for parking at parking lot j is also now a function of the time period. The users can now be distributed at different time periods based on the corresponding demand price and transaction cost functions which are now based on the time period of the day. The corresponding demand price functions are:

$$\rho_{jk}(d) = -b_{11}d_{11} - b_{12}d_{12} \ldots - b_{1k}d_{1k} \ldots - b_{j1}d_{j1} \ldots - b_{jk}d_{jk} \ldots - b_{n1}d_{n1} \ldots - b_{nk}d_{jk} + b_0 .$$

Similarly the transaction cost functions are based on the time period of the day and are given by:

$$c_{ijk}(Q) = e_{i11}Q_{i11} + e_{i22}Q_{i22} \ldots + e_{i2k}Q_{i2k} \ldots + e_{in1}Q_{in1} \ldots + e_{ink}Q_{ink} + e_{i0} .$$

The solution to this problem is the equilibrium parking supply and demand price for each parking lot and each user demand group, for each time period of the day, respectively. This also results in the corresponding flows for each user demand group to each parking lot, the transportation (transaction) costs, and the parking supply for each parking facility. The formulation to the above problem will consequently be presented in the next sections.

2 VI Formulation of the Spatial Price Equilibrium Problem

In this section we provide a summary of the general VI mathematical formulation and characteristics. The spatial price equilibrium problem can be formulated as a VI in the case where the feasible region K is convex and compact subset of R^n.If F is a vector function on K, then x* in K satisfies the VI if $F(x^*) \cdot (x - x^*) \geq 0$ for all x in K. This is equivalent to: If x^* is in the interior of K (Figure 1_a) then $F(x^*) = 0$, otherwise if x^* is on the boundary of K, then $F(x^*)$ is perpendicular to the boundary of K at x^* and $F(x^*)$ points inward (see Figure 1_b). In rare cases, $F(x^*)$ could be equal to zero even though x^* is on the boundary.

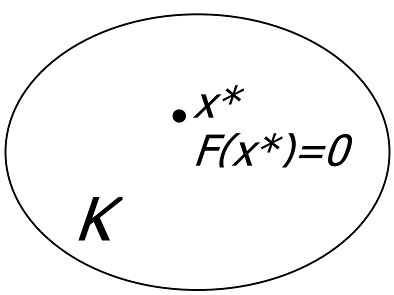


Figure 1_a

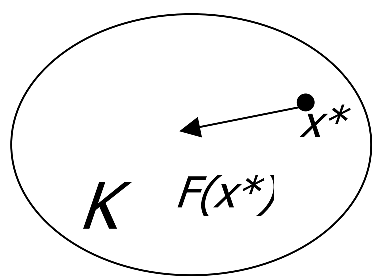


Figure 1_b

The existence of a solution to the VI formulation where F is continuous, K is convex and compact in R^n is based on fixed-point theory. A map $P \circ (I - \gamma F)$ is constructed from the set K onto itself, where P is the orthogonal projection operator on the boundary of the feasible region K and γ is any positive number greater than zero. From analysis we know that such a map has a fixed point [Nagurney, 1993]. Finding a solution to this VI problem is equivalent to finding a fixed point $P(x^* - \gamma F(x^*)) = x^*$ on K.

In the implementation of the parking spatial price equilibrium problem the feasible region is R_+^n, the non-negative orthant of R^n. F is assumed to be of the form $F(x) = Mx + b$ where M is a positive definite matrix not necessarily symmetric (Note that given that M is positive definite implies F is strongly monotone). It is shown that for a number R sufficiently large, there is no solution to the VI outside $B_R \cap R_+^n$, where B_R is the closed ball of radius R. This restricted region $B_R \cap R_+^n$ is now convex and compact.

Proof. Consider $x_R \in R_+^n$, $\|x_R\| = R$, then for $x \in B_R \cap R_+^n$,

$$F(x_R) \cdot (x - x_R) = (Mx_R + b) \cdot (x - x_R) = -x_R^T M x_R + x^T M x_R + b \cdot (x - x_R) < 0,$$

for large R since $x_R^T M x_R$ is positive definite and dominates the remaining terms (see Figure 2). We therefore cannot possibly have a solution x^* to the VI since $\|x_R\| \geq R$.

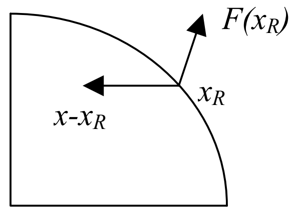


Figure 2. Feasible Region $B_R \cap R_+^2$

Since $B_R \cap R_+^n$ is convex and compact we have a solution to the VI problem using fixed point theory as previously described.

Uniqueness. As for uniqueness, suppose x_1^*, x_2^* are two solutions which satisfy $F(x_1^*) \cdot (x - x_1^*) \geq 0$ and $F(x_2^*) \cdot (x - x_2^*) \geq 0$, respectively for all $x \in B_R \cap R_+^n$. Choose $x = x_2^*$, and $x = x_1^*$ in the first and second inequalities, respectively. Adding the inequalities we obtain

$$(F(x_1^*) - (F(x_2^*)) \cdot (x_2^* - x_1^*) \geq 0, \quad \text{or}$$

$$(F(x_1^*) - (F(x_2^*)) \cdot (x_1^* - x_2^*) \leq 0.$$

On substituting $F(x) = Mx + b$ we have

$$M(x_1^* - x_2^*) \cdot (x_1^* - x_2^*) = (x_1^* - x_2^*)^T M (x_1^* - x_2^*) \leq 0.$$

Since M is positive definite $x_1^* - x_2^* = 0$ or $x_1^* = x_2^*$. We therefore have uniqueness.

To determine whether M is positive definite, since M is not necessarily symmetric, we observe that

$$x^T M x = x^T M_s x$$

where $M_s = \dfrac{M + M^T}{2}$, the symmetric part of M. A necessary and sufficient condition for M to be positive definite is that all eigenvalues of M_s are positive [Strang, 1988]. We will apply this property in the examples to follow.

2.1 Solution Algorithm – The Barrier Method

As pointed out previously, the solution to the VI complementarity problem lies on one of the coordinate planes and the vector field F at that point is perpendicular to the coordinate plane. For very small dimensions the problem could be solved algebraically. However, as the dimension field increases the problem becomes intractable to solve. Therefore, other methodologies are needed to direct the search towards a feasible VI solution. The barrier method [Vanderbei, 2001] is one of the more widely used methodologies employed to solve similar types of problems and it is used here. A description of the barrier method follows.

We will first describe the implementation to the symmetric case followed by the asymmetric case. For $F(x) = (F_1(x), \ldots\ldots, F_n(x))$, where $x = (x_1, \ldots\ldots, x_n)$, the symmetric case is defined generally as

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}.$$

This implies that for $F(x) = Mx + b$, M is symmetric. Because of the symmetry we can find a potential function ϕ such that $F = \nabla\phi$. In our case $\phi = \frac{1}{2} x^T Mx + x^T b$, which is a quadratic form and has a minimum. However, the minimum may not be in the feasible region. Adding a barrier to ϕ forces the minimum to be relocated within the feasible region. The new function is

$$\phi_B(x;\mu) = \phi(x) - \mu\left(\sum_{i=1}^{n} \ln x_i\right).$$

As $\mu \to 0$ the $\min \phi_B(x;\mu) \to x^*$, which is the solution to the VI problem. This can be seen since the minimum of ϕ_B, is found to satisfy $\nabla\phi_B = 0$, or

$\nabla\phi_B(x;\mu) = \nabla\phi(x) - \mu\left(\frac{1}{x_1}, \ldots\ldots\ldots, \frac{1}{x_n}\right)^T = 0.$ Component wise we have

$(\nabla\phi(x))_i - \frac{\mu}{x_i} = 0$ or $x_i(\nabla\phi(x))_i - \mu = 0$ for $i = 1, \ldots\ldots, n$.

And as $\mu \to 0$ we have $x_i(\nabla\phi(x))_i = 0$, the Kuhn-Tucker conditions, which states that either $x_i = 0$ or $(\nabla\phi)_i = 0$.

Asymmetric Case. The asymmetric case is defined generally as

$\frac{\partial F_i}{\partial x_j} \neq \frac{\partial F_j}{\partial x_i}$, implying that M is asymmetric.

Decomposing M into the symmetric and skew-symmetric parts $M = M_s + M_{sk}$,

where $M_s = \frac{M + M^T}{2}$ and $M_{sk} = \frac{M - M^T}{2}$, respectively

Now *F(x)*, can be expressed as, $F(x) = (M_s + M_{sk})x + b = M_s x + (M_{sk}x + b)$.

We can think of F as being almost symmetric with the exception of the term $M_{sk}x+b$. We will define the following sequence $\{x^n\}$, following the general iterative scheme described in [Nagurney, 1993] however implementing the barrier method. Consider

$$g(x^n, x^{n-1}) = M_s x^n + (M_{sk} x^{n-1} + b).$$

(Note that since for $g(x, y) = M_s x + M_{sk} y + b$, then (i) $g(x, x) = F(x)$ and (ii) $\nabla_x g = M_s$ is symmetric and positive definite [Nagurney, 1993; section 2.1])

Since M_s is symmetric, the potential ϕ of g as a function of x^n is given by

$$\phi(x^n, x^{n-1}) = \frac{1}{2}(x^n)^T M_s x^n + x^{n^T}(M_{sk} x^{n-1} + b)$$

and satisfies the condition $g = \nabla\phi$, where the gradient is taken with respect to x^n.

Similarly, as before, we add the barrier

$$\phi_B(x^n, x^{n-1}; \mu) = \phi(x^n, x^{n-1}) - \mu(\sum_{i=1}^{n} \ln x_i^n).$$

For a fixed μ and x^{n-1}, we find x^n, which minimizes ϕ_B. The sequence $\{x^n\}$ thus obtained will converge to some limit x_μ. As in the symmetric case, x_μ will satisfy

$$(\nabla\phi(x_\mu))_i - \mu\frac{1}{(x_\mu)_i} = 0 \text{ for } i = 1, \ldots.., n .$$

The above convergence is not obvious for a general F, however in our special case, where $g(x^n, x^{n-1}) = M_s x^n + (M_{sk} x^{n-1} + b)$, the minimization condition is that

$$M_s x^n + (M_{sk} x^{n-1} + b) - \mu(\frac{1}{x_1^n}, \ldots\ldots., \frac{1}{x_n^n})^T = 0 .$$

We will show that μ small the conditions for the convergence of the sequence $\{x^n\}$ we require $\left\|M_s^{-1} M_{sk}\right\| < 1$.

Proof. We have for two successive iterations

$$M_s x^n + (M_{sk} x^{n-1} + b) - \mu(\frac{1}{x_1^n}, \ldots\ldots., \frac{1}{x_n^n})^T = 0 ,$$

$$M_s x^{n-1} + (M_{sk} x^{n-2} + b) - \mu(\frac{1}{x_1^{n-1}}, \ldots\ldots., \frac{1}{x_n^{n-1}})^T = 0 .$$

Subtracting the second term from the first term

$$M_s(x^n - x^{n-1}) + M_{sk}(x^{n-1} - x^{n-2}) - \mu(\frac{1}{x_1^n} - \frac{1}{x_1^{n-1}}, \ldots\ldots., \frac{1}{x_n^n} - \frac{1}{x_n^{n-1}})^T = 0 \text{, or}$$

$$M_s(x^n - x^{n-1}) + \mu(\frac{x_1^n - x_1^{n-1}}{x_1^n x_1^{n-1}}, \ldots\ldots., \frac{x_n^n - x_n^{n-1}}{x_n^n x_n^{n-1}})^T = -M_{sk}(x^{n-1} - x^{n-2}) \text{, or}$$

$$(M_s + \mu D)(x^n - x^{n-1}) = -M_{sk}(x^{n-1} - x^{n-2}),$$

where D is the diagonal matrix with entries, $d_{ii} = \frac{1}{x_i^n x_i^{n-1}}$.

Since M_s is diagonally dominant and all the diagonal elements of D are positive (μ is also positive) then $M_s + \mu D$ will be even more diagonally dominant. It is well known that a diagonally dominant matrix has an inverse [Strang, 1988]. Taking the inverse, we have

$$x^n - x^{n-1} = -(M_s + \mu D)^{-1} M_{sk}(x^{n-1} - x^{n-2}).$$

Taking the norm of both sides and using properties of norms,

$$\left\| x^n - x^{n-1} \right\| \le \left\| (M_s + \mu D)^{-1} M_{sk} \right\| \left\| x^{n-1} - x^{n-2} \right\|.$$

Convergence follows if $\left\| (M_s + \mu D)^{-1} M_{sk} \right\| \le c < 1$, where c is a constant. For μ small we require $\left\| M_s^{-1} M_{sk} \right\| < 1$.

Numerical Algorithms

We solved the problem $M_s x^n + (M_{sk} x^{n-1} + b) - \mu(\frac{1}{x_1^n}, \ldots\ldots, \frac{1}{x_n^n})^T = 0$ using the Newton method iteratively and the sequence converged to the solution. We could have used the diagonalization method which solves, $Dx^n + (M - D)x^{n-1} + b - \mu(\frac{1}{x_1^n}, \ldots\ldots, \frac{1}{x_n^n})^T = 0$. Where the diagonal matrix D is given by $D = (m_{ii})$ is expected to produce the same results.

The barrier solution algorithm

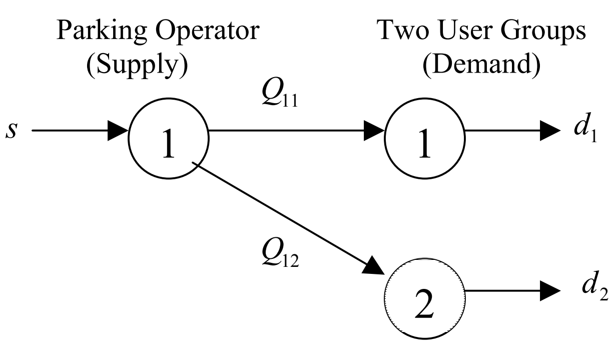
Step 1. Choose a small μ. Select an initial point $x^0 \in R_+^n$.

Step 2. Determine x^1 which minimizes $\phi_B(x, x^0; \mu)$.

Step 3. Repeat step 2 to find a sequence of points and then terminate when a tolerance is reached.

Step 4. Check the final result to see if the limit is on one of the coordinate planes and that F evaluated at that point is perpendicular to that coordinate plane.

2.2 Example 1**. Two-dimensional Problem**



s: parking space supply of parking lot 1

Q_{1j}: flow from parking lot 1 to destination of user's group j

d_j: parking space demand of user group j

Supply Price Function: $\pi_1 = s + 2$

Where, $s = Q_{11} + Q_{12}, d_1 = Q_{11}, d_2 = Q_{12}$

Equilibrium Principle: If the supplier's price is greater than the jth demand price,

$\pi_1 > \rho_j$, there is **no transaction**, and $Q_{1j} = 0$ (no flow; no customers of user group j will park at parking lot 1). Otherwise,

$\pi_1 = \rho_j$, and there is a **transaction,** and $Q_{1j} > 0$ (flow; customers of user group j will park at parking lot 1).

For this example there are four possibilities:

$Q_{11} > 0, Q_{12} > 0$	There are flows for both 1 and 2
$Q_{11} = 0, Q_{12} > 0$	There is a flow to 2 only
$Q_{11} > 0, Q_{12} = 0$	There is a flow to 1 only
$Q_{11} = 0, Q_{12} = 0$	There is no flow either to 1 or 2

The feasible region is $R_+^2 = \{Q_{11} \geq 0, Q_{12} \geq 0\}$, the first quadrant.

Case 1.

Demand Price Functions:

User group 1: $\rho_1(d) = -3d_1 - 0d_2 + 6$,

User group 2: $\rho_2(d) = -d_1 - 2d_2 + 8$.

The difference between the supply and the demand prices is

$$F(Q) = (\pi_1 - \rho_1, \pi_1 - \rho_2) = (4Q_{11} + Q_{12} - 4, 2Q_{11} + 3Q_{12} - 6)$$

Solving for *F(Q)* = 0, we find $Q^* = (Q_{11}, Q_{12}) = (.6, 1.6)$, an interior point of R_+^2. This means that there is a 0.6 flow to destination 1 (user group 1), and 1.6 flow to destination 2 (user group 2) from parking lot one, respectively.

The parking supply is the sum of the flows, s = 2.2, and the corresponding supply price is $\pi_1 = 4.2$. Similarly, the corresponding demand prices (to have an equilibrium) have the same value as π_1 and are $\rho_1 = 4.2$ and $\rho_2 = 4.2$, respectively.

Case 2. Same as before except we assign different marginal demands.

User group 1 demand price function: $\rho_1(d) = -3d_1 - 0d_2 + 9$,

User group 2 demand price function: $\rho_2(d) = -d_1 - 2d_2 + 3$.

Now, $F(Q) = (4Q_{11} + Q_{12} - 7, 2Q_{11} + 3Q_{12} - 1)$.

Solution. Solving for *F(Q)* = 0 we find Q = (0, -1) which is outside the feasible region R_+^2. In this case we observe that Q^* = (7/4, 0) is on the boundary of the feasible region that corresponds to $F(Q^*)$ = (0, 5/2). The vector $F(Q^*)$ is perpendicular to the Q_{11} axis and points into the feasible region. Now the supply price is π_1 = 15/4 and the demand prices for groups 1 and 2 are $\rho_1 = 15/4, \rho_2 = 5/4$, respectively. There is flow from user group 1 to parking lot one, $\pi_1 = \rho_1$ and no flow from user group 2, $\pi_1 > \rho_2$.

Implementation of the barrier method. The above result will be shown numerically by implementing the barrier method.

We have that $F(Q) = MQ + b$ where $M = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$ and $b = -\begin{pmatrix} 7 \\ 1 \end{pmatrix}$. Note that M is asymmetric.

The symmetric part of M is $M_s = \begin{pmatrix} 4 & 3/2 \\ 3/2 & 3 \end{pmatrix}$ and the skew-symmetric part of M is

$M_{sk} = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}$. The matrix M is positive definite since $x^T Mx = x^T M_s x$, and the eigenvalues of M_s are positive, $\lambda = (7 \pm \sqrt{10})/2$.

Define the following sequence $\{Q^n\}$ as

$$g(Q^n, Q^{n-1}) = (4Q_{11}^n + 3/2Q_{12}^n + (-1/2Q_{12}^{n-1} - 7),\ 3/2Q_{11}^n + 3Q_{12}^n + (1/2Q_{11}^{n-1} - 1))\text{. The}$$

potential ϕ of g satisfying $g = \nabla\phi$ as a function of Q^n is

$$\phi = 2(Q_{11}^n)^2 + 3/2Q_{11}^n Q_{12}^n + 3/2(Q_{12}^n)^2 - Q_{11}^n(1/2Q_{12}^{n-1} + 7) + Q_{12}^n(1/2Q_{11}^{n-1} - 1)\,.$$

We add a barrier function to ϕ, where μ is small and positive,

$$\phi_B = \phi - \mu(\ln Q_{11}^n + \ln Q_{12}^n)\,.$$

The minimum of ϕ_B is found iteratively for fixed Q^{n-1} and μ as,

$$4Q_{11}^n + 3/2Q_{12}^n + (-1/2Q_{12}^{n-1} - 7) - \mu/Q_{11}^n = 0,$$

$$3/2Q_{11}^n + 3Q_{12}^n + (1/2Q_{11}^{n-1} - 1) - \mu/Q_{12}^n = 0.$$

Since the leading term is diagonally dominant, we can solve for Q^n for a fixed Q^{n-1} and μ, using Newton's method. We now show that the sequence $\{Q^n\}$ converges. We use the property the norm property of matrices, $\|A\| = \sqrt{\lambda_{\max}(A^T A)}$, where $\lambda_{\max}$ is the maximum eigenvalues of $A^T A$ [Strang, 1988]. Using this formula, we evaluate numerically the norm of $M_s^{-1}M_{sk}$ to be *.2606*, which is less than one.

The solution to the nonlinear equations is shown in Table 1 for different μ. Note that for μ =.0001, we have the solution Q^* of the VI problem.

Table 1. Summary of solutions based on μ.

μ	Q_{11}	Q_{12}
.1000	1.7547	.0381
.0100	1.7504	.0039
.0010	1.7501	.0004
.0001	**1.7500**	**.0000**

The software MATLAB v6.1 was used to implement the barrier method, using the Newton-Raphson method to solve the set of nonlinear equations. The diagonalization method was also implemented using MATLAB and yielded identical results.

2.3 Example 2. Four-dimensional Problem

We implement the barrier method to the spatial price equilibrium sample problem presented in [Nagurney, 1993]. This problem has two suppliers (parking facilities) and two demands(user groups). The corresponding supply price, demand price, and transaction cost functions are given below.

$$\pi_1(s) = 5s_1 + s_2 + 2,$$
$$\pi_2(s) = s_1 + 2s_2 + 3.$$

The user group demand price functions are:

$$\rho_1(d) = -2d_1 - d_2 + 28.75,$$
$$\rho_2(d) = -d_1 - 4d_2 + 41.$$

The corresponding transaction functions are:

$$c_{11}(Q) = Q_{11} + .5Q_{12} + 1,$$
$$c_{12}(Q) = 2Q_{12} + Q_{22} + 1.5,$$
$$c_{21}(Q) = 2Q_{11} + 3Q_{21} + 15,$$
$$c_{22}(Q) = Q_{12} + 2Q_{22} + 10.$$

Flow conservation constraints:

Supply:

$$s_1 = Q_{11} + Q_{12}$$
$$s_2 = Q_{21} + Q_{22}$$

Demand:

$$d_1 = Q_{11} + Q_{21}$$
$$d_2 = Q_{12} + Q_{22}$$

The feasible region is R_+^4. The difference function *F(Q)* is $F(Q) = MQ + b$, where:

$$M = \begin{pmatrix} 8 & 6.5 & 3 & 2 \\ 6 & 11 & 2 & 6 \\ 5 & 2 & 7 & 3 \\ 2 & 6 & 3 & 8 \end{pmatrix} \quad \text{and} \quad b = -\begin{pmatrix} 25.75 \\ 37.50 \\ 10.75 \\ 28.00 \end{pmatrix}.$$

The zero of *F(Q)* is (2.5899, 0.6761, -1.8022, 3.0213) which is outside the feasible region R_+^4. We expect the solution to be on one of the coordinate planes. Here,

$$M_s = \begin{pmatrix} 8 & 6.25 & 4 & 2 \\ 6.25 & 11 & 2 & 6 \\ 4 & 2 & 7 & 3 \\ 2 & 6 & 3 & 8 \end{pmatrix} \quad \text{and} \quad M_{sk} = \begin{pmatrix} 0 & .25 & -1 & 0 \\ -.25 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of M_s are all positive, λ_1, λ_2, λ_3, and λ_4, equal to .7839, 5.6300, 6.6755, and 20.9105, respectively, which makes *M* to be positive definite, implying existence and uniqueness.

As before, construct the sequence $\{Q^n\}$, the potential ϕ, and the potential with the barrier added, ϕ_B, where

$$\phi_B = \phi - \mu(\ln Q_{11}^n + \ln Q_{12}^n + \ln Q_{21}^n + \ln Q_{22}^n)$$

Step 1. Decompose $F(Q)$ into a symmetric part, which is a function of Q^n and a non symmetric part which is a function of Q^{n-1}.

Step 2. The potential ϕ of $F(\nabla\phi = F)$ as a function of Q^n is found since we have symmetry

Step 3. We add a barrier function to ϕ, where μ is small and positive.

$$\phi_B = \phi - \mu(\ln Q_{11}^n + \ln Q_{12}^n + \ln Q_{21}^n + \ln Q_{22}^n)$$

The minimum of ϕ_B satisfies the equation,

$$M_s Q^n + (M_{sk} Q^{n-1} + b) - \mu\left(\frac{1}{Q_1^n},\ldots\ldots,\frac{1}{Q_n^n}\right)^T = 0.$$

Iterating, we have convergence since $\|M_s^{-1} M_{sk}\| = .8387$, which is less than one.

The limit of the minimum of ϕ_B is given in Table 2 for various values of μ. As μ goes to zero we observe that the limit the VI solution, $Q^* = (1.5, 1.5, 0, 2)$, given in [Nagurney, 1993]. The solution Q^* is on the coordinate plane $Q_{21} = 0$ and $F(Q^*) = (0, 0, 5.75, 0)$ is perpendicular to the plane.

Table 1a. Summary of solutions based on μ.

μ	Q_{11}	Q_{12}	Q_{21}	Q_{22}	s_1	s_2	d_1	d_2
.1000	1.4976	1.5067	.0171	1.9954	3.0043	2.0125	1.5147	3.5021
.0100	1.4997	1.5002	.0017	1.9995	3.0009	2.0012	1.5014	3.5007
.0010	1.5000	1.5000	.0002	2.0000	3.0000	2.0002	1.5002	3.5000
.0001	**1.5000**	**1.5000**	**.0000**	**2.0000**	**3.0000**	**2.0000**	**1.5000**	**3.5000**

Table 1b. Summary of solutions based on μ.

μ	c_{11}	C_{12}	c_{21}	c_{22}	π_1	π_2	ρ_1	ρ_2
.1000	3.2510	6.5088	18.0465	15.4975	19.0340	10.0293	22.2185	25.4769
.0100	3.2503	6.5019	18.0045	15.5002	19.0057	10.0033	22.2465	25.4958
.0010	3.2500	6.5000	18.0006	15.5000	19.0002	10.0004	22.2496	25.4998
.0001	**3.2500**	**6.5000**	**18.0000**	**15.5000**	**19.0000**	**10.0000**	**22.2500**	**25.5000**

Using Newton's method the results summarized in Tables 1a and 1b were obtained.

3 Conclusions

The parking space pricing between parking facility owners and travelers has been formulated as a spatial price equilibrium variational inequality problem. The parking facility owners and the travelers who want to park at a specific geographic area reach an equilibrium that is based on the functional form of the respective parking supply price and the user group demand price. No attempt was made in this study to identify the form of the supply and demand price functions. The supply, demand, and transaction cost are defined as functions of the corresponding supply parking spaces of all the competing parking facilities, user demand groups, and link flows, respectively.

The contribution of the corresponding parking supply at each parking lot, number of users in each demand group, and the link flows to each other's parking supply price, user group demand price, and link flows are assumed to be asymmetric. Therefore the corresponding VI mathematical problem for the parking spatial price equilibrium falls into the asymmetric category. We showed the necessary conditions to produce a unique solution.

An algorithm has been developed based on the barrier method that forces the solution to stay in the feasible region, which is the first implementation of this method to solve VI problems. The algorithm was implemented on two small parking problems and showed very promising results.

The solution algorithm will be tested on large systems with many parking lots and customer groups. Furthermore, the algorithm developed will be compared to the diagonalization (relaxation) algorithm for large problems using the steepest descent method as well as other approaches. Generalize the problem where the supply price, demand price and transaction cost functions are nonlinear. The parking lot supply price functions, the user group demand functions and the travel cost functions need to be defined through market research analysis.

4 References

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