A Construction Technique for Heteroclinic Solutions to Continuous and Differential-Difference Damped Wave Equations

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Abstract. In this paper, we give a systematic method for generating continuous and differential-difference damped wave equations for which explicit travelling wave solutions can be obtained. We demonstrate the procedure with several examples. In some specific cases, we recover the well-known solutions of the continuous Nagumo and sine-Gordon equations.

1. Introduction

Occurring naturally in many areas of science are systems that exhibit a spatially discrete structure which affects the evolution and dynamics of such systems. Some of the fields of research in which we see the effects of a spatially discrete structure include crystal growth and liquid-solid materials in materials science [5, 8], chemical reaction theory [14, 27], cellular neural networks and optical memory in image processing [7, 18], relativistic quantum mechanics [33, 36], and myelinated axons and the myocardium in biology [2, 13, 24, 26]. While continuum models often do an adequate job of modelling the behavior of such systems, at times they fail to reflect the effects of the discrete structure of the medium.

In this study, we examine equations of the form

$$\alpha \frac{\partial u}{\partial t} + (1-\alpha)\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + f(u), \quad x \in \mathbb{R}, \quad t > 0, \tag{1}$$

$$\alpha \frac{du_n}{dt} + (1-\alpha)\frac{d^2 u_n}{dt^2} = u_{n-1} - 2u_n + u_{n+1} + f(u_n), \quad n \in \mathbb{Z}, \quad t > 0, \tag{2}$$

where $\alpha \in [0,1]$. If $\alpha = 0$, then (1) is the undamped wave equation with a nonlinear forcing term, and (2) is an undamped nonlinear coupled oscillator system. If $\alpha = 1$, then (1) is a nonlinear diffusion equation, and (2) is a system of first-order nonlinear ordinary differential equations.

These two equations include several specific equations that arise in applications, namely, the spatially continuous and spatially discrete versions of the Allen-Cahn, the sine-Gordon, and the Nagumo equations. In materials science, a spatially discrete model is often the result of an underlying spatial lattice that occurs in many applications, of which specific examples include crystal growth, interface motion in crystalline materials, spinoidal decomposition, and grain boundary movement in thin films. Modelling such processes with the spatially discrete Allen-Cahn equation allows the material being modelled to be described in terms of its underlying crystalline lattice. Discrete effects, such as crystallographic pinning and lattice anisotropy, occur naturally in spatially discrete models of materials, but often are added to spatially continuous models in an *ad hoc* manner.

In solid state physics, the Frenkel-Kontorova (or discrete sine-Gordon) equation [20] and its continuous counterpart, the sine-Gordon equation, have been used to study such phenomena as motion of a dislocation in a crystal, adsorbate layers on a crystal surface, ionic conductors, glassy materials, charge density wave transport, chains of coupled Josephson functions, and sliding friction. An effect that the discrete sine-Gordon equation represents is a strongly nonlinear mobility, where below some field or force threshold, the mobility is zero, and above this threshold, it is nonzero. The continuous sine-Gordon equation does not have this effect.

In the field of biology, spatially discrete systems occur in neuophysiology. For example, signal propagation in the nervous system occurs along nerve axons in the form of propagating membrane electrical activity called action potentials. Some axons are coated with myelin (a fairly insoluble coating), which contains gaps, called the nodes of Ranvier, where the nerve fiber is exposed. The myelin increases the effective resistance of the membrane and decreases its capacitance, thus effectively reducing the electrotonic length of the fiber, i.e., making the fiber electrically shorter. As a result, instead of a smoothly propagating wave, saltatory conduction occurs, i.e., the action potential hops from node to node as it travels down the fiber. Although the Nagumo equation (including McKean's version) can model a system that admits active wave propagation, when this equation is applied to nerve fibers, the effects of myelination are lost. The discrete Nagumo equation corrects for this.

In this paper, we present a systematic construction method for finding nonlinearities f such that (1) and (2) exhibit explicit travelling wave solutions. A *travelling wave solution* of (1) (respectively, (2)) is a solution of the form $u(x,t) = v(x+ct)$ (respectively, $u_n(t) = v(n+ct)$). Letting $z = x + ct$ or $z = n + ct$, we can rewrite (1) and (2) as the single equation

$$\alpha c v'(z) + (1-\alpha)c^2 v''(z) = (\mathcal{D}v)(z) + f(v(z)), \tag{3}$$

where

$$(\mathcal{D}v)(z) = \begin{cases} v''(z), & \text{if continuous,} \\ v(z-1) - 2v(z) + v(z+1), & \text{if discrete.} \end{cases} \tag{4}$$

Note that z is a continuous, real variable in both cases. For the nonlinearity f, we assume that it satisfies

$$f(0) = f(a) = f(1) = 0 \tag{5}$$

for *at least one* a in $(0, 1)$. The assumption that $f(a) = 0$ is not crucial to the success of the method that we propose, but it allows us to choose a particular wave speed c as a function of a and the other parameters that appear in f. We will look for monotonic solutions of (3)–(5) satisfying the boundary conditions

$$v(-\infty) = 0, \quad v(+\infty) = 1. \tag{6}$$

For the continuous case, (3) is a nonlinear second-order ordinary differential equation. Here our focus will be on travelling wavefront solutions [16, 17], examples of which arise in genetics [3, 19, 25, 28] and fluid mechanics [37].

For the discrete case, (3) is a differential-shift equation with both forward and backward shifts, i.e., a functional differential equation of mixed type. Work has been done on general systems of such mixed functional equations, including the work of Rustichini [31, 32] and Mallet-Paret [29]. Propagation failure for one-dimensional spatially discrete nonlinear diffusion equations was studied by Keener [24]. Zinner provided existence and stability results for travelling wave solutions of spatially discrete nonlinear diffusion equations [38]–[40]. In [21], Gao looked at varying the diffusion coefficient of the discrete diffusion operator. In [6], Cahn et al. gave propagation failure and lattice anisotropy results for travelling wave solutions of two-dimensional spatially discrete nonlinear diffusion equations. (See also the analysis of Fath [15] for one-dimensional spatially discrete nonlinear diffusion equations.) Mallet-Paret [30] looked at the global structure of solutions to lattice-differential nonlinear diffusion equations. Shen's work includes (3) with $\alpha = 1$ and the nonlinearity $f = f(u, t)$ almost periodic in t [34, 35]. In [22], Johnston studied bifurcation phenomena for several bistable nonlinearities. Elmer and Van Vleck have produced extensive work on differential-difference diffusion equations of bistable type [9]–[12].

As far as we know, the search for exact solutions to (3), (4) is still largely anecdotal. Bressloff and Rowlands [4] studied the discrete case of (3), (4) with $\alpha = 1$ and found a class of nonlinearities exhibiting exact kink-type solutions. The idea behind the method we present in this paper, which is reminiscent of the method of Bressloff and Rowlands, is to assume a specific form for v that satisfies the monotonicity and boundary conditions in (6). The assumption should be expressed in terms of exponentials in such a way that after differentiation, the derivatives of v can be expressed in terms of v again. In effect, we are solving an inverse problem since, instead of looking for v given a nonlinearity f, we assume that we have the functional form for v satisfying certain conditions, and then we look for the corresponding nonlinearity.

It turns out that even though the functional form for v is relatively simple, the nonlinearity it produces can be algebraically complicated, especially in the discrete problem. Nevertheless, the nonlinearities that we find have arbitrary parameters that we are free to specify, so we can make these nonlinearities as close as we wish to any given nonlinearity which also satisfies (5).

Two issues that we will not address in this paper are the stability of the travelling wave solutions and propagation failure. These issues are related, and they will be discussed in a future paper.

The paper is organized as follows. In Section 2, we describe the method for finding exact solutions to (3)–(6) in general terms. In Section 3, we demonstrate the procedure with three examples. We remark that each example applies to both continuous and discrete cases; however, the corresponding nonlinearities and the wave speeds are different for each case. We also plot some of the solutions and nonlinearities that we find. Finally, in Section 4, we give brief concluding remarks.

2. Method for Finding Exact Travelling Wave Solutions

Here we describe a general method for constructing exact solutions of (3)–(6). In this paper, we restrict our attention to monotonic solutions, i.e., travelling wave front solutions of (1), (2). Assume that v has the following canonical form

$$v(z) = G(e^{bz}), \quad b > 0, \tag{7}$$

where G is an increasing, twice-differentiable function defined on $(0, +\infty)$ and satisfies $G(0) = 0, G(+\infty) = 1$. Explicit computations yield

$$\begin{aligned} v'(z) &= be^{bz}G'(e^{bz}) = bH(v(z))G'[H(v(z))], \\ v''(z) &= b^2e^{bz}G'(e^{bz}) + b^2e^{2bz}G''(e^{bz}), \\ &= b^2H(v(z))G'[H(v(z))] + b^2H(v(z))^2G''[H(v(z))], \\ v(z-1) &= G(e^{-b}e^{bz}) = G[e^{-b}H(v(z))], \\ v(z+1) &= G(e^{b}e^{bz}) = G[e^{b}H(v(z))], \end{aligned}$$

where $H(v(z)) = G^{-1}(v(z))$. Solving for f in (3), we obtain

$$\begin{aligned} f(v) = {} & \alpha cbH(v)G'[H(v)] \\ & + (1-\alpha)c^2\left\{b^2H(v)G'[H(v)] + b^2H(v)^2G''[H(v)]\right\} - \mathcal{L}(v) \end{aligned} \tag{8}$$

where the function $\mathcal{L}$ is given by

$$\mathcal{L}(v) = \begin{cases} b^2H(v)G'[H(v)] + b^2H(v)^2G''[H(v)], & \text{if continuous,} \\ G[e^{-b}H(v)] + G[e^{b}H(v)] - 2v, & \text{if discrete.} \end{cases} \tag{9}$$

Thus, v in (7) satisfies (3), (4), (6), (8), and (9).

A specific choice of G determines the nonlinearity f, which then specifies the equation (3). Note that once a choice of G has been made, we obtain the same travelling wave solution for different continuous and discrete equations.

From our choice of the restriction on G that is used to construct v, the nonlinearity f automatically satisfies $f(0) = f(1) = 0$. The wave speed c remains undetermined; however, if we assume that f satisfies (5), then we can solve for the wave speed c by setting $f(a) = 0$. If $\alpha = 1$ in (8), then there exists a unique wave speed since the equation $f(a) = 0$ is linear in c. However, if $\alpha \in [0, 1)$, then the equation $f(a) = 0$ is quadratic in c, so there could be two possible values for the wave speed. Generally, one speed corresponds to a stable wave profile, whereas the other is the speed of an unstable profile, and therefore, is not observed physically.

3. Examples of Exact Travelling Wave Solutions

In Section 2, we sketched the method for specifying a continuous or discrete equation for which we can construct an exact travelling wave solution. Here, we demonstrate the implementation of the method by assuming some particular forms for the solution v.

3.1. First Example. Let

$$v(z) = \left[\frac{e^{bz}}{1+e^{bz}}\right]^r, \quad b, r > 0.$$

Note that v is monotonically increasing and satisfies the boundary conditions in (6). Hence,

$$G(\eta) = \left[\frac{\eta}{1+\eta}\right]^r,$$

which is twice-differentiable and increasing for $\eta \in (0, +\infty)$, satisfies $G(0) = 0, G(+\infty) = 1$, and has inverse G^{-1} defined by

$$G^{-1}(\eta) = H(\eta) = \frac{\eta^{1/r}}{1-\eta^{1/r}}.$$

Substituting the above expressions into (8), (9) and simplifying, we obtain

$$f(v) = bcrv(1-v^{1/r})\left\{bc(1-\alpha)[-v^{1/r} + r(1-v^{1/r})] + \alpha\right\} - \mathcal{L}(v) \tag{10}$$

where

$$\mathcal{L}(v) = \begin{cases} b^2 rv(1-v^{1/r})\left[-v^{1/r} + r(1-v^{1/r})\right], & \text{if continuous,} \\ \frac{v}{\left[e^b + (1-e^b)v^{1/r}\right]^r} + \frac{v}{\left[e^{-b} + (1-e^{-b})v^{1/r}\right]^r} - 2v, & \text{if discrete.} \end{cases} \tag{11}$$

To determine the wave speed, we impose the condition

$$0 = f(a) = b^2 r(1-\alpha)a(1-a^{1/r})[-a^{1/r} + r(1-a^{1/r})]c^2 + br\alpha a(1-a^{1/r})c - \mathcal{L}(a) \tag{12}$$

yielding a quadratic equation in c. If $\alpha = 1$, then (12) reduces to a linear equation for c, and we obtain

$$c = \begin{cases} b\left[r - a^{1/r}(1+r)\right], & \text{if continuous,} \\ \frac{1}{br(1-a^{1/r})}\left\{\frac{1}{\left[e^b + a^{1/r}(1-e^b)\right]^r} + \frac{1}{\left[e^{-b} + a^{1/r}(1-e^{-b})\right]^r} - 2\right\}, & \text{if discrete.} \end{cases} \tag{13}$$

On the other hand, if $\alpha \in [0, 1)$ and $r \neq a^{1/r}/(1-a^{1/r})$, then we can easily solve the quadratic equation for c. In either case, once we have c, we substitute its value into (10), (11) and simplify to obtain the nonlinearity f.

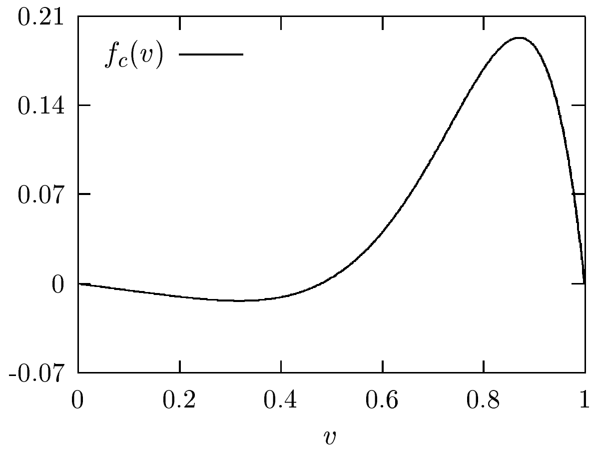

Figure 1. Profile of (14) with $r = 0.25$, $b = 1.79$, and $a = 0.48$.

If we specify the arbitrary parameters such that $\alpha = 1$ and $b = 1/[r(1+r)]^{1/2}$, where r is any positive number, then the nonlinearity obtained from (10)–(13) in the *continuous* case is given by

$$f(v) = v(1 - v^{1/r})(v^{1/r} - a^{1/r}). \tag{14}$$

A profile of f (denoted by f_c) with $r = 0.25$, $b = 1.79$, and $a = 0.48$ is shown in Figure 1. The corresponding nonlinearity for the discrete case also can be obtained although the form is not as compact for arbitrary r as in the continuous case.

For the case when $\alpha = 1$, $r = 1$, and $b = 1/\sqrt{2}$, the solution v and (10)–(13) simplify to

$$v(z) = \frac{e^{z/\sqrt{2}}}{1 + e^{z/\sqrt{2}}}, \tag{15}$$

$$f(v) = \begin{cases} v(1-v)(v-a), & \text{if continuous,} \\ \frac{v(1-v)}{(1-a)}\left[\frac{1}{e^{1/\sqrt{2}}+a(1-e^{1/\sqrt{2}})} + \frac{1}{e^{-1/\sqrt{2}}+a(1-e^{-1/\sqrt{2}})} - 2\right] & \\ \quad - \frac{v}{e^{1/\sqrt{2}}+(1-e^{1/\sqrt{2}})v} - \frac{v}{e^{-1/\sqrt{2}}+(1-e^{-1/\sqrt{2}})v} + 2v, & \text{if discrete,} \end{cases} \tag{16}$$

and

$$c = \begin{cases} \frac{1-2a}{\sqrt{2}}, & \text{if continuous,} \\ \frac{\sqrt{2}}{1-a}\left[\frac{1}{e^{1/\sqrt{2}}+a(1-e^{1/\sqrt{2}})} + \frac{1}{e^{-1/\sqrt{2}}+a(1-e^{-1/\sqrt{2}})} - 2\right], & \text{if discrete.} \end{cases} \tag{17}$$

Note that the continuous case of (3), (4) with $\alpha = 1$ reduces to the well-known Nagumo equation which has the exact solution given by (15) [23]. Profiles of f for the continuous

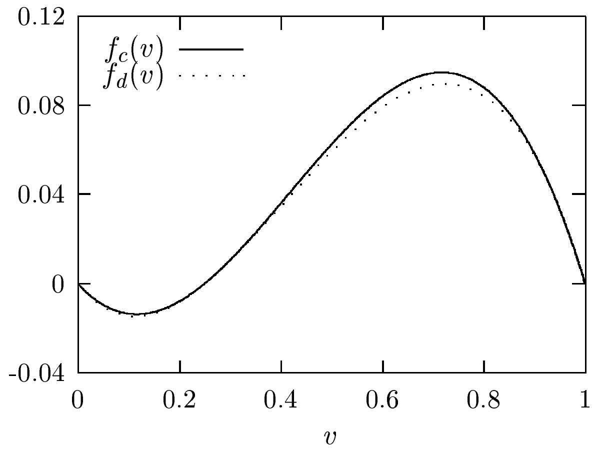

Figure 2. Profiles of (16) for the continuous (f_c) and discrete (f_d) cases with $a = 0.25$.

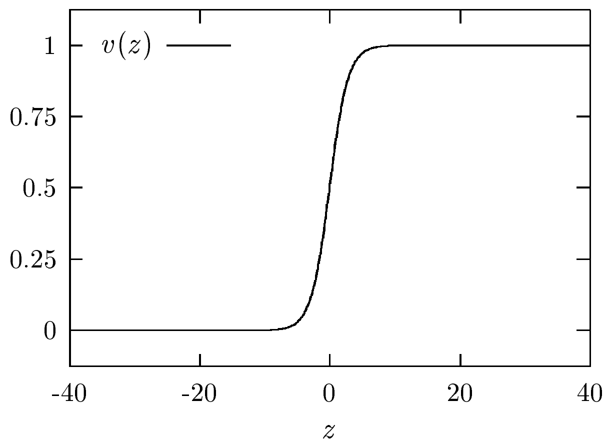

Figure 3. Profile of (15).

(denoted by f_c) and discrete (denoted by f_d) cases with $a = 0.25$ are shown in Figure 2. Note the remarkable closeness of the nonlinearities for the same parameter values. Furthermore, both the continuous and discrete equations have the same monotonic solution given by (15), shown in Figure 3.

3.2. Second Example. Let

$$v(z) = \frac{2}{\pi}\tan^{-1}(e^{bz}), \quad b > 0. \tag{18}$$

Note that v is monotonically increasing and satisfies the boundary conditions in (6). Hence,

$$G(\eta) = \frac{2}{\pi}\tan^{-1}(\eta),$$

which is twice-differentiable and increasing for $\eta \in (0, +\infty)$, satisfies $G(0) = 0, G(+\infty) = 1$, and has inverse G^{-1} defined by

$$G^{-1}(\eta) = H(\eta) = \tan\left(\frac{\pi\eta}{2}\right).$$

Substituting the above expressions into (8), (9) and simplifying, we obtain

$$f(v) = \frac{bc}{\pi}\left[\alpha + bc(1-\alpha)\cos(\pi v)\right]\sin(\pi v) - \mathcal{L}(v) \tag{19}$$

where

(20)

$$\mathcal{L}(v) = \begin{cases} \frac{b^2}{2\pi}\sin(2\pi v), & \text{if continuous,} \\ \frac{2}{\pi}\tan^{-1}\left[e^{-b}\tan\left(\frac{\pi v}{2}\right)\right] + \frac{2}{\pi}\tan^{-1}\left[e^{b}\tan\left(\frac{\pi v}{2}\right)\right] - 2v, & \text{if discrete.} \end{cases}$$

To determine the wave speed, we impose the condition

$$0 = f(a) = \frac{b^2}{2\pi}(1-\alpha)\sin(2\pi a)c^2 + \frac{b}{\pi}\alpha\sin(\pi a)c - \mathcal{L}(a) \tag{21}$$

yielding a quadratic equation in c. If $\alpha = 1$, then (21) reduces to a linear equation for c, and we obtain

(22)

$$c = \begin{cases} b\cos(\pi a), & \text{if continuous,} \\ \frac{2}{b\sin(\pi a)}\left\{\tan^{-1}\left[e^{-b}\tan\left(\frac{\pi a}{2}\right)\right] + \tan^{-1}\left[e^{b}\tan\left(\frac{\pi a}{2}\right)\right] - \pi a\right\}, & \text{if discrete.} \end{cases}$$

On the other hand, if $\alpha \in [0, 1)$, then we can easily solve the quadratic equation for c. In either case, once we have c, we substitute its value into (19), (20) and simplify to obtain the nonlinearity f. Profiles of f for the continuous (denoted by f_c) and discrete (denoted by f_d) cases are shown in Figure 4, where $\alpha = 1$, $a = 0.4$, and $b = 0.25$. Again, note the remarkable closeness of the nonlinearities for the same parameter values.

As mentioned earlier, the assumption that $f(a) = 0$ is not essential for our method. In fact, for the *continuous* case with $\alpha = 0$, (21) is satisfied for *any* wave speed c and any positive b. In particular, if

$$b = \left[\frac{2\pi}{1-c^2}\right]^{1/2} \quad \text{and} \quad c \in (-1, 1),$$

then (19), (20) become

(23)

$$f(v) = \begin{cases} -\sin(2\pi v), & \text{if continuous,} \\ 2v - \frac{2}{\pi}\tan^{-1}\left[e^{-\sqrt{2\pi/(1-c^2)}}\tan\left(\frac{\pi v}{2}\right)\right] & \\ \quad - \frac{2}{\pi}\tan^{-1}\left[e^{\sqrt{2\pi/(1-c^2)}}\tan\left(\frac{\pi v}{2}\right)\right] + \frac{c^2}{1-c^2}\sin(2\pi v), & \text{if discrete.} \end{cases}$$

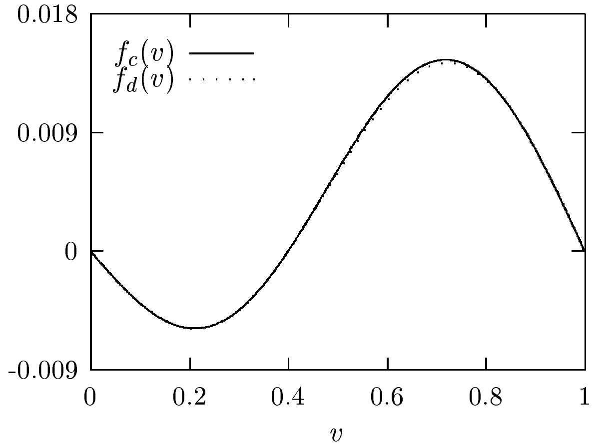

Figure 4. Profiles of (19), (20), and (22) for the continuous (f_c) and discrete (f_d) cases with $a = 0.4$ and $b = 0.25$.

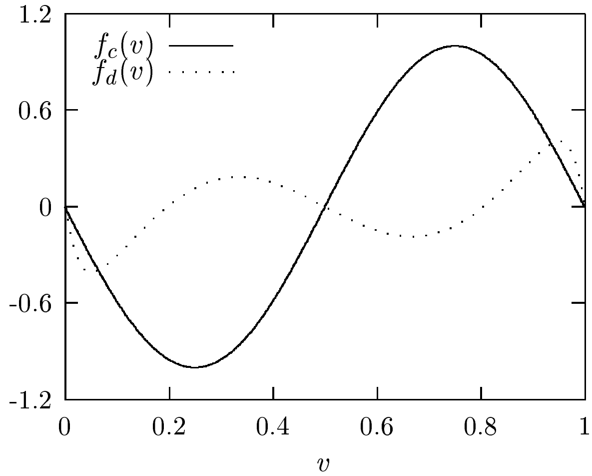

Figure 5. Profiles of (23) for the continuous (f_c) and discrete (f_d) cases with $c = 0.6$ and $b = 3.13$.

Here, b was chosen so that the coefficient of $\sin(2\pi v)$ in the continuous case becomes equal to -1. Note that the continuous case of (3), (4) with $\alpha = 0$ reduces to the well-known sine-Gordon equation whose kink solution is given by (18) [1]. The profile of this solution is similar to that in Figure 3. Profiles of f for the continuous (denoted by f_c) and discrete (denoted by f_d) cases are shown in Figure 5 where $c = 0.6$ and $b = 3.13$. It is not clear why in this case the nonlinearities for the continuous and discrete cases are very different.

3.3. Third Example. Let

$$v(z) = \tanh(e^{bz}), \quad b > 0. \tag{24}$$

Note that v is monotonically increasing and satisfies the boundary conditions in (6). Hence,

$$G(\eta) = \tanh(\eta),$$

which is twice-differentiable and increasing for $\eta \in (0, +\infty)$, satisfies $G(0) = 0, G(+\infty) = 1$, and has inverse G^{-1} defined by

$$G^{-1}(\eta) = H(\eta) = \tanh^{-1}(\eta).$$

Substituting the above expressions into (8), (9) and simplifying, we obtain

(25)
$$f(v) = bc(1 - v^2)\tanh^{-1}(v)\left[\alpha + bc(1-\alpha) - 2bc(1-\alpha)v\tanh^{-1}(v)\right] - \mathcal{L}(v)$$

where

(26)
$$\mathcal{L}(v) = \begin{cases} b^2(1-v^2)\tanh^{-1}(v)\left[1 - 2v\tanh^{-1}(v)\right], & \text{if continuous,} \\ \tanh\left[e^{-b}\tanh^{-1}(v)\right] + \tanh\left[e^{b}\tanh^{-1}(v)\right] - 2v, & \text{if discrete.} \end{cases}$$

To determine the wave speed, we impose the condition

(27)
$$0 = f(a) = b^2(1-a^2)\tanh^{-1}(a)\left[1 - 2a\tanh^{-1}(a)\right](1-\alpha)c^2 + b(1-a^2)\tanh^{-1}(a)\alpha c - \mathcal{L}(a)$$

yielding a quadratic equation in c. If $\alpha = 1$, then (27) reduces to a linear equation for c, and we obtain

(28)
$$c = \begin{cases} b\left[1 - 2a\tanh^{-1}(a)\right], & \text{if continuous,} \\ \frac{1}{b(1-a^2)\tanh^{-1}(a)}\left\{\tanh\left[e^{-b}\tanh^{-1}(a)\right] + \tanh\left[e^{b}\tanh^{-1}(a)\right] - 2a\right\}, & \text{if discrete.} \end{cases}$$

On the other hand, if $\alpha \in [0, 1)$, then we can easily solve the quadratic equation for c. In either case, once we have c, we substitute its value into (25), (26) and simplify to obtain the nonlinearity f. The profile of the solution (24) is similar to that in Figure 3. Profiles of f for the continuous (denoted by f_c) and discrete (denoted by f_d) cases are shown in Figure 6, where $\alpha = 1$, $a = 0.4$, and $b = 0.25$. As with the previous two examples, the nonlinearities are quite close for the same parameter values.

4. Conclusion

In this paper, we have given a general construction procedure for finding nonlinearities f in (1) and (2) such that explicit travelling wave solutions can be obtained. We demonstrated the procedure with three examples for v. Certainly, these are not the only ones; the only consideration we have to keep in mind is whether or not the inversion of G can be performed explicitly. In some specific cases, we recovered the well-known solutions of the continuous Nagumo and sine-Gordon equations. We would like to emphasize

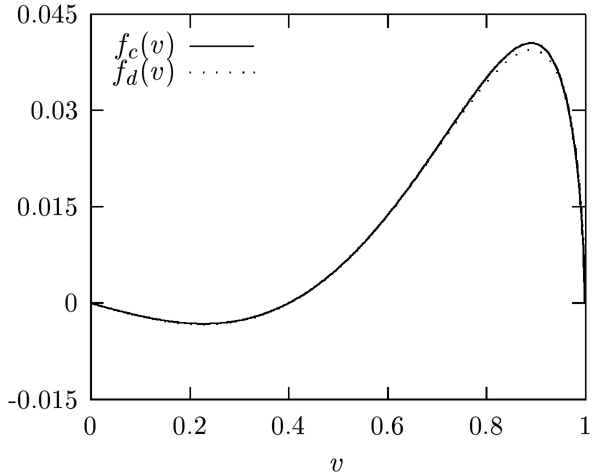

Figure 6. Profiles of (25), (26), and (28) for the continuous (f_c) and discrete (f_d) cases with $a = 0.4$ and $b = 0.25$.

that although the nonlinearities we found may have complicated functional forms, especially in the discrete case, qualitatively they look very similar to nonlinearities having a simpler functional form. The functional forms for the solutions and wave speeds, however, are not so complicated, and the solutions are identical for the continuous and discrete equations with the same parameter values. In future papers, we will examine the stability of these solutions as well as the interaction of wave fronts.

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