4-WEBS IN THE PLANE AND THEIR LINEARIZABILITY

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CAMS Report 0203-02, Fall 2002

Center for Applied Mathematics and Statistics

NJIT

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Abstract

We investigate the linearizability problem for different classes of 4-webs in the plane. In particular, we prove that a 4-web MW with equal curvature forms of its 3-subwebs and a covariantly nonconstant basic invariant is always linearizable, and such a 4-web with a constant basic invariant is linearizable if and only if it is parallelizable. We also consider four classes of the so-called almost parallelizable 4-webs $APW_a, a = 1, 2, 3, 4$ (for them the curvature $K = 0$ and the basic invariant is covariantly constant on the web foliation X_a), and prove that a 4-web APW_a is linearizable if and only if it coincides with a 4-web MW_a of the corresponding special class of 4-webs MW. The existence theorems are proved for all the classes of 4-webs considered in the paper.

0 Introduction

Let W be a 4-web given by 4 one-parameter foliations $X_a, a = 1, 2, 3, 4$, of curves on a two-dimensional manifold M^2. The web W is linearizable (rectifiable) if it is equivalent to a linear 4-web, i.e., a 4-web formed by four one-parameter foliations of straight lines.

It is well-known that the geometry of a 4-web W in the plane is completely determined by the curvature K of its 3-subweb [1, 2, 3], defined by its first three foliations X_1, X_2, and X_3, and by the basic invariant a and their covariant derivatives (see [G 93], or [G 77], [G 80], [G 88], Section 7.1, where 4-webs of codimension $r, r \geq 1$, on M^{2r} were considered).

There are many special classes of 4-webs of codimension r, $r > 1$, on M^{2r} and only a few special classes of 4-webs in the plane (of codimension one). The reason for this is that in the case of codimension $r = 1$, most of the classes coincide with the class of parallelizable 4-webs.

The known (different) special classes of 4-webs in the plane are: parallelizable 4-webs, 4-webs whose basic invariant is covariantly constant on one of the web foliations, and 4-webs with equal curvature forms of their 3-subwebs and not constant basic invariant (see [G 88], Section 7.2, [G 93], and [G 97]) and similar webs with a constant basic invariant (see [Na 96] and [Na 98]). These classes of webs are characterized by some invariant conditions imposed on the curvature K, the basic invariant a, and their covariant derivatives. There is also the class of 4-webs of maximal rank (see [H 94]) but the relations between K, a, and their covariant derivatives for this class are unknown.

A criterion of linearizability is very important in web geometry and in its applications. For 3-webs and 4-webs, the problem of linearizability was posed by Blaschke (see [B 55], $$ 17 and 42). Blaschke claimed that because of complexity of calculations involving high order differential neighborhoods, it is hopeless to find such a criterion.

Recently the conditions of linearizability for d-webs, $d \geq 4$, on 2-dimensional manifolds were found by Akivis, Goldberg, and Lychagin in [AGL]. For 4-webs, these conditions were presented in two different forms: in the form of two relations expressing the covariant derivatives K_1 and K_2 of the curvature K in terms of the curvature K itself, the basic invariant a and its covariant derivatives a_i, a_{ij}, a_{ijk} of the first three orders, and in the form of two partial differential equations of the fourth order for the web functions $z = f(x, y)$ and $u = g(x, y)$.

In the current paper we consider special classes of 4-webs in the plane and investigate the linearizability problem for them using the first form of linearizability conditions found in [AGL].

There are two different classes of 4-webs with equal curvature forms of its 3-subwebs: 4-webs MW whose basic invariant is covariantly nonconstant and 4-webs NW whose basic invariant is constant. We prove that for 4-webs MW the curvatures and the curvature forms of all their 3-subwebs vanish, i.e., all 3-subwebs are hexagonal (see also [G 93] and [G 97]). 4-webs all 3-subwebs of which are hexagonal were introduced by Mayrhofer (see [M 28]). We arrived to Mayrhofer's 4-webs MW solving the problem of finding 4-webs with a nonconstant basic invariant and equal curvature forms of its 3-subwebs. Mayrhofer was first who proved that such 4-webs are equivalent to 4-webs all foliations of which are pencils of straight lines. There are different proofs of Mayrhofer's theorem: see [M 28], [M 29], [R 28], [BB 38], §10. All these proofs are rather complicated. We think that the reason for this is that the authors of these papers did not use the invariant characterization of 4-webs MW similar to the conditions (42)–(45). Using the linearizability condition of 4-webs indicated above, we present a straightforward short proof of the fact that *a 4-web MW with a covariantly nonconstant basic invariant and equal curvature forms of its 3-subwebs and is always linearizable.*

As to the 4-webs with a constant basic invariant and equal curvature forms of its 3-subwebs and (they were studied by Nakai in [Na 96] and [Na 98]), using the same linearizability condition, we prove that *such 4-webs are linearizable if and only if they are parallelizable.*

We also consider four classes of the so-called almost parallelizable 4-webs APW_a, $a = 1, 2, 3, 4$ (for them the curvature $K = 0$ and the basic invariant is covariantly constant on the web foliation X_a). Using again the same linearizability condition, we prove that *a 4-web APW_a is linearizable if and only if it coincides with a 4-web MW_a of the corresponding special class of 4-webs MW.*

The existence theorems are proved for all classes of 4-webs considered in the paper. Note that some of the classes of 4-webs, for example, almost parallelizable 4-webs APW_a were not known earlier.

1 Basic Notions and Equations

1. 4-webs in the plane. Let M^2 be a two-dimensional analytic manifold.

Definition 1. A 4-*web* W is given in an open domain D of M^2 by four foliations $X_a,\ a = 1, 2, 3, 4$, of curves if the tangent lines to these curves through any point $p \in D$ are in general position, i.e., the tangents to the curves of different foliations are distinct and two curves of different foliations have in D at most one common point.

Definition 2. Two 4-webs W and $\widetilde{W}$ with domains $D \subset M^2$ and $\widetilde{D} \subset \widetilde{M}^2$ are *locally equivalent* if there exists a local diffeomorphism $\phi : D \rightarrow \widetilde{D}$ of their domains such that $\phi(X_a) = \widetilde{X_a}$.

In this definition a *local diffeomorphism* ϕ is a differentiable mapping $\phi : D \rightarrow \widetilde{D}$ which induces at each point $p \in D$ a diffeomorphism of some open neighborhood of p onto an open neighborhood of $\widetilde{p} = \phi(p)$.

All functions and differential forms which we will introduce on M^2 will be real of class $C^k, k \geq 3$, or holomorphic in the complex case. Since in the complex analytic case we will not use the conjugate variables or the Cauchy–Riemann equations, it will not be necessary to distinguish the real and complex case.

2. Basic equations of a 3-web. Suppose that (x, y) are local coordinates in $D \subset M^2$ and the foliations $X_a, a = 1, 2, 3, 4$, of a 4-web W are given by the equations

$$u_a(x, y) = \text{const}, \quad a = 1, 2, 3, 4, \tag{1}$$

where $u_a(x, y)$ are differentiable functions.

In order to find the basic equations of a 4-web W, we first consider its 3-subweb [1, 2, 3] defined by the foliations $X_\alpha, \alpha = 1, 2, 3$. If we multiply each of the total differentials du_α of the functions $u_\alpha(x, y)$ from (1) by a factor $g_\alpha(x, y) \neq 0$, we obtain the Pfaffian forms

$$\omega_\alpha = g_\alpha du_\alpha, \quad \alpha = 1, 2, 3. \tag{2}$$

The differential equation $\omega_\alpha = 0$ defines the foliation X_α of curves. The factors g_α can be chosen in such a way that for all the points $(x, y) \in D$ and for all the directions dx/dy, we have the equation

$$\omega_1 + \omega_2 + \omega_3 = 0 \tag{3}$$

(see [B 55], §6).

It follows from (2) that

$$d\omega_\alpha = \omega_\alpha \wedge \theta_\alpha, \tag{4}$$

where

$$\theta_\alpha = -d(\ln g_\alpha).$$

Taking the exterior derivatives of (3) and using (3) and (4), we obtain

$$\omega_1 \wedge (\theta_1 - \theta_3) + \omega_2 \wedge (\theta_2 - \theta_3) = 0.$$

Application of Cartan's lemma to this equation implies

$$\begin{cases} \theta_1 - \theta_3 = a\omega_1 + b\omega_2, \\ \theta_2 - \theta_3 = b\omega_1 + c\omega_2. \end{cases} \tag{5}$$

It follows from (5) that

$$\theta_1 - (a-b)\omega_1 = \theta_3 + b(\omega_1 + \omega_2) = \theta_2 - (c-b)\omega_2 = \theta$$

or

$$\begin{cases} \theta_1 = \theta + (a-b)\omega_1, \\ \theta_2 = \theta + (c-b)\omega_2, \\ \theta_3 = \theta + b\omega_3. \end{cases} \tag{6}$$

Equations (6) allow us to write equations (4) in the form

$$d\omega_\alpha = \omega_\alpha \wedge \theta, \quad \alpha = 1, 2, 3. \tag{7}$$

The form θ is called the *connection form* of the three-web [1, 2, 3].

Exterior differentiation of (7) leads to the cubic exterior equation $d\theta \wedge \omega_\alpha = 0$ from which we easily conclude that

$$d\theta = K\Omega, \tag{8}$$

where K is the *curvature* and

$$\Omega = \omega_1 \wedge \omega_2 = \omega_2 \wedge \omega_3 = \omega_3 \wedge \omega_1 \tag{9}$$

is the *surface element* of the three-web [1, 2, 3]. The form $\Theta = K\Omega$ is called the *curvature form* of the three-web [1, 2, 3]. The equations (8) and (9) are the *structure equations* of the 3-subweb [1, 2, 3] (see [B 55], §8). Equations (7) and (8) prove that the *affine connection* γ_{123} is associated with 3-subweb [1, 2, 3], and θ and K are the *connection form* and the *curvature* of this connection.

3. Basic equations of a 4-web. The fourth foliation of curves of a 4-web W can be given by the equation $\omega_4 = 0$, where

$$-\omega_4 = a\omega_1 + \omega_2 = 0 \tag{10}$$

(see [G 88], Section 7.1, or [G 93]).

The quantity a in equation (10) is called the *basic invariant* of the 4-web W. It satisfies the conditions

$$a \neq 0, 1 \tag{11}$$

(see [G 93], or [G 77], [G 80], [G 88], Section 7.1).

Equations $\omega_a = 0$, $a = 1, 2, 3, 4$, defining the foliations X_a, are preserved under the following concordant transformations of the form $\underset{a}{\omega}$:

$$'\omega_a = s\,\omega_a,$$

and only under such transformations. Under such transformations the basic invariant a is not changed, $'a = a$.

It is easy to show (see [G 93], or [G 77], [G 80], [G 88], Section 7.1) that *at a point $p \in M^2$, the basic invariant a of the web W equals the cross-ratio of the four tangent to the leaves F_2, F_1, F_3, F_4 of the foliations X_2, X_1, X_3, X_4 passing through the point p.*

The basic invariant a of the 4-web W is an absolute invariant of W. It satisfies the differential equation

$$da = a_1\,\omega_1 + a_2\,\omega_2, \tag{12}$$

where $a_i \in C^\infty(M^2),\ i = 1, 2$ are the first covariant derivatives of a.

4. Prolongations of the basic equations. In what follows, we will need prolongations of equations (8). Exterior differentiation of (8) by means of (7) gives the following exterior cubic equation:

$$\nabla K \wedge \omega_1 \wedge \omega_2 = 0, \tag{13}$$

where $\nabla K = dK - 2K\theta$. It follows from (13) that

$$\nabla K = K_1\omega_1 + K_2\omega_2, \tag{14}$$

where $K_i \in C^\infty(M^2),\ \ i = 1, 2$. Equation (4) shows that the curvature K is a relative invariant of weight two.

We will also need two prolongations of equations (12). Taking exterior derivatives of equations (12), we obtain the following exterior quadratic equation:

$$\nabla a_1 \wedge \omega_1 + \nabla a_2 \wedge \omega_2 = 0, \tag{15}$$

where $\nabla a_i = da_i - a_i\theta$, $i = 1, 2$. Applying Cartan's lemma to equation (15), we find that

$$\nabla a_1 = a_{11}\omega_1 + a_{12}\omega_2,\ \ \nabla a_2 = a_{12}\omega_1 + a_{22}\omega_2, \tag{16}$$

where $a_{ij} \in C^\infty(M^2),\ i, j = 1, 2$.

Taking exterior derivatives of equations (16), we obtain the following exterior quadratic equations:

$$\nabla a_{11}\wedge\omega_1 + \nabla a_{12}\wedge\omega_2 + a_1 K\omega^1\wedge\omega^2 = 0,\ \ \nabla a_{12}\wedge\omega_1 + \nabla a_{22}\wedge\omega_2 + a_2 K\omega_1\wedge\omega_2 = 0, \tag{17}$$

where $\nabla a_{ij} = da_{ij} - 2a_{ij}\theta$, $i, j = 1, 2$. Applying Cartan's lemma to equations (17), we find that

$$\left\{\begin{array}{l} \nabla a_{11} = a_{111}\omega_1 + (a_{112} + \dfrac{2}{3}a_1 K)\omega_2, \\ \nabla a_{12} = (a_{112} - \dfrac{1}{3}a_1 K)\omega_1 + (a_{122} + \dfrac{1}{3}a_2 K)\omega_2, \\ \nabla a_{22} = (a_{122} - \dfrac{2}{3}a_2 K)\omega_1 + a_{222}\omega_2, \end{array}\right. \tag{18}$$

where $a_{ijk} \in C^\infty(M^2),\ i, j, k = 1, 2$.

It follows from (16) and (18) that a_i and a_{ij} are relative invariants of weight one and two, respectively.

5. Connection forms, curvatures, and curvature forms of 3-subwebs of a 4-web. Next we will indicate the formulas for the connection forms, the curvatures, and the curvature forms of all 3-subwebs [1, 2, 3] [1, 2, 4], [1, 3, 4] and [2, 3, 4] of a 4-web W.

For the 3-subweb [1, 2, 3], the indicated quantities are:

$$\underset{123}{\theta} = \theta, \quad \underset{123}{K} = K, \quad \underset{123}{\Theta} = K\omega_1 \wedge \omega_2. \tag{19}$$

As was shown in [G 93] (see also [G 88], Section 7.1), for the 3-subweb [1, 2, 4] these quantities are

$$\begin{aligned} \underset{124}{\theta} &= \theta - \frac{a_2}{a}\omega_2, \\ \underset{124}{K} &= \frac{1}{a}\Big(K - \frac{a_{12}}{a} + \frac{a_1 a_2}{a^2}\Big), \\ \underset{124}{\Theta} &= a\underset{124}{K}\omega_1 \wedge \omega_2; \end{aligned} \tag{20}$$

for the 3-subweb [1, 3, 4], they are:

$$\begin{aligned} \underset{134}{\theta} &= \theta + \frac{a_2}{1-a}(\omega_1 + \omega_2), \\ \underset{134}{K} &= \frac{1}{a-1}\Big[K + \frac{a_2(a_1 - a_2)}{(1-a)^2} + \frac{a_{12} - a_{22}}{1-a}\Big], \\ \underset{134}{\Theta} &= (a-1)\underset{134}{K}\omega_1 \wedge \omega_2; \end{aligned} \tag{21}$$

and for the 3-subweb [2, 3, 4], they are

$$\begin{aligned} \underset{234}{\theta} &= \theta + \frac{a_1 - a_2}{a}\omega_2 + \frac{a_1}{1-a}(\omega_1 + \omega_2), \\ \underset{234}{K} &= \frac{1}{a(a-1)}\Big[K + \frac{(2a-1)a_1(a_1 - a_2)}{a^2(1-a)^2} + \frac{a_{11} - a_{12}}{a(1-a)}\Big], \\ \underset{234}{\Theta} &= a(a-1)\underset{234}{K}\omega_1 \wedge \omega_2. \end{aligned} \tag{22}$$

2 Special Classes of 4-Webs in the Plane

1. Parallelizable and linearizable 4-webs.

Definition 3. A 4-web W formed by four foliations of parallel straight lines is said to be a *parallel 4-web*. A 4-web which is equivalent to the parallel 4-web is called *parallelizable.*

For a 3-web in M^2 the definitions of parallel and parallelizable 3-webs are similar.

A parallel 4-web defined by the foliations X_a, $a = 1, 2, 3, 4$, can be given by the functions

$$u_1 = x, \quad u_2 = y, \quad u_3 = x + y, \quad u_4 = ax + y$$

(cf. [G 93] for parallel 3-webs).

Theorem 1. *A 4-web W is parallelizable if and only if the curvature K vanishes, and the basic invariant is constant in M^2, i.e., if and only if the following conditions are satisfied:*

$$K = 0, \quad a_1 = 0, \quad a_2 = 0. \tag{23}$$

For the proof see [G 77] or [G 80], or [G 88], Section 7.2.

Definition 4. A 4-web W formed by 4 foliations of straight lines is said to be *linear*. A 4-web which is equivalent to a linear 4-web is called *linearizable* (*rectifiable*).

It is obvious that a parallelizable 4-web is linearizable but the converse is not true.

In [AGL] the following criterion of linearizability for 4-webs in the plane was found.

Theorem 2. *A 4-web W is linearizable if and only if the curvature K, its covariant derivatives K_1 and K_2, the basic invariant a, and its covariant derivatives a_i, a_{ij}, a_{ijk} of the first three orders satisfy the following equations:*

$$\begin{aligned}
K_1 = {} & \frac{1}{a - a^2}\Big[\frac{1}{3}\Big(a_1(1-a) + aa_2\Big)K - a_{111} + (2+a)a_{112} - 2aa_{122}\Big] \\
& + \frac{1}{(a-a^2)^2}\bigg\{\Big[(4-6a)a_1 + (-2+3a+a^2)a_2\Big]a_{11} \\
& + \Big[(-6+7a+2a^2)a_1 + (2a-3a^2)a_2\Big]a_{12} + \Big[(2a(1-a)a_1 - 2a^2a_2\Big]a_{22}\bigg\} \\
& + \frac{1}{(a-a^2)^3}\bigg\{(-3+8a-6a^2)(a_1)^3 - 2a^3(a_2)^3 \\
& + (6-15a+9a^2+2a^3)(a_1)^2a_2 + (-2a+6a^2-3a^3)a_1(a_2)^2\bigg\},
\end{aligned} \tag{24}$$

$$\begin{aligned}
K_2 = {} & \frac{1}{a - a^2}\Big[\frac{1}{3}\Big(a_1 + a_2(a-1)\Big)K + 2a_{112} - (2a+1)a_{122} + aa_{222}\Big] \\
& + \frac{1}{(a-a^2)^2}\bigg\{\Big[(2a_1 + (2a-2)a_2\Big]a_{11} + \Big[(-5+6a)a_1 + (2-3a-2a^2)a_2\Big]a_{12} \\
& + \Big[(1-a-2a^2)a_1 + 2a^2a_2\Big]a_{22}\bigg\} + \frac{1}{(a-a^2)^3}\bigg\{(4a-2)(a_1)^3 + a^3(a_2)^3 \\
& + (5-12a+6a^2)(a_1)^2a_2 + (-2+5a-3a^2-2a^3)a_1(a_2)^2\bigg\}.
\end{aligned} \tag{25}$$

Theorem 3. *Linearizable* 4*-webs exist, and the general solution of the system of equations defining such webs depends on four arbitrary functions of one variable.*

Proof. There is an indirect proof of this theorem. A linearizable 4-web is equivalent to a linear 4-web. The latter is formed by four one-parameter families of curves. Each of these families has its envelope (a curve). So to define a web foliation, one needs to take a curve. A curve is defined by one function of one variable. Hence a 4-web in the plane depends on four functions of one variable.

We present here also an analytic proof of this theorem. Taking exterior derivatives of (18) and applying (4), (8), (12), (14), (16), (18), (24), and (25), we get three exterior quadratic equations

$$\begin{aligned} &\nabla a_{111} \wedge \omega_1 + \nabla a_{112} \wedge \omega_2 + (\ldots)\omega^1 \wedge \omega^2 = 0, \\ &\nabla a_{112} \wedge \omega_1 + \nabla a_{122} \wedge \omega_2 + (\ldots)\omega^1 \wedge \omega^2 = 0, \\ &\nabla a_{122} \wedge \omega_1 + \nabla a_{222} \wedge \omega_2 + (\ldots)\omega^1 \wedge \omega_2 = 0, \end{aligned} \tag{26}$$

where the coefficients $(\ldots)$ depend on K, a, a_i, and a_{ij}. If we substitute the expressions for K_1 and K_2 from (24) and (25) into equation (14) and take exterior derivatives of the resulting equation using the same (4), (8), (12), (14), (16), (18), (24), and (25), we get one more exterior quadratic equation of the form

$$\begin{aligned} &[-\nabla a_{111} + (2+a)\nabla a_{112} - 2a\nabla a_{122}] \wedge \omega^1 \\ &+ [2\nabla a_{112} - (2a+1)\nabla a_{122} + a\nabla a_{222}] \wedge \omega_2 + (\ldots)\omega^1 \wedge \omega^2 = 0, \end{aligned} \tag{27}$$

where the coefficients $(\ldots)$ again depend on K, a, a_i, and a_{ij}.

The system defining linearizable 4-webs consists of the Pfaffian equation (14) (in which K_1 and K_2 are replaced by their values (24) and (25)), (3), (16), (18), and exterior quadratic equations (26), (27). For this system we have 4 unknown forms ∇a_{ijk}. Hence $q = 4$. Since the exterior quadratic equations (26) and (27) are obviously independent, and their number is four, we have $s_1 = 4$. Thus $s_2 = q - s_1 = 0$, and the Cartan number $Q = s_1 + 2s_2 = 4$. On the other hand, if we apply Cartan's lemma to (26), we find the forms ∇a_{ijk} in terms of ω_i, and there will be 5 coefficients in the expansions of these forms. Substituting these expansions into (27), we find one of these coefficients. So, the dimension N of the space of integral elements over a point equals four, $N = 4$. Thus $Q = N$, and by Cartan's test (see [BCGGG 91], p. 120), the system is in involution, and its general solution depends on four functions of one variable. □

2. 4-webs whose basic invariant is covariantly constant on one of web foliations.

Theorem 4. *The basic invariant a of a* 4*-web is covariantly constant on the web foliation X_1, X_2, X_3, or X_4 if and only if there is respectively the following relation between the covariant derivatives a_1 and a_2 of this invariant:*

$$a_2 = 0, \tag{28}$$

$$a_1 = 0, \tag{29}$$

$$a_1 = a_2, \tag{30}$$

$$a_1 = aa_2. \tag{31}$$

Proof. In fact, it follows from (1.3) that conditions (29)–(31) are necessary and sufficient for the following congruences:

$$\begin{aligned}
da &\equiv 0 \pmod{\omega_1}, \\
da &\equiv 0 \pmod{\omega_2}, \\
da &\equiv 0 \pmod{\omega_1 + \omega_2}, \\
da &\equiv 0 \pmod{a\omega_1 + \omega^2},
\end{aligned}$$

respectively. □

Theorem 5. *4-webs whose basic invariant is covariantly constant on one of web foliations exist, and the general solution of the system of equations defining such webs depends on one arbitrary function of two variables.*

Proof. Suppose, for example, that the basic invariant is covariantly constant on the foliation X_2 defined by the equation $\omega_2 = 0$. Then condition (29) holds. For such a 4-web, we have the Pfaffian equations (12) and (16). By (29), the former becomes

$$da = a_2\omega_2. \tag{32}$$

Taking the exterior derivatives of these two Pfaffian equations, we arrive at the following two exterior quadratic equations:

$$\begin{aligned}
&\nabla K_1 \wedge \omega_1 + \nabla K_2 \wedge \omega_2 = 0, \\
&\nabla a_2 \wedge \omega_2 = 0,
\end{aligned} \tag{33}$$

where $\nabla K_i = dK_i - 3K_i\theta$, $\nabla a_2 = da_2 - a_2\theta$. Thus, we have three unknown functions, and $q = 3$. Since there are two exterior quadratic equations, we have $s_1 = 2$. As a result, $s_2 = q - s_1 = 1$, and the Cartan number $Q = s_1 + 2s_2 = 4$. Applying Cartan's lemma to the exterior quadratic equations (33), we find that

$$\begin{aligned}
\nabla K_1 &= K_{11}\omega_1 + K_{12}\omega^2, \\
\nabla K_2 &= K_{12}\omega_1 + K_{22}\omega^2, \\
\nabla a_2 &= a_{22}\omega_2.
\end{aligned}$$

Hence the dimension N of the space of integral elements over a point equals 4, $N = 4$. Therefore, $Q = N$, and by Cartan's test, our system of equations is in involution, and its general solution depends on one arbitrary function of two variables.

The proof in other cases is similar. The difference is that in the cases when a 4-web satisfies equations (28), (30), and (31), equation (32) has respectively the form

$$da = a_1\omega_1,$$

$$da = a_1(\omega_1 + \omega_2),$$

and

$$da = a_2(a\omega_1 + \omega_2),$$

and the last equation (33) becomes

$$\nabla a_1 \wedge \omega_1 = 0,$$

$$\nabla a_1 \wedge (\omega_1 + \omega_2) = 0,$$

and

$$\nabla a_2 \wedge (a\omega_1 + \omega_2) = 0,$$

respectively. □

3. Almost parallelizable 4-webs.

Definition 5. We say that a 4-web W is *almost parallellizable* if its 3-subweb $[1, 2, 3]$ is parallelizable (i.e., $K = 0$), and its basic invariant is covariantly constant on one of the web foliations.

So, according to subsection 2.2, there are four classes of almost parallelizable 4-webs characterized by the following conditions:

$$K = 0, \ \ a_1 = 0, \tag{34}$$

$$K = 0, \ \ a_2 = 0, \tag{35}$$

$$K = 0, \ \ a_1 = a_2, \tag{36}$$

$$K = 0, \ \ a_1 = aa_2. \tag{37}$$

We denote the 4-webs of these 4 classes by APW_a, $a = 1, 2, 3, 4$.

Theorem 6. *The almost parallelizable* 4*-webs* APW_a *exist, and the general solution of the system of equations defining such webs depends on one arbitrary function of one variable.*

Proof. In fact, in each of the cases (34)–(37), the condition $K = 0$ implies $K_1 = 0$, $K_2 = 0$. Thus, we have $q = 1$. For a 4-web APW_2, in (33) (and in similar systems corresponding to 4-webs APW_1, APW_3, and APW_4) only the second equation remains. Hence Cartan's characters are $s_1 = 1$ and $s_2 = 0$. It is easy to see that in this case $Q = N = 1$, and by Cartan's test, our system of equations is involutive, and its general solution depends on one arbitrary function of one variable. □

The following theorem establishes the values of the curvatures $\underset{abc}{K}$ for the 4-webs APW_a.

Theorem 7. *The curvatures* $\underset{abc}{K}$ *for the* 4*-webs* APW_a *are*

(i) *For the* 4*-webs* APW_1*:*

$$\underset{123}{K} = \underset{124}{K} = \underset{134}{K} = 0, \quad \underset{234}{K} = \frac{(1-2a)a_1^2 - (a-a^2)a_{11}}{(a-a^2)^3}; \tag{38}$$

(ii) *For the* 4*-webs* APW_2*:*

$$\underset{123}{K} = \underset{124}{K} = \underset{234}{K} = 0, \quad \underset{134}{K} = \frac{a_2^2 + (1-a)a_{22}}{(1-a)^3}; \tag{39}$$

(iii) *For the* 4*-webs* APW_3*:*

$$\underset{123}{K} = \underset{134}{K} = \underset{234}{K} = 0, \quad \underset{124}{K} = \frac{a_1^2 - aa_{11}}{a^3}; \tag{40}$$

(iv) *For the* 4*-webs* APW_2*:*

$$\underset{123}{K} = 0, \quad \underset{124}{K} = -\frac{aa_{22}}{a^2}, \quad \underset{134}{K} = \frac{a_{22}}{1-a}, \quad \underset{234}{K} = \frac{a_{22}}{a-a^2}. \tag{41}$$

Proof. The proof is straightforward: it follows from formulas (20)—(22) and conditions (34)—(37). □

4. 4-webs with a nonconstant basic invariant and equal curvature forms of their 3-subwebs. It is important to emphasize that we assume here that *the basic invariant of a* 4*-web is not covariantly constant on* M_2, i.e., its covariant derivatives a_1 and a_2 do not vanish simultaneously. This assumption shows that the class of 4-webs we are going to consider is completely different from the class considered by Nakai in [Na 96] and [Na 98]: although Nakai also considered 4-webs with equal curvature forms of their 3-subwebs but he assumed that the basic invariant a of a 4-web, is covariantly constant on M_2, i.e., he assumed that $a_1 = a_2 = 0$. We will consider 4-webs with a constant basic invariant and equal curvature forms of their 3-subwebs in subsection 6.

Theorem 8. *Suppose that the basic invariant* a *of a* 4*-web* W *is not constant. Then* W *has the curvature forms of its* 3*-subwebs* $[a, b, c]$ *being equal if and only if all* 3*-subwebs* $[a, b, c]$ *are parallelizable, i.e., if*

$$\underset{abc}{K} = 0, \quad \underset{abc}{\Theta} = 0, \quad a, b, c = 1, 2, 3, 4, \tag{42}$$

and the second covariant derivatives a_{ij} *of the basic invariant* a *are expressed in terms of the invariant* a *itself and its first covariant derivatives* a_i *as follows:*

$$a_{11} = \frac{a_1[(1-2a)a_1 + aa_2]}{a-a^2}, \tag{43}$$

$$a_{12} = \frac{a_1 a_2}{a}, \tag{44}$$

$$a_{22} = \frac{a_2(a_1 - a a_2)}{a - a^2}. \tag{45}$$

Proof. In fact, by equation (20), it follows that if $a \neq 0$, then the condition $\underset{124}{\Theta} = \underset{123}{\Theta}$ holds if and only if equation (44) is valid.

In a similar manner, by (21) and (22), it follows that if $a \neq 0$, then the conditions $\underset{134}{\Theta} = \underset{123}{\Theta}$ and $\underset{234}{\Theta} = \underset{123}{\Theta}$ hold if and only if equations (45) and (43) are valid, respectively.

To prove (42), we substitute the values of a_{ij} from (43)–(45) into equations (16). This gives

$$\begin{aligned} da_1 &= a_1\theta + \frac{\big[(1-2a)a_1 + a a_2\big]a_1}{a - a^2}\omega_1 + \frac{a_1 a_2}{a}\omega_2, \\ da_2 &= a_2\theta + \frac{1}{a - a^2}[a_1 a_2(1-a)\omega_1 + a_2(a_1 - a a_2)\omega_2]. \end{aligned} \tag{46}$$

Exterior differentiation of either of equations (46) implies $K = 0$. The latter condition and equations (43)–(45) lead to (42).

Thus, each of 3-subwebs $[a, b, c]$ is parallelizable. However, by Theorem 1, this does not imply the parallelizability of the 4-web W—the latter web is parallelizable if and only if all 3-subwebs $[a, b, c]$ are parallelizable and the basic invariant a is covariantly constant in the connection γ_{123}, i.e., if and only if we have conditions (42) and $a_1 = a_2 = 0$. □

We denote the 4-webs with equal curvature forms of their 3-subwebs and a nonconstant basic invariant by MW.

4-webs all 3-subwebs of which are parallelizable (hexagonal) were introduced by Mayrhofer (see [M 28]). This is the reason that we denoted them by MW (Mayrhofer's 4-webs). We arrived to Mayrhofer's 4-webs MW solving the problem of finding 4-webs with a nonconstant basic invariant and equal curvature forms of its 3-subwebs. Mayrhofer was first who proved that such 4-webs are equivalent to 4-webs all foliations of which are pencils of straight lines. There are different proofs of Mayrhofer's theorem: see [M 28], [M 29], [R 28], [BB 38], §10. However, all these proofs are rather complicated. We think that the reason for this is that the authors of these papers did not use the invariant characterization of 4-webs MW similar to the conditions (42)–(45).

Now we prove the existence theorem for webs NW.

Theorem 9. *The 4-webs MW exist. The system of Pfaffian equations defining such webs is completely integrable, and its general solution depends on three arbitrary constants.*

Proof. For a web MW, we have Pfaffian equations (12) and (16), where the a_{ij} are expressed by formulas (43)–(45). If we take exterior derivatives of equations (16) in which a_{ij} are replaced by their values (43)–(45), we arrive at the identities. So, for the system in question, we have $s_0 = 3$, $s_1 = 0$. The system is completely integrable, and its general solution depends on three arbitrary constants. $\square$

In the next theorem we present a straightforward short proof of the fact that 4-webs MW are always linearizable by applying a linearizability condition of 4-webs found in [AGL]. This theorem gives the first nontrivial application of linearizability conditions (24) and (25) for 4-webs.

Theorem 10. *A* 4*-web* MW *is linearizable.*

Proof. To prove this result, we must prove that conditions (24) and (25) of linearizability are satisfied identically for the 4-webs MW. To check these conditions, we substitute the values of the invariants a_{ij} and a_{ijk} into them. As to a_{ij}, their values are given by (43)–(45).

In order to find a_{ijk}, we differentiate (43)–(45) by using (12) and (16) and compare the result with equations (18). Equating the coefficients of ω_1 and ω_2 in the resulting equations, we find that

$$\begin{aligned}
a_{111} &= \frac{1}{(a-a^2)^2}\Big[(6a^2-6a+1)a_1^3 + (4a-6a^2)a_1^2a_2 + a^2a_1a_2^2\Big],\\
a_{112} &= \frac{1}{(a-a^2)^2}\Big[(1-2a)(1-a)a_1^2a_2 + (a-a^2)a_1a_2^2\Big] = \frac{a_2a_{11}}{a},\\
a_{122} &= \frac{a_1a_2(a_1-aa_2)}{a^2(1-a)} = \frac{a_1a_{22}}{a},\\
a_{222} &= \frac{1}{(a-a^2)^2}\Big(a_1^2a - 2aa_1a_2^2 + a^2a_2^3\Big).
\end{aligned} \tag{47}$$

Next, by (42), for 4-webs MW, we have $K = 0$. This and (14) imply $K_1 = K_2 = 0$. Substituting the values of a_{ij} from (43)–(45) and the values of a_{ijk} from (47) into (24) and (25), after lengthy straightforward calculations, we come to the identities: the coefficients of a_1^3, $a_1^2a_2$, $a_1a_2^2$ and a_2^3 vanish. Hence any 4-web MW is linearizable. $\square$

5. Special classes of 4-webs *MW*. Although for 4-webs MW the basic invariant a is not covariantly constant on M^2, it can be covariantly constant on one of the foliations X_a of a web MW. If this is the case, we have four special classes of 4-webs MW. These four classes are intersections of four classes of almost parallelizable 4-webs APW_a and the class of 4-webs MW. We denote webs of these classes by MW_a.

Theorem 11. *The* 4*-webs* MW_a *are characterized by the following conditions:*

$$
\begin{aligned}
&\text{Webs } MW_1: && K=0,\ a_2=0,\ a_{11}=\frac{(1-2a)a_1^2}{a-a^2},\ a_{12}=0,\ a_{22}=0;\\
&\text{Webs } MW_2: && K=0,\ a_1=0,\ a_{11}=0,\ a_{12}=0,\ a_{22}=-\frac{a_2^2}{1-a};\\
&\text{Webs } MW_3: && K=0,\ a_1=a_2,\ a_{11}=a_{12}=a_{22}=\frac{a_1^2}{a};\\
&\text{Webs } MW_4: && K=0,\ a_1=aa_2,\ a_{11}=2aa_2^2,\ a_{12}=a_2^2,\ a_{22}=0.
\end{aligned}
\tag{48}
$$

Proof. The proof follows from (34)–(37) and (43)–(45). □

Note that by (48) and (38)–(41), for any 4-web MW_a, all the curvatures $\underset{abc}{K}$ vanish (as it should be according to Theorem 8 (see (42)).

Theorem 12. *The* 4*-webs* $MW_a, a = 1,2,3,4$, *exist. The system of Pfaffian equations defining such webs is completely integrable, and its general solution depends on two arbitrary constants.*

Proof. In fact, by (48), in the system defining each of the webs MW_a (see proof of Theorem 9), only two Pfaffian equations are independent. □

The next theorem gives a relation between the 4-webs APW_a and MW_a.

Theorem 13. *A* 4*-web* $APW_a, a = 1,2,3,4$, *is a* 4*-web* MW_a *if and only if it is linearizable.*

Proof. The necessity is obvious: any MW_a is APW_a, and by Theorem 10, any MW_a is linearizable.

We prove the sufficiency separately for different $a = 1,2,3,4$.

(i) *Linearizable* 4*-webs* APW_1. Note that it follows from (34), (16), and (18) that for a 4-web APW_1 not only $a_2 = 0$ but also $a_{12} = a_{22} = a_{112} = a_{122} = a_{222} = 0$. Since by hypothesis our 4-web APW_1 is linearizable, the linearizability conditions (24) and (25) must be satisfied identically. Substituting

$$K = K_1 = K_2 = 0,\ \ a_1 = a_{11} = a_{12} = a_{111} = a_{112} = a_{122} = 0$$

into (25) and (24), we find that

$$a_{11} = -\frac{(1-2a)a_1^2}{a-a^2}$$

and

$$a_{111} = -\frac{(6a^2-6a+1)a_1^3}{(a-a^2)^2}.$$

Comparing with (48), we see that *a linearizable* 4*-web* APW_1 is a 4-web MW_1. As to the second condition obtained from (24), it is easy to see that it is a differential consequence of the first condition.

(ii) *Linearizable* 4*-webs* APW_2. Note that it follows from (34), (16), and (18) that for a 4-web APW_2 not only $a_1 = 0$ but also $a_{11} = a_{12} = a_{111} = a_{112} = a_{122} = 0$. Since by hypothesis, our 4-web APW_2 is linearizable, the linearizability conditions (24) and (25) must be satisfied identically. Substituting

$$K = K_1 = K_2 = 0, \;\; a_1 = a_{11} = a_{12} = a_{111} = a_{112} = a_{122} = 0$$

into (24) and (25), we find that

$$a_{22} = -\frac{a_2^2}{1-a}$$

and

$$a_{222} = -\frac{a_2^3}{(1-a)^2}.$$

Comparing with (48), we see that *a linearizable* 4*-web* APW_2 is a 4-web MW_2. As to the second condition obtained from (25), it is easy to see that it is a differential consequence of the first condition.

(iii) *Linearizable* 4*-webs* APW_3. Note that it follows from (34), (16), and (18) that for a 4-web APW_3, the condition $a_1 = a_2$ implies $a_{11} = a_{12} = a_{22}$ and $a_{111} = a_{112} = a_{122} = a_{222}$. Since by hypothesis, our 4-web APW_3 is linearizable, the linearizability conditions (24) and (25) must be satisfied identically. Substituting

$$K = K_1 = K_2 = 0, \;\; a_1 = a_2, a_{11} = a_{12} = a_{22}, \;\; a_{111} = a_{112} = a_{122} = a_{222}$$

into (24) and (25), we find that

$$a_{111} = \frac{4a_1 a_{11}}{a} - \frac{3a_1^3}{a^2}$$

and

$$a_{111} = -\frac{2a_1 a_{11}}{a^2} - \frac{a_1^3}{a^3}.$$

It follows from these two equations that

$$a_{11} = -\frac{a_1^2}{a}.$$

Comparing this with (48), we see that *a linearizable* 4*-web* APW_3 is a 4-web MW_3. As to the expression for a_{111}, it is easy to see that this expression can be obtained by differentiation from the expression of a_{11}.

(iv) *Linearizable* 4*-webs* APW_4. Note that it follows from (34), (16), and (18) that for a 4-web APW_4, the condition $a_1 = aa_2$ implies

$$a_{11} = a^2 a_{22} + 2a_2^2, \;\; a_{12} = aa_{22} + a_2^2$$

and

$$a_{111} = a^3 a_{222} + (7a^2 a_2 + 2aa_2)a_{22} + 6aa_2^3,$$
$$a_{112} = a^2 a_{222} + 6aa_2 a_{22} + 2a_2^3,$$
$$a_{122} = aa_{222} + 3a_2 a_{22}.$$

Since by hypothesis, our 4-web APW_4 is linearizable, the linearizability conditions (24) and (25) must be satisfied identically. Substituting

$$K = K_1 = K_2 = 0, \;\; a_1 = aa_2$$

and the expressions for $a_{11}, a_{12}, a_{111}, a_{112}, a_{122}$ written above into (24) and (25), we find that (25) is identically satisfied, and (24) reduces to

$$a_{22} = 0.$$

As a result, we see from the expressions for a_{11} and a_{12} that they become

$$a_{11} = 2aa_2^2, \;\; a_{12} = a_2^2.$$

Comparing the results with (48), we see that *a linearizable* 4*-web* APW_4 is a 4-web MW_4.

□

The next theorem gives another criterion for a 4-web APW_a to be a 4-web MW_a.

Theorem 14. *A* 4*-web* APW_a, $a = 1, 2, 3, 4$, *is a* 4*-web* MW_a *if and only if all its* 3*-subwebs are parallelizable.*

Proof. By Theorem 2.10, for 4-webs APW_1, APW_2, and APW_3, three of the 3-subwebs are parallelizable. It follows from (38)–(40) that the fourth 3-subweb of these 4-webs is parallelizable if and only if we have

$$a_{22} = -\frac{a_2^2}{1-a}, \;\; a_{11} = \frac{(1-2a)a_1^2}{a-a^2}, \;\; a_{11} = a_{12} = a_{22} = \frac{a_1^2}{a},$$

respectively. Comparing these relations with (48), we see that 4-webs APW_1, APW_2, and APW_3 are 4-webs MW_1, MW_2, and MW_3, respectively.

For a 4-web APW_4, by (41), only the 3-subweb $[1, 2, 3]$ is parallelizable. But relations (41) show also that other 3-subwebs are parallelizable if and only if

$$a_{22} = 0.$$

But as we already showed in the proof of Theorem 13, the condition $a_{22} = 0$ implies

$$a_{11} = 2aa_2^2, \;\; a_{12} = a_2^2,$$

and by (48), our 4-web APW_4 is a 4-web MW_4. □

6. 4-webs with a constant basic invariant and equal curvature forms of their 3-subwebs. For such 4-webs, we have

$$a_1 = a_2 = 0. \tag{49}$$

By (14) and (16), conditions (49) imply

$$a_{ij} = 0, \quad a_{ijk} = 0, \quad i, j, k = 1, 2. \tag{50}$$

Now formulas (20)–(22) imply

$$\underset{123}{K} = \underset{124}{K} = \underset{134}{K} = \underset{234}{K} = K$$

and

$$\underset{123}{\Theta} = \underset{124}{\Theta} = \underset{134}{\Theta} = \underset{234}{\Theta} = K\omega_1 \wedge \omega_2,$$

i.e., the curvatures and the curvature forms of all 3-subwebs are equal without vanishing.

We denote such 4-webs by NW (Nakai's webs—see [Na 96] and [Na 98]).

Theorem 15. *The* 4*-webs* NW *exist, and the general solution of the system of equations defining such webs depends on one arbitrary function of two variables.*

Proof. In fact, for 4-webs NW, equation (12) becomes $da = 0$. Taking exterior derivatives of (14), we arrive at the exterior quadratic equation

$$\nabla K_1 \wedge \omega_1 + \nabla K_2 \wedge \omega_2 = 0,$$

where $\nabla K_i = dK_i - 3K_i\theta$. It is easy to see that in this case, we have $q = 2$, the Cartan characters $s_1 = 1$, $s_2 = 1$, and $Q = s_1 + 2s_2 = 3$.

Next, it follows from the quadratic equation above that

$$\nabla K_1 = K_{11}\omega_1 + K_{12}\omega^2, \quad \nabla K_2 = K_{12}\omega_1 + K_{22}\omega^2.$$

Hence the dimension N of the space of integral elements over a point equals three, $N = 3$. Therefore, $Q = N$, and by Cartan's test, our system of equations is in involution, and its general solution depends on one arbitrary function of two variables. □

It is easy to see that, in general, a 4-web NW is not linearizable since conditions (24) and (25) do not hold for such a web.

However, equations (24) and (25) produce the simple conditions for linearizability of 4-webs NW. Namely, by (49) and (50), it follows from (24) and (25) that

$$K_1 = 0, \quad K_2 = 0. \tag{51}$$

Now we are able to prove the following theorem describing linearizable 4-webs NW.

Theorem 16. *If a* 4*-web* NW *is linearizable, then it is parallelizable.*

Proof. In fact, for a linearizable 4-web NW, we have equations (51). They and equation (14) imply

$$dK - 3K\theta = 0.$$

Taking exterior derivative of this equation and applying (8), we find that

$$-2K^2\omega_1 \wedge \omega_2 = 0.$$

It follows that $K = 0$, and by Theorem 1, this and (49) prove that a linearizable 4-web NW is parallelizable. $\square$

Theorem 16 implies the following corollaries.

Corollary 17. *There exists no linearizable nonparallelizable* 4*-webs* NW*.*

Corollary 18. *Parallelizable* 4*-webs* NW *exist. They are defined by one completely integrable equations* $da = 0$*. The set of such parallelizable* 4*-webs* NW *depends on one constant* (*the constant basic invariant* a)*.*

Proof. In fact, since the 4-web NW is parallelizable, we have $K = K_1 = K_2 = 0$. So, the only remaining equation defining parallelizable 4-webs NW is the equation $da = 0, a \neq 0, 1$. This proves the corollary. $\square$

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