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Calculus of Variations

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Traveling wave solutions of harmonic heat flow

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Abstract We prove the existence of a traveling wave solution of the equation $u_t = \Delta u + |\nabla u|^2 u$ in an infinitely long cylinder of radius R, which connects two locally stable and axially symmetric steady states at $x_3 = \pm\infty$. Here u is a director field with values in $\mathbb{S}^2 \subset \mathbb{R}^3$: $|u| = 1$. The traveling wave has a singular point on the cylinder axis. Letting $R \to \infty$ we obtain a traveling wave defined in all space.

Keywords Harmonic map · Director field · Traveling wave · Singularity · Calculus of variations · Bistable potential

1 Introduction

Let u be a unit vector in $\mathbb{R}^3$ defined on the disk $D_R \subset \mathbb{R}^2$ of radius R. Considering the Dirichlet integral $\int_{D_R} |\nabla u|^2 dx$ for $u \in H^1(D_R; \mathbb{S}^2)$, the corresponding Euler-Lagrange equation is (see [24])

$$\Delta u + |\nabla u|^2 u = 0 \quad \text{in } D_R := \left\{(x_1, x_2) : x_1^2 + x_2^2 < R^2\right\}. \tag{1}$$

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Given a constant $b > \frac{1}{R}$, we associate to Eq. (1) the boundary condition

$$u(x_1, x_2) = u_b(x_1, x_2) := \left(\frac{2bx_1}{1+b^2R^2}, \frac{2bx_2}{1+b^2R^2}, \frac{1-b^2R^2}{1+b^2R^2} \right) \text{ for } (x_1, x_2) \in \partial D_R. \tag{2}$$

Setting $r := \sqrt{x_1^2 + x_2^2}$, the following two functions are solutions of (1)–(2):

$$u_+(x_1, x_2) := \left(\frac{2bx_1}{1+b^2r^2}, \frac{2bx_2}{1+b^2r^2}, \frac{1-b^2r^2}{1+b^2r^2} \right),$$
$$u_-(x_1, x_2) := \left(\frac{2bR^2x_1}{r^2+b^2R^4}, \frac{2bR^2x_2}{r^2+b^2R^4}, \frac{r^2-b^2R^4}{r^2+b^2R^4} \right).$$

Observe that $|u_+| = |u_-| = 1$, and that, since $bR > 1$,

$$\int_{D_R} |\nabla u_-|^2 dx = \frac{8\pi}{1+b^2R^2} < \int_{D_R} |\nabla u_+|^2 dx = \frac{8b^2R^2\pi}{1+b^2R^2}. \tag{3}$$

More precisely, u_- is a global minimizer of the Dirichlet integral in $H^1(D_R; \mathbb{S}^2)$ subject to the boundary condition (2), while u_+ is a local minimizer.

In the present paper we consider traveling wave solutions of the equation

$$u_t = \Delta u + |\nabla u|^2 u \quad \text{in } D_R \times \mathbb{R} \tag{4}$$

which connect u_- at $x_3 = -\infty$ to u_+ at $x_3 = \infty$:

$$u(x_1, x_2, x_3, t) = v(x_1, x_2, x_3 - ct) \in \mathbb{S}^2,$$

where $c \in \mathbb{R}$ and the function $v = v(x_1, x_2, z)$ is a solution of the problem

$$\begin{cases} \Delta v + cv_z + |\nabla v|^2 v = 0 \text{ and } |v| = 1 & \text{in } D_R \times \mathbb{R} \\ v(x_1, x_2, \pm\infty) = u_\pm(x_1, x_2) & \text{for } (x_1, x_2) \in D_R \\ v = u_b & \text{on } \partial D_R \times \mathbb{R}. \end{cases}$$

In other words, the traveling wave is a connecting orbit between the two harmonic maps u_- and u_+.

In view of the energy inequality (3) and the bistable character of the Dirichlet integral, we expect that there exists a solution v for a certain *positive* wave speed, c_R. Actually, a similar result holds for bistable Ginzburg-Landau systems [18], which can be considered as approximations of our problem (see [24]). In the present paper we shall construct a traveling wave with speed $c_R > 0$. In a forthcoming paper [3] we shall use this construction to prove the existence of a traveling wave for *all* wave speeds $c \in \mathbb{R}$, a most surprising result which is undoubtfully counterintuitive, in particular if $c < 0$. As we shall see in [3], this result is intimately related to a nonuniqueness property of initial-boundary value problems for Eq. (4) (see also [4, 5, 22]).

Before stating our main results we observe that the asymptotic states $u_\pm$ are axially symmetric and can be written as

$$u_\pm(x_1, x_2) = \left(\frac{x_1}{r}\sin\theta_\pm(r), \frac{x_2}{r}\sin\theta_\pm(r), \cos\theta_\pm(r)\right),$$

where

$$\begin{aligned} \theta_+(r) &:= 2\arctan(br) && \text{for } 0 < r \le R,\\ \theta_-(r) &:= 2\arctan\left(\frac{bR^2}{r}\right) = \pi - 2\arctan\left(\frac{r}{bR^2}\right) && \text{for } 0 < r \le R. \end{aligned}$$

Therefore it is natural to consider axially symmetric traveling waves:

$$v(x_1, x_2, z) = \left(\frac{x_1}{r}\sin\theta(r,z), \frac{x_2}{r}\sin\theta(r,z), \cos\theta(r,z)\right),$$

where θ, the so-called *angle function*, is a solution of the problem

$$(\mathrm{I}_{c,R}) \quad \begin{cases} \theta_{rr} + \dfrac{1}{r}\theta_r + \theta_{zz} + c\theta_z - \dfrac{\sin(2\theta)}{2r^2} = 0 & \text{in } (0,R)\times\mathbb{R}\\ \theta(R,z) = 2\arctan(bR) & \text{for } z \in \mathbb{R}\\ \theta(r,\pm\infty) = \theta_\pm(r) & \text{for } 0 < r < R. \end{cases}$$

The axial symmetry implies that $v(0,0,z) = (0,0,\pm1)$. Since $v(0,0,\pm\infty) = u_\pm(0,0) = (0,0,\pm1)$, we expect that any solution of problem $\mathrm{I}_{c,R}$ has at least one singular point at the cylinder axis (although, in principle, the singularity could also occur at $z = \pm\infty$). This is confirmed by the following result:

Theorem 1.1 *Let $b, R > 0$ be such that $bR > 1$. Then there exists $c_R > 0$ such that Problem $I_{c_R,R}$ has a solution, θ_R, which satisfies:*
(i) θ_R is real analytic in $[0,R]\times\mathbb{R}\setminus\{(0,0)\}$;
(ii) $\theta_R(0,z) = \pi$ if $z < 0$, $\theta_R(0,z) = 0$ if $z > 0$;
(iii) θ_R is strictly decreasing with respect to z in $(0,R)\times\mathbb{R}$;
(iv) the limits of θ_R to $\theta_\pm$ as $z \to \pm\infty$ are uniform with respect to r.

The translation invariance of Problem $\mathrm{I}_{c,R}$ with respect to z implies that θ_R belongs to a one-parameter family of solutions of Problem $\mathrm{I}_{c,R}$. If $bR < 1$, the energy inequality (3) is reversed and, due to the symmetry of the problem, Theorem 1.1 continues to hold with $c_R < 0$.

Our second main result concerns the limit problem as $R \to \infty$:

$$(\mathrm{I}_{c,\infty}) \quad \begin{cases} \theta_{rr} + \dfrac{1}{r}\theta_r + \theta_{zz} + c\theta_z - \dfrac{\sin(2\theta)}{2r^2} = 0 & \text{in } \mathbb{R}^+\times\mathbb{R}\\ \theta(r,\infty) = 2\arctan(br) & \text{for } r > 0\\ \theta(r,-\infty) = \pi & \text{for } r > 0. \end{cases}$$

Observe that in this case the equilibrium solution $2\arctan(br)$ is no longer isolated and belongs to the continuum $\{2\arctan(ar) : a \ge 0\}$.

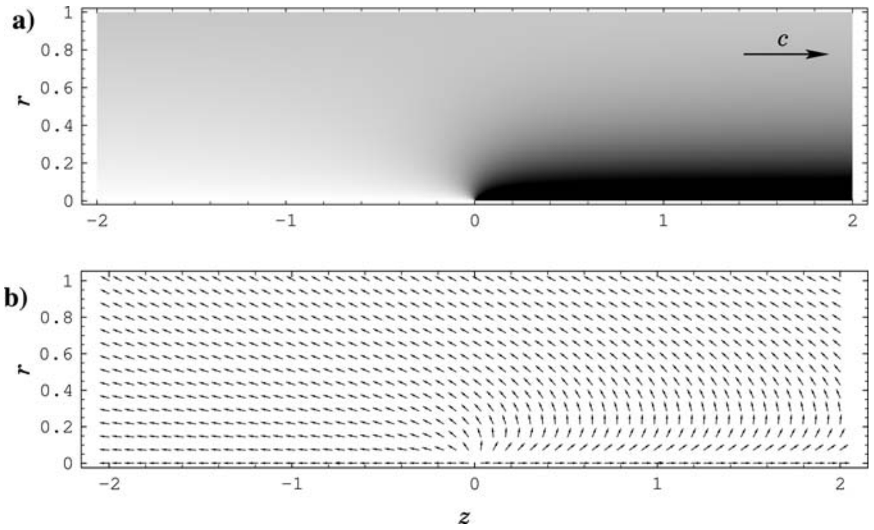

Fig. 1 Qualitative form of the traveling wave solution from Theorem 1.1. In **a**, the angle variable θ as a function of z and r is shown as a density plot, with black corresponding to $\theta = 0$ and white to $\theta = \pi$. In **b**, the corresponding vector field is plotted. The wave is moving from left to right

Theorem 1.2 *Let $b > 0$ and let c_R be defined by Theorem 1.1 for all $R > \frac{1}{b}$. Then $c_R \to c_\infty$ as $R \to \infty$ for some $c_\infty \in \mathbb{R}^+$ and Problem $I_{c_\infty,\infty}$ has a solution, θ_∞, which satisfies:*
(i) θ_∞ is real analytic in $\mathbb{R}^+ \times \mathbb{R} \setminus \{(0,0)\}$;
(ii) $\theta_\infty(0, z) = \pi$ if $z < 0$, $\theta_\infty(0, z) = 0$ if $z > 0$;
(iii) θ_∞ is strictly decreasing with respect to z in $\mathbb{R}^+ \times \mathbb{R}$;
(iv) the limits of θ_∞ as $z \to \pm\infty$ are uniform with respect to r.

Let us note that the problem of existence of traveling wave solutions for scalar reaction-diffusion equations has been studied in great detail (see, e.g. [1, 28]). In particular, problems in infinite cylinders with Dirichlet boundary data were treated in [9, 14, 27]. In our case, the situation is complicated by the fact that the nonlinearity in Problem $\mathrm{I}_{c_R,R}$ becomes singular as $r \to 0$. This is why a variational approach to this problem can be particularly useful.

The proof of Theorem 1.1 is based on the solution of a constrained minimization problem which is similar to one introduced by Lucia, Muratov and Novaga in the context of Ginzburg-Landau problems in cylinders [18–20] (see also the work [14] of Heinze for a related approach). Methods introduced in [2] will be used to handle some specific technicalities related to director fields and axial symmetry. In Sect. 2 we introduce and solve the constrained minimization problem, and in Sect. 3 we prove Theorem 1.1. In Sect. 4 we consider the limit $R \to \infty$ and prove Theorem 1.2.

2 The constrained minimization problem

In the following, we follow closely the arguments of [19]. Formally the equation for $\theta(r, z)$,

$$\theta_{rr} + \frac{1}{r}\theta_r + \theta_{zz} + c\theta_z - \frac{\sin(2\theta)}{2r^2} = 0 \quad \text{in } (0, R) \times \mathbb{R}, \tag{5}$$

is the Euler-Lagrange equation of the functional

$$\int_{\mathbb{R}} dz \int_0^R \frac{1}{2} r e^{cz} \left(\theta_r^2 + \theta_z^2 + \frac{\sin^2\theta}{r^2} - (\theta_+')^2 - \frac{\sin^2\theta_+}{r^2} \right) dr.$$

The terms of the integrand containing $\theta_+(r) = 2\arctan(br)$ have been added to make the functional finite for certain functions θ behaving like θ_+ as $z \to \infty$. More precisely, setting $f = \theta - \theta_+$ and denoting by $L^2_{c,r}((0, R) \times (-\infty, M))$, with $-\infty < M \le \infty$, the set of Lebesgue measurable functions f on $(0, R) \times (-\infty, M)$ such that

$$\int_{-\infty}^M dz \int_0^R r e^{cz} f^2 dr < \infty,$$

we define the sets

$$\begin{aligned} Y^M_{c,R} = \Bigg\{ & f \in L^2_{c,r}((0, R) \times (-\infty, M));\ f_r,\ f_z,\ \frac{\sin f}{r} \in L^2_{c,r}((0, R) \times (-\infty, M)), \\ & \qquad f(R, z) = 0 \text{ for a.e.} z \in \mathbb{R} \Bigg\}, \\ Y_{c,R} = \big\{ & f \in L^2_{c,r}((0, R) \times \mathbb{R});\ f \in Y^M_{c,R} \text{ for all } M < \infty \big\}. \end{aligned}$$

For all $f \in Y_{c,R}$ we define the functional

$$\Phi_{c,R}(f) := \lim_{M\to\infty} \Phi^M_{c,R}(f) := \lim_{M\to\infty} \int_{-\infty}^M dz \int_0^R r e^{cz} \left(\frac{1}{2} f_r^2 + \frac{1}{2} f_z^2 + V(r, f) \right) dr,$$

where

$$V(r, f) := \frac{1}{2r^2}(\sin^2(\theta_+ + f) - f \sin(2\theta_+) - \sin^2\theta_+).$$

It follows from the following proposition that $\Phi_{c,R} : Y_{c,R} \to (-\infty, \infty]$ is well-defined. Before stating it we recall an auxiliary result which we shall use several times (see [2], Lemma A.1).

Lemma 2.1 *For all* $w \in H^1_{\text{loc}}(0, \infty) \subset C((0, \infty))$ *and* $0 < \rho_1 < \rho_2$

$$\int_{\rho_1}^{\rho_2} \frac{r}{2} \left(w_r^2 + \frac{\sin^2 w}{r^2} \right) dr \ge |\cos(w(\rho_2)) - \cos(w(\rho_1))|.$$

If $k \in \mathbb{Z}$, $a \in \mathbb{R}$ *and* $w(r) = k\pi + 2\arctan(ar)$, *then, for all* $0 \le \rho_1 < \rho_2$,

$$\begin{aligned} \int_{\rho_1}^{\rho_2} \frac{r}{2} \left(w_r^2 + \frac{\sin^2 w}{r^2} \right) dr &= |\cos(w(\rho_2)) - \cos(w(\rho_1))| \\ &= \left| \frac{1 - a^2\rho_2^2}{1 + a^2\rho_2^2} - \frac{1 - a^2\rho_1^2}{1 + a^2\rho_1^2} \right|. \end{aligned}$$

Proposition 2.2 *Let* $f \in Y_{c,R}$ *and* $\theta = f + \theta_+$. *Then*
(i) for a.e. $z \in \mathbb{R}$

$$\int_0^R r\left(\frac{f_r^2}{2} + V(r, f)\right) dr = \int_0^R \frac{r}{2}\left(\theta_r^2 + \frac{\sin^2\theta}{r^2} - (\theta_+')^2 - \frac{\sin^2\theta_+}{r^2}\right) dr\ ;$$

(ii) there exists $C > 0$ *such that for all* $0 < M < \infty$ *and* $b > \frac{1}{R}$

$$\begin{aligned}\Sigma_{c,R}^M(f) := & \int_{-\infty}^M e^{cz} dz \int_0^R \frac{r}{2}\left(\theta_r^2 + \frac{\sin^2\theta}{r^2} - (\theta_+')^2 - \frac{\sin^2\theta_+}{r^2}\right) dr \\ & \geq -Cb^2\|f\|^2, \end{aligned} \tag{6}$$

where $\|\cdot\|$ *is the norm in the weighted space* $L^2_{c,r}((0, R) \times \mathbb{R})$*;*

(iii) $\Sigma_{c,R}(f) := \lim_{M\to\infty} \Sigma_{c,R}^M(f)$ *exists, and* $-Cb^2\|f\|^2 \leq \Sigma_{c,R}(f) \leq +\infty$.

Proof (i) For a.e. $z \in \mathbb{R}$, $f(\cdot, z) \in C([0, R]) \cap H_r^1(0, R)$ and $\frac{\sin f(r,z)}{r} \in L_r^2(0, R)$ (see [26]). Here the subscript r in L_r^2 and H_r^1 indicates that the usual L^p or Sobolev spaces are to be considered with the weight function r. Since $\theta(\cdot, z) \in C([0, R]) \cap H_r^1(0, R)$ and $\frac{\sin\theta(r,z)}{r} \in L_r^2(0, R)$ for a.e. $z \in \mathbb{R}$, integration by parts yields that

$$\int_0^R r f_r(r, z)\theta_+' \, dr = -\int_0^R \frac{\sin(2\theta_+) f}{2r}\, dr \quad \text{for a.e. } z \in \mathbb{R}.$$

Hence (i) follows from the definition of θ and V.

(ii) Since $bR > 1$, $\theta(R, z) = 2\arctan(bR) > \frac{\pi}{2}$. Let $\rho_b \in (0, R)$ be such that $\theta_+(\rho_b) = \frac{\pi}{3}$. For every $z \in \mathbb{R}$ we define $A_z = \{r \in [3^{-1/2}\rho_b, \rho_b]\ ;\ |f(r, z)| > \frac{\pi}{6}\}$ and, denoting by μ the 1-dimensional Lebesgue measure,

$$B = \left\{z \in \mathbb{R}\ :\ \mu(A_z) \geq \rho_b\left(1 - \frac{1}{\sqrt{3}}\right)\right\} = \left\{z \in \mathbb{R}\ :\ \mu(A_z) = \rho_b\left(1 - \frac{1}{\sqrt{3}}\right)\right\}.$$

A simple computation shows that $\|f\|^2 \geq \frac{\pi^2\rho_b^2}{36\sqrt{3}}(1 - 3^{-1/2})\int_B e^{cz}\, dz$, where $\|\cdot\|$ is the norm in $L^2_{c,r}((0, R) \times \mathbb{R})$. Hence $\int_B e^{cz}\, dz \leq Kb^2\|f\|^2$ for some $K > 0$. On the other hand, if $z \notin B$ there exists $\rho(z) \in [\rho_b 3^{-1/2}, \rho_b]$ such that $|f(\rho(z), z)| \leq \frac{\pi}{6}$, and therefore $|\theta(\rho(z), z)| \leq \frac{\pi}{2}$. Since $\theta(R, z) > \frac{\pi}{2}$, there exists $r(z) \in [0, R]$ such that $\theta(r(z), z) = \frac{\pi}{2}$. For a.e. $z \in \mathbb{R}$, $f(0, z)$ is a multiple of π [26], and, using Lemma 2.1 and computing the following integral separately over $(0, r(z))$ and $(r(z), R)$, we obtain that

$$\int_0^R \frac{r}{2}\left(\theta_r^2(r, z) + \frac{\sin^2(\theta(r, z))}{r^2} dr\right) dr \geq \frac{2b^2R^2}{1 + b^2R^2} \quad \text{for a.e. } z \in \mathbb{R}\setminus B.$$

By Lemma 2.1

$$\begin{aligned}\Sigma(z) &:= \int_0^R \frac{r}{2}\left(\theta_r^2(r,z) + \frac{\sin^2(\theta(r,z))}{r^2} - (\theta_+')^2 - \frac{\sin^2\theta_+}{r^2}\right) dr \\ &= \int_0^R \frac{r}{2}\left(\theta_r^2(r,z) + \frac{\sin^2(\theta(r,z))}{r^2}\right) dr - \frac{2b^2R^2}{1+b^2R^2} \quad \text{for a.e. } z \in \mathbb{R}.\end{aligned}$$

Hence $\Sigma(z) \geq 0$ for a.e. $z \notin B$, and, for every $M > 0$,

$$\Sigma_{c,R}^M(f) \geq \int_{(-\infty,M]\cap B} \Sigma(z)e^{cz}dz \geq \frac{-2b^2R^2}{1+b^2R^2}\int_{(-\infty,M]\cap B} e^{cz}dz \geq -2Kb^2\|f\|^2 .$$

(iii) For every $M > 0$ we have that

$$\Sigma_{c,R}^M(f) = \int_{(-\infty,M]\setminus B} \Sigma(z)e^{cz}\,dz + \int_{(-\infty,M]\cap B}\left(G(z) - \frac{2b^2R^2}{1+b^2R^2}\right)e^{cz}\,dz,$$

where $G(z) = \int_0^R \frac{r}{2}\left(\theta_r^2(r,z) + \sin^2(\theta(r,z))r^{-2}\right) dr$. Since G is nonnegative, Σ is nonnegative in $\mathbb{R} \setminus B$, and $\int_B e^{cz}\,dz \leq Kb^2\,\|f\|^2 < \infty$, the result follows at once. □

Observe that, reasoning as in the proof of (ii), we obtain that

$$\begin{aligned}\Phi_{c,R}(f) - \Sigma_{c,R}^M(f) &\geq \Sigma_{c,R}(f) - \Sigma_{c,R}^M(f) \\ &= \lim_{N\to\infty}\left(\Sigma_{c,R}^N(f) - \Sigma_{c,R}^M(f)\right) \geq -Cb^2\|f\|^2.\end{aligned} \tag{7}$$

Corollary 2.3 *The functional $\Phi_{c,R} : Y_{c,R} \to (-\infty,\infty]$ is well defined and $\Phi_{c,R}(f) \geq -Cb^2\|f\|^2$ for all $f \in Y_{c,R}$, where $\|\cdot\|$ is the norm in $L_{c,r}^2((0,R)\times\mathbb{R})$. If $\theta = f + \theta_+$, then*

$$\Phi_{c,R}(f) = \int_{\mathbb{R}} e^{cz}dz \int_0^R \frac{r}{2}\left(\theta_r^2 + \theta_z^2 + \frac{\sin^2\theta}{r^2} - (\theta_+')^2 - \frac{\sin^2\theta_+}{r^2}\right) dr\,. \tag{8}$$

Setting

$$\Gamma_{c,R}(f) := \int_{\mathbb{R}} dz \int_0^R \frac{r}{2} e^{cz} f_z^2 dr,$$

Corollary 2.3 and the following result imply that $\Phi_{c,R}$ is bounded from below on the set

$$\mathcal{X}_{c,R} := \{f \in Y_{c,R};\, \Gamma_{c,R}(f) = 1\}.$$

Lemma 2.4 *For all* $f \in Y_{c,R}$ *such that* $f_z \in L^2_{c,r}((0,R)\times\mathbb{R})$

$$\Gamma_{c,R}(f) \geq \frac{c^2}{8}\|f\|^2\,,$$

where $\|\cdot\|$ *is the norm in* $L^2_{c,r}((0,R)\times\mathbb{R})$. *Moreover, for all* $z \in \mathbb{R}$

$$\int_0^R rf^2(r,z)\,dr \;\leq\; \frac{2e^{-cz}}{c}\Gamma_{c,R}(f). \tag{9}$$

Proof For a.e. $r \in (0,R)$ the function $f(r,z)e^{\frac{cz}{2}}$ belongs to $H^1(\mathbb{R})$ and hence vanishes as $z \to \pm\infty$. Therefore, integrating by parts and using Hölder's inequality, we obtain that $\frac{c}{2}\|f\|^2 \leq \|f\|\cdot\|f_z\| = \|f\|\sqrt{2\Gamma_{c,R}(f)}$. The proof of (9) is equally simple: given any $z \in \mathbb{R}$, from the inequality

$$\int_z^\infty e^{cy}dy\int_0^R r\left(\sqrt{c}f + \frac{f_z}{\sqrt{c}}\right)^2 dr \geq 0$$

we derive that

$$\begin{aligned} &c\int_z^\infty dy\int_0^R re^{cy}f^2dr + \frac{1}{c}\int_z^\infty dy\int_0^R re^{cy}f_z^2dr\\ &\geq -\int_z^\infty dy\int_0^R re^{cy}\frac{\partial}{\partial y}(f^2)dr = \int_0^R re^{cz}f^2(r,z)\,dr + c\int_z^\infty dy\int_0^R re^{cy}f^2dr \end{aligned}$$

and this last inequality implies (9). □

In the remainder of this section we shall solve the following constrained minimization problem for all $c > 0$:

(MP) Find $h \in \mathcal{X}_{c,R}$ such that $\Phi_{c,R}(h) = \mathcal{I}_{c,R} := \inf_{f\in\mathcal{X}_{c,R}} \Phi_{c,R}(f)\,.$

Lemma 2.5 *There exists* $M_0 = M_0(b,c)$ *such that for any* $M \geq M_0$ *and* $f \in \mathcal{X}_{c,R}$

$$\Phi_{c,R}(f) \geq \Phi^M_{c,R}(f)\,.$$

Proof Given any $f \in \mathcal{X}_{c,R}$, we define ρ_b, A_z, $\Sigma(z)$ $(z \in \mathbb{R})$ and B as in the proof of Proposition 2.2. It follows from (2.9) that

$$\frac{\pi^2\rho_b^2}{108} \leq \int_{\frac{\rho_b}{\sqrt{3}}}^{\rho_b} rf^2(r,z)dr \leq \frac{2e^{-cz}}{c}$$

for $z \in B$. Since $\rho_b = \frac{1}{\sqrt{3}b}$, this implies that there exists $M_0 = M_0(b,c)$ such that $B \subseteq (-\infty, M_0]$. At the same time, for any $z \in \mathbb{R}\setminus B$ we have that $\Sigma(z) \geq 0$ (see proof of Proposition 2.2). So, if $M \geq M_0$, then

$$\Phi_{c,R}(f) - \Phi^M_{c,R}(f) \geq \Sigma_{c,R}(f) - \Sigma^M_{c,R}(f) = \int_M^\infty e^{cz}\Sigma(z)dz \geq 0\,.$$

□

Proposition 2.6 *Let $\{h_n\}$ be a minimizing sequence for $\Phi_{c,R}$ in $\mathcal{X}_{c,R}$. Then there exist $h \in Y_{c,R}$ and a subsequence, which we denote again by $\{h_n\}$, such that $h_n \to h$ a.e. in $(0, R) \times \mathbb{R}$ and, for every $M > 0$,*

$$h_{nr} \rightharpoonup h_r, \quad h_{nz} \rightharpoonup h_z, \quad \frac{\sin(h_n + \theta_+)}{r} \rightharpoonup \frac{\sin(h + \theta_+)}{r} \tag{10}$$

in $L^2_{c,r}((0, R) \times (-\infty, M))$ as $n \to \infty$, and $\Phi_{c,R}(h) \leq \mathcal{I}_{c,R}$.

Proof Since $\Gamma_{c,R}(h_n) = 1$, the functions h_{nz} and, by Lemma 2.4, h_n are uniformly bounded in $L^2_{c,r}((0, R) \times \mathbb{R})$. Fixing $M > 0$, we claim that

$$h_{nr} \text{ and } \frac{\sin(h_n + \theta_+)}{r} \text{ are uniformly bounded in } L^2_{c,r}((0, R) \times (-\infty, M)). \tag{11}$$

By (7) and Lemma 2.4,

$$\begin{aligned}\int_{-\infty}^{M} dz \int_0^R \frac{re^{cz}}{2} h_{nr}^2 \, dr &= \Sigma^M_{c,R}(h_n) - \int_{-\infty}^{M} dz \int_0^R re^{cz} V(r, h_n) \, dr \\ &\leq \Phi_{c,R}(h_n) + K_1 + \int_{-\infty}^{M} dz \int_0^R e^{cz} \left(\frac{h_n \sin(2\theta_+)}{2r} + \frac{\sin^2 \theta_+}{2r} \right) dr \, ,\end{aligned}$$

where K_1 is a constant depending only on b and c. Observe that $\frac{h_n \sin(2\theta_+)}{2r} = h_n(r\theta_+^{'})'$ and the $L^2_r(0, R)$-norm of $\frac{\sin \theta_+}{r}$ and $\theta_+^{'}$ are bounded by 2. Hence, integrating by parts and applying Hölder's inequality, we obtain that

$$\int_{-\infty}^{M} dz \int_0^R re^{cz} h_{nr}^2 \, dr \leq C_1 \left(1 + \sqrt{\int_{-\infty}^{M} dy \int_0^R re^{cz} h_{nr}^2 \, dr} \right),$$

where C_1 depends on b, c, M and $\mathcal{I}_{c,R}$.

Similarly, it follows from the equality

$$\begin{aligned}\int_{-\infty}^{M} dz \int_0^R re^{cz} \frac{\sin^2(h_n + \theta_+)}{2r^2} dr &= \Sigma_c(h_n, M) - \frac{1}{2} \int_{-\infty}^{M} dz \int_0^R re^{cz} h_{nr}^2 dr \\ &+ \int_{-\infty}^{M} e^{cz} dz \int_0^R \frac{\sin(2\theta_+) h_n}{2r} dr + \int_{-\infty}^{M} e^{cz} dz \int_0^R \frac{\sin^2(\theta_+)}{2r} dr \, ,\end{aligned}$$

that also $\frac{\sin(h_n + \theta_+)}{r}$ is uniformly bonded in $L^2_{c,r}((0, R) \times (-\infty, M))$, and we have proved (11).

In view of the uniform bounds on h_n, it follows from a standard diagonal procedure that, up to a subsequence, there exists a limit function $h \in Y_{c,R}$ (observe that, by the compactness of the usual trace operator, h vanishes at $r = R$ for a.e. $z \in \mathbb{R}$).

If for every $M > 0$ and $f \in Y^M_{c,R}$ we define the functional

$$\begin{aligned} E^M_{c,R}(f) &= \int_{-\infty}^{M} e^{cz}dz \int_0^R \frac{r}{2}\left(f_r^2 + f_z^2 + \frac{\sin^2 f}{r^2} - (\theta_+')^2 - \frac{\sin^2\theta_+}{r^2}\right) dr \\ &= \int_{-\infty}^{M} e^{cz}dz \int_0^R \frac{r}{2}\left(f_r^2 + f_z^2 + \frac{\sin^2 f}{r^2}\right) dr - \frac{2b^2R^2e^{cM}}{c(1+b^2R^2)}\,, \end{aligned}$$

then, by Proposition 2.2, $\Phi^M_{c,R}(h) = E^M_{c,R}(h+\theta_+)$ and $\Phi^M_{c,R}(h_n) = E^M_{c,R}(h_n+\theta_+)$ for every $n \in \mathbb{N}$. By Fatou's Lemma

$$\int_{-\infty}^{M} e^{cz}dz \int_0^R \frac{\sin^2(h+\theta_+)}{2r} dr \le \liminf_{n\to\infty} \int_{-\infty}^{M} e^{cz}dz \int_0^R \frac{\sin^2(h_n+\theta_+)}{2r} dr\,,$$

therefore for all $M > 0$

$$\Phi^M_{c,R}(h) = E^M_{c,R}(h+\theta_+) \le \liminf_{n\to\infty} E^M_{c,R}(h_n+\theta_+) = \liminf_{n\to\infty} \Phi^M_{c,R}(h_n)\,,$$

and, by Lemma 2.5, for all $M > M_0$

$$\Phi^M_{c,R}(h) \le \liminf_{n\to\infty} \Phi_{c,R}(h_n) = \mathcal{I}_{c,R}.$$

The thesis follows then from the definition of $\Phi_{c,R}$. □

Since $\Gamma_{c,R}(h) \le 1$, we don't know whether h is a solution of Problem (MP). The following result provides a criterion for the existence of a minimizer.

Lemma 2.7 *If* $\Phi_{c,R}(w) \le 0$ *(resp.* < 0*) and* $\Gamma_{c,R}(w) > 0$ *for some* $w \in Y_{c,R}$*, then* $\mathcal{I}_{c,R} \le 0$ *(*$\mathcal{I}_{c,R} < 0$*) and Problem (MP) has a solution.*

Proof Reasoning along the lines of [19], we set $a := -c^{-1}\log(\Gamma_{c,R}(w))$ and $w_a(r,z) := w(r, z-a)$. Then $\Gamma_{c,R}(w_a) = e^{ca}\Gamma_{c,R}(w) = 1$ and $\Phi_{c,R}(w_a) = e^{ca}\Phi_{c,R}(w) \le 0$ (resp. < 0). Hence $w_a \in \mathcal{X}_{c,R}$ and $\mathcal{I}_{c,R} \le 0$ (resp. $\mathcal{I}_{c,R} < 0$).

If $\mathcal{I}_{c,R} = 0$, w_a itself is a minimizer.

If $\mathcal{I}_{c,R} < 0$, we use the function h defined by Proposition 2.6 to construct a minimizer: since $0 < \Gamma_{c,R}(h) \le 1$, $d := -c^{-1}\log(\Gamma_{c,R}(h)) \ge 0$; setting $h_d(r,z) := h(r, z-d)$ we have that $\Gamma_{c,R}(h_d) = 1$ and $\Phi_{c,R}(h_d) = e^{cd}\Phi_{c,R}(h) \le \Phi_{c,R}(h) \le \mathcal{I}_{c,R}$. Hence h_d is a solution of Problem (MP). □

Proposition 2.8 *Let* $b, R > 0$ *be such that* $bR > 1$*. Then there exists* $c^*_R = c^*_R(b)$ *such that for every* $c \in (0, c^*_R)$ *Problem (MP) has a solution and* $\mathcal{I}_{c,R} < 0$*.*

Proof In view of Lemma 2.7, it is enough to prove that there exists $c^*_R > 0$ such that for all $0 < c < c^*_R$ there exists $f \in Y_{c,R}$ such that $\Phi_{c,R}(f) < 0$ and $\Gamma_{c,R}(f) > 0$.

We define the function

$$\vartheta(r,z) := \max\left(2\arctan(br), 2\arctan\left(\frac{A(z)}{r}\right)\right) \quad \text{for } (r,z) \in [0,R]\times\mathbb{R}\,,$$

where

$$A(z) = \begin{cases} 0 & \text{if } z \geq 1 \\ bR^2(1-z)^2 & \text{if } 0 < z < 1 \\ bR^2 & \text{if } z \leq 0\,. \end{cases}$$

Observe that $\vartheta(r,z) = 2\arctan(br)$ if $z \geq 1$, $\vartheta(r,z) = 2\arctan(bR^2r^{-1})$ if $z \leq 0$, and, for $z \in (0,1)$,

$$\vartheta(r,z) = \begin{cases} 2\arctan\left(\dfrac{A(z)}{r}\right) & r < \sqrt{\frac{A(z)}{b}} \\ 2\arctan(br) & r \geq \sqrt{\frac{A(z)}{b}}\,. \end{cases}$$

Since $(A')^2A^{-1} = 4bR^2$ in $(0,1)$, one easily checks that the function $f := \vartheta - 2\arctan(br)$ belongs to $Y_{c,R}$. It follows from (8) that

$$\begin{aligned} \Phi_{c,R}(f) &= \int_{-\infty}^{0} 2e^{cz}\frac{1-b^2R^2}{1+b^2R^2}\,dz + \int_0^1 2e^{cz}\frac{1-bA(z)}{1+bA(z)}\,dz \\ &\quad + \int_0^1 e^{cz}(A'(z))^2\left(\log\left(1+\frac{1}{bA(z)}\right) - \frac{1}{1+bA(z)}\right)dz \\ &\leq \frac{2}{c}\left(\frac{1-b^2R^2}{1+b^2R^2} + e^c - 1\right) + \int_0^1 e^{cz}\frac{(A'(z))^2}{bA(z)}\,dz \\ &= \frac{2}{c}\left(\frac{1-b^2R^2}{1+b^2R^2} + (e^c-1)(1+2R^2)\right). \end{aligned}$$

Hence there exists $c_R^* = c_R^*(b) > 0$ such that $\Phi_{c,R}(f) < 0$ for all $c \in (0, c_R^*)$. □

Now we are ready to prove the main result of this section:

Theorem 2.9 *Let $b, R > 0$ be such that $bR > 1$.*
(i) For all $c > 0$ the constrained minimization problem (MP) has a solution, $h_{c,R}$.
(ii) There exists $c_R = c_R(b) > 0$ such that $\mathcal{I}_{c_R,R} = 0$, and

$$\mathcal{I}_{c,R} = 1 - \left(\frac{c_R}{c}\right)^2 \quad \textit{for all } c > 0. \tag{12}$$

Proof Let $c, \tilde{c} > 0$ and let $T : \mathcal{X}_{\tilde{c},R} \longrightarrow \mathcal{X}_{c,R}$ be the map defined by $T(f)(r,z) \equiv f(r, \frac{c}{\tilde{c}}z + \beta)$, where $\beta = \frac{1}{\tilde{c}}\log\left(\frac{c}{\tilde{c}}\right)$. One easily verifies that T is well-defined and bijective, and that

$$\Phi_c(T(f)) = 1 + \left(\frac{\tilde{c}}{c}\right)^2(\Phi_{\tilde{c}}(f) - 1) \quad \text{for all } f \in \mathcal{X}_{\tilde{c},R}. \tag{13}$$

Let $c_R^*(b)$ be defined by Proposition 2.8, let $\tilde{c} \in (0, c_R^*)$ and let $h_{\tilde{c},R}$ be a minimizer of $\Phi_{\tilde{c},R}$ on $\mathcal{X}_{\tilde{c},R}$. Since T is bijective, relation (13) implies that $T(h_{\tilde{c},R})$ is a minimizer of $\Phi_{c,R}$ on $\mathcal{X}_{c,R}$, and that

$$h_{c,R}(r,z) = h_{\tilde{c},R}\left(r, \frac{c}{\tilde{c}}z + \frac{1}{\tilde{c}}\log\left(\frac{c}{\tilde{c}}\right)\right) \quad \text{if } c > 0.$$

In particular

$$\mathcal{I}_{c,R} = 1 + \left(\frac{\tilde{c}}{c}\right)^2 \left(\mathcal{I}_{\tilde{c},R} - 1\right). \tag{14}$$

Since $\mathcal{I}_{\tilde{c},R} < 0$, it follows from (14) that there exists $c_R > \tilde{c}$ such that $\mathcal{I}_{c_R,R} = 0$. Replacing $\tilde{c}$ by c_R in (14), we obtain (12). □

Corollary 2.10 *Let c_R be defined by Theorem 2.9. Then $\Phi_{c_R,R}(w) \geq 0$ for all $w \in Y_{c_R,R}$.*

The proof is immediate: if $\Phi_{c_R,R}(w) < 0$ for some $w \in Y_{c_R,R}$, then $\Gamma_{c_R,R}(w) > 0$ and, by Lemma 2.7, $\mathcal{I}_{c_R,R} < 0$. On the other hand, by definition, $\mathcal{I}_{c_R,R} = 0$ and we have found a contradiction.

3 Proof of Theorem 1.1

In this section we shall prove our first main result:

Theorem 3.1 *Let $bR > 1$ and let c_R and $h_{c,R}$ be defined by Theorem 2.9. Then there exists $z_R \in \mathbb{R}$ such that the function*

$$\theta_R(r,z) := \theta_+(r) + h_{c_R,R}(r, z + z_R)$$

satisfies all properties listed in Theorem 1.1.

We shall often omit the subscripts of c_R, $h_{c_R,R}$, $\Gamma_{c_R,R}$ and $\Phi_{c_R,R}$.

The proof of Theorem 3.1 consists of several steps. First we introduce some function spaces. Let V be the Hilbert space

$$V := \left\{\eta : \frac{\eta}{r} \in L^2_{c,r}((0,R)\times\mathbb{R});\ \eta_r, \eta_z \in L^2_{c,r}((0,R)\times\mathbb{R});\ \eta(R,z) = 0 \text{ for a.e.}\right\}$$

with scalar product

$$\langle u, v\rangle_V = \int_{\mathbb{R}} dz \int_0^R r e^{cz}\left(\frac{uv}{r^2} + u_r v_r + u_z v_z\right) dr\,.$$

We remark that if $\eta \in V$, then $\eta(0,z) = 0$ for a.e. $z \in \mathbb{R}$ (see [26]). For each $M > 0$ let S_M be the subspace of V containing the functions $\eta \in C^1([0,R]\times\mathbb{R})$ such that $\mathrm{supp}(\eta) \subseteq [0,R]\times(-\infty, M]$, $\eta(R,z) = 0$ for $z \in \mathbb{R}$, and $\|\eta\|_V < \infty$. Let V_M be the closure of S_M in V. Then V_M is a Hilbert space with scalar product $\langle\cdot,\cdot\rangle_V$.

Lemma 3.2 *Let $c = c_R$ and $h = h_{c_R,R}$. For all $\varepsilon \in (0, 1)$ there exist $M > 0$ and $\eta \in V_M$ such that $\langle h_z, \eta_z\rangle > 2(1-\varepsilon)\Gamma(h)$ and $\|\eta_z\|^2 < (1+\varepsilon)^2\|h_z\|^2$, where $\langle\cdot\,,\,\cdot\rangle$ and $\|\cdot\|$ are the scalar product and norm in $L^2_{c,r}((0, R)\times\mathbb{R})$.*

Proof Since $\|h_z\|^2 = 2\Gamma(h)$, we have that $2(1-\varepsilon)\Gamma(h) - \langle h_z, \eta_z\rangle = \langle h_z, h_z - \eta_z\rangle - \varepsilon\|h_z\|^2 \le \|h_z\|\,(\|h_z - \eta_z\| - \varepsilon\|h_z\|)$. Hence it is sufficient to show that for all $\varepsilon \in (0, 1)$ there exist $M > 0$ and $\eta \in V_M$ such that $\|h_z - \eta_z\| < \varepsilon\|h_z\|$. Let $\{g_n\} \subset C^1_0((0, R)\times\mathbb{R})$ be a sequence such that $g_n \to h_z$ in $L^2_{c,r}((0, R)\times\mathbb{R})$. For every $n \in \mathbb{N}$ we define

$$\eta_n(r, z) := -\int_z^\infty g_n(r, t)dt \quad (r, z) \in [0, R]\times\mathbb{R}\,.$$

For all $n \in \mathbb{N}$ there exists $M_n > 0$ such that $\eta_n = 0$ in $[0, R]\times(M_n, \infty)$. Moreover, $\eta_n(R, z) = 0$ for $z \in \mathbb{R}$ and $\|\eta_n\|_V < \infty$. Therefore $\eta_n \in V_{M_n}$ for all n, and, since $(\eta_n)_z = g_n$, the proof is complete. □

Proposition 3.3 *Let $c = c_R$ and $h = h_{c_R,R}$. Then h is a distributional solution of the equation*

$$h_{zz} + ch_z + h_{rr} + \frac{h_r}{r} - \frac{\sin(2h + 2\theta_+) - \sin(2\theta_+)}{2r^2} = 0 \quad in\ (0, R)\times\mathbb{R}. \tag{15}$$

Proof For $M > 0$ we define the following functionals on V_M:

$$F_M(\eta) = \Phi(h+\eta) \quad\text{and}\quad G_M(\eta) = \Gamma(h+\eta)\,.$$

G_M is locally Lipschitz continuous on V_M, Frechet differentiable in zero and its differential in 0 is $\nabla G_M(\eta) = \langle h_z, \eta_z\rangle$. By Lemma 3.2, $\nabla G_M \not\equiv 0$ on V_M if M is large enough.

Also F_M is differentiable in 0 and its differential in 0 is

$$\nabla F_M(\eta) = \int_{\mathbb{R}} dz \int_0^R re^{cz}\left(\nabla h\nabla\eta + \frac{\sin(2h + 2\theta_+) - \sin(2\theta_+)}{2r^2}\eta\right) dr\,.$$

Let $\mathcal{G}_M := \{\eta \in V_M : G_M(\eta) = G_M(0)\} = \{\eta \in V_M : \Gamma(h+\eta) = 1\}$. Since $\eta + h \in \mathcal{X}_{c_R,R}$ for all $\eta \in \mathcal{G}_M$, we have that $\Phi(h) \le \Phi(h+\eta)$ if $\eta \in \mathcal{G}_M$. By the Lagrange's multiplier theorem and the inclusion $V_M \subseteq V_{M'}$ for $M' > M$, there exists $\lambda \in \mathbb{R}$ such that $\nabla F_M = \lambda\nabla G_M$ on V_M for all $M > 0$. In particular, for all $\eta \in C^1_0((0, R)\times\mathbb{R})$ we have that

$$\int_{\mathbb{R}} dz \int_0^R re^{cz}\left(h_r\eta_r + (1-\lambda)h_z\eta_z + \frac{\sin(2h + 2\theta_+) - \sin(2\theta_+)}{2r^2}\eta\right) dr = 0, \tag{16}$$

i.e. h is a distributional solution of the equation

$$(1-\lambda)(h_{zz} + ch_z) + h_{rr} + \frac{h_r}{r} - \frac{\sin(2h + 2\theta_+) - \sin(2\theta_+)}{2r^2} = 0 \quad \text{in } (0, R)\times\mathbb{R}.$$

It remains to prove that $\lambda = 0$. By Lemma 3.2, applied with $\varepsilon = \frac{1}{2}$, there exist $M > 0$ and $\eta \in V_M$ such that $\langle h_z, \eta_z \rangle > \Gamma(h) = 1$ and $\|\eta_z\| < \frac{3}{2}\|h_z\| = \frac{3}{\sqrt{2}}$, where the scalar product and the norm are taken in $L^2_{c,r}((0, R) \times \mathbb{R})$.

First we suppose that $\lambda > 0$. Let $a < 0$ and $\eta_a := a\eta$. Then

$$\nabla F_M(\eta_a) = \lambda \nabla G_M(\eta_a) = \lambda a \langle h_z, \eta_z \rangle < \lambda a < 0,$$

whence

$$\Phi(h + \eta_a) < \Phi(h) + \lambda a + \|\eta\|_V \, o(a) \quad \text{as } a \to 0^-.$$

Since $\Phi(h) = 0$, we can choose $a < 0$ so small that $\Phi(h + \eta_a) < 0$. On the other hand, since $h + \eta_a \in Y_{c_R,R}$ it follows from Corollary 2.10 that $\Phi(h + \eta_a) \geq 0$ and we have found a contradiction.

Hence $\lambda \leq 0$. Arguing by contradiction we suppose that $\lambda < 0$. Reasoning as before, with $a > 0$ instead of $a < 0$, the result follows at once. □

Standard regularity theory (see [17]) implies

Corollary 3.4 *Let $c = c_R$ and $h = h_{c_R,R}$. Then h is real analytic in $(0, R] \times \mathbb{R}$, h is a classical solution of (15) in $(0, R] \times \mathbb{R}$, and $h(R, z) = 0$ for every $z \in \mathbb{R}$.*

Proposition 3.5 *Let $h = h_{c_R,R}$. Then*

$$0 < h(r, z) < \varphi(r) := \pi - 2 \arctan\left(\frac{r}{bR^2}\right) - 2 \arctan(br) \tag{17}$$

for $0 < r \leq R$ and $z \in \mathbb{R}$.

We observe that, properly speaking, Proposition 3.5, as well as some Propositions to be announced below, should state that “we may assume that h satisfies....”, meaning that we can choose the minimizing sequence $\{h_n\}$ such that its limit h satisfies the required property.

Proof Let $f_1(r, z) = \max(0, h(r, z))$ and $f_2(r, z) = \min(f_1(r, z), \varphi(r))$. Reasoning as in the proof of Theorem 4.1(ii) in [2], it follows that $f_i \in Y_{c_R,R}$, $\Gamma(f_2) \leq \Gamma(f_1) \leq \Gamma(h)$, $\Sigma(f_2) \leq \Sigma(f_1) \leq \Sigma(h) \Rightarrow \Phi(f_2) \leq \Phi(f_1) \leq \Phi(h) = 0$. At the same time $\Gamma(f_i) > 0$, since $\Gamma(f_i) = 0$ would imply $f_i \equiv 0$ and then we would have $0 = \Sigma(f_i) \leq \Sigma(h) = \Phi(h) - \Gamma(h) = -1$, which is clearly absurd.

Arguing as in the first part of the proof of Lemma 2.7, there exists a constant k such that the function $f_2(r, z - k)$ belongs to $\mathcal{X}_{c_R,R}$ and is a minimizer of Problem (MP). Replacing $h(r, z)$ by $f_2(r, z - k)$ we have shown that $0 \leq f_2(r, z - k) \leq \varphi(r)$. The strict inequalities follow from the strong maximum principle. □

Lemma 3.6 *Let $h = h_{c_R,R}$. Then $h(\cdot, z) \to 0$ in $C^2_{\mathrm{loc}}((0, R])$ as $z \to \infty$.*

Proof Let $\rho \in (0, R)$ be fixed and let $W_\rho(z) = \int_\rho^R h^2(r, z)\, dr$. It follows from Lemma 2.4 that $\int_{\mathbb{R}} e^{cz} W_\rho(z)\, dz \leq \frac{8}{c^2 \rho}$, whence $\int_0^\infty W_\rho(z)\, dz < \infty$. Standard Schauder estimates (see [17]) imply that there exists $K = K(c, b, \lambda, \rho) > 0$ such that

$$\|h\|_{C^4([\rho,R]\times\mathbb{R})} \leq K. \tag{18}$$

Hence W_ρ is uniformly Lipschitz continuous in $\mathbb{R}$ and $h(\cdot, z) \to 0$ in $L^2(\rho, R)$ as $z \to \infty$. The convergence in $C^2([\rho, R] \times \mathbb{R})$ follows from (18) and the arbitrariness of ρ completes the proof. ☐

Lemma 3.7 *Let* $h = h_{c_R,R}$. *Then* $h_z < 0$ *in* $(0, R) \times \mathbb{R}$.

Proof Arguing as in Sect. 5 of [2], we apply a one-dimensional rearrangement technique (with respect to z) to $\theta(r, z) = h(r, z) + \theta_+(r)$, or, equivalently, to h. This procedure yields a function $\tilde{h} \in Y_{c,R}$ which is nondecreasing with respect to z and satisfies $0 < \Gamma(\tilde{h}) \leq 1$ and $\Phi(\tilde{h}) \leq \Phi(h) = 0$. It is better to remark that $\Gamma(\tilde{h}) > 0$ follows from the observation that $\forall r \in (0, R)$ $\sup_{z\in\mathbb{R}} \tilde{h}(r, z) = \sup_{z\in\mathbb{R}} h(r, z) > 0$, while $\Gamma(\tilde{h}) = 0$ would imply $\tilde{h} \equiv 0$. Arguing as in the proof of Lemma 2.7, a suitable translation of $\tilde{h}$ with respect to z yields a minimizer of Problem (MP) which is decreasing in z. The strict monotonicity follows from the strong maximum principle. ☐

Proposition 3.8 *Let* $h = h_{c_R,R}$. *Then there exists* $z_R \in \mathbb{R}$ *such that* $h(0, z) = \pi$ *if* $z < z_R$ *and* $h(0, z) = 0$ *if* $z > z_R$.

Proof Since $h(0, z) = \lim_{r\to 0^+} h(r, z) = k(z)\pi$ for some $k(z) \in \mathbb{Z}$ for a.e. $z \in \mathbb{R}$ [26], Theorem 3.5 implies that $k(z)$ is either 0 or 1. Hence, by Lemma 3.7, there are three possibilities for the behavior of $h(0, z)$:

(A) $h \in C([0, R] \times \mathbb{R})$ and $h(0, z) = 0$ for all $z \in \mathbb{R}$;
(B) there exists $z_R \in \mathbb{R}$ such that $h(0, z) = \pi$ if $z < z_R$ and $h(0, z) = 0$ if $z > z_R$;
(C) $h \in C([0, R] \times \mathbb{R})$ and $h(0, z) = \pi$ for all $z \in \mathbb{R}$.

We have to prove that cases (A) and (C) do not occur.

Arguing by contradiction, we first suppose that case (C) occurs. Let $\theta = h + \theta_+$, $a > b$ and $0 < \rho < 1$. By Lemma 3.6, there exists z_ρ such that $0 < \theta(\frac{\rho}{a}, z) < 2 \arctan \rho$ for all $z \geq z_\rho$. Since $\theta(0, z) = \pi$ for all $z \in \mathbb{R}$, it follows from Lemma 2.1 that

$$\int_{z_\rho}^{\infty} e^{cz} dz \int_0^R \frac{r}{2} \left(\theta_r^2 + \frac{\sin^2 \theta}{r^2} - (\theta_+')^2 - \frac{\sin^2 \theta_+}{r^2} \right) dr$$

$$\geq \int_{z_\rho}^{\infty} 2 \left(\frac{1 - \rho^2}{1 + \rho^2} \right) e^{cz} dz = \infty.$$

Hence $\Phi(h) = \infty$ and we have found a contradiction.

It remains to exclude case A: suppose that $h \in C([0, R] \times \mathbb{R})$ and $h(0, z) = \theta(0, z) = 0$ for all $z \in \mathbb{R}$. Then, by Lemma 2.1,

$$\int_0^R \left(\frac{1}{2} |h_r|^2 + V(r, h) \right) dr = \int_0^R \left(\theta_r^2 + \frac{\sin^2(\theta)}{r^2} - (\theta_+')^2 - \frac{\sin^2(\theta_+)}{r^2} \right) dr \geq 0$$

for a.e. $z \in \mathbb{R}$. Hence $\Phi(h) \geq \Gamma(h) = 1$. But $\Phi(h) = 0$ and we have found a contradiction. ☐

Proposition 3.9 *Let* $h = h_{c_R,R}$. *Then*

$$h(\cdot, z) \to \begin{cases} 0 & as\ z \to \infty \\ \varphi & as\ z \to -\infty \end{cases}$$

in $C^2_{\mathrm{loc}}((0, R])$ *and uniformly in* $[0, R]$, *where* $\varphi(r)$ *is defined by (17).*

Proof The convergence to 0 as $z \to \infty$ is an immediate consequence of Proposition 3.8 and Lemmas 3.6 and 3.7.

Since $h_z \leq 0$, the limit $H(r) := \lim_{z\to-\infty} h(r, z)$ is well-defined for all $r \in [0, R]$ and satisfies $0 \leq H \leq \varphi$ and $H(R) = 0$. By (18), $h(\cdot, z) \to H$ in $C^2_{\mathrm{loc}}((0, R])$ as $z \to -\infty$, and, for all $r \in (0, R]$, $h_z(r, z)$ and $h_{zz}(r, z)$ vanish as $z \to -\infty$. Hence $H \in C^2((0, R])$ and satisfies

$$H_{rr} + \frac{H_r}{r} - \frac{\sin(2H + 2\theta_+) - \sin(2\theta_+)}{2r^2} = 0 \quad \text{in } (0, R).$$

It follows from Proposition 3.8 and Lemma 3.7 that H is continuous down to $r = 0$ and $H(0) = \pi$. Setting $\theta_- = H + \theta_+$, we have that θ_- is a classical solution of

$$\begin{cases} \psi_{rr} + \frac{\psi_r}{r} - \frac{\sin(2\psi)}{2r^2} = 0 & \text{in } (0, R) \\ \psi(0) = \pi, \quad \psi(R) = 2\arctan(bR) & \\ \theta_+(r) \leq \psi(r) \leq \varphi(r) + \theta_+(r) & \text{in } [0, R]\,. \end{cases} \tag{19}$$

This problem has a unique solution, $\pi - 2\arctan(b^{-1}R^{-2}r)$, and hence $H = \varphi$ in $(0, R)$.

As before, the uniform convergence to φ in $[0, R]$ follows from Proposition 3.8 and Lemma 3.7. □

Theorem 3.1 follows almost at once from Propositions 3.4, 3.8 and 3.9, Corollary 3.4 and Lemma 3.7. We omit the proof of the real analyticity of $h_{c_R,R}$ in $[0, R] \times \mathbb{R} \setminus \{(0, z_R)\}$. It is similar to the proof of Proposition 6.3 in [2].

4 Proof of Theorem 1.2

In this section we consider the limit of $h_{c_R,R}$ as $R \to \infty$ to construct a solution of Problem $\mathrm{I}_{c_\infty,\infty}$, where c_∞ is the limit of c_R as $R \to \infty$. Here c_R and $h_{c_R,R}$ are defined by Theorem 2.9 (throughout this section we shall assume that $b > 0$ is fixed and $bR > 1$).

We first prove the existence of the limit speed c_∞.

Lemma 4.1 *The wave speed* c_R *is nondecreasing with respect to* R *and*

$$c_\infty := \lim_{R\to\infty} c_R < \infty.$$

Proof Let $0 < \rho < R$ and

$$w(r,z) = \begin{cases} h_{c_\rho,\rho}(r,z) & \text{if } 0 \le r \le \rho, z \in \mathbb{R} \\ 0 & \text{if } \rho < r \le R, z \in \mathbb{R}. \end{cases}$$

Since $w \in Y_{c_\rho,R}$ and $\Phi_{c_\rho,R}(w) = \Phi_{c_\rho,\rho}(h_{c_\rho,\rho}) = 0$, it follows from Lemma 2.7 that $\mathcal{I}_{c_\rho,R}(w) \le 0$. Hence, by (12), $c_\rho \le c_R$.

It remains to show that $c_R \le C$ for a constant C which does not depend on R. By Proposition 2.2, there exists a constant K such that

$$0 = \Phi_{c_R,R}(h_{c_R,R}) = \Sigma_{c_R,R}(h_{c_R,R}) + 1 \ge -Kb^2\|h_{c_R,R}\|^2 + 1$$

for all b and R, and by Lemma 2.4,

$$0 \ge \frac{-8Kb^2}{c_R^2} + 1 \Rightarrow c_R \le \sqrt{8K}b\,.$$

□

The following result can be viewed as a stronger version of Proposition 3.8.

Lemma 4.2 *There exist* $z_-^*, z_+^* \in \mathbb{R}$ *and* $0 < r^* < \frac{1}{b}$ *such that for all* $R > \frac{1}{b}$

$$h_{c_R,R} + \theta_+ > \frac{\pi}{2} \quad \text{in } [0,R] \times (-\infty, z_-^*) \tag{20}$$

and

$$h_{c_R,R} + \theta_+ < \frac{\pi}{2} \quad \text{in } [0,r^*] \times (z_+^*, \infty) \tag{21}$$

Proof To prove (20) we argue by contradiction and suppose that for all $n \in \mathbb{N}$ there exist $R_n > \frac{1}{b}$, $r_n \in [0, R_n]$ and $z_n \to -\infty$ as $n \to \infty$ such that

$$\theta_n(r_n, z_n) \le \frac{\pi}{2},$$

where $\theta_n = h_{c_{R_n},R_n} + \theta_+$. The monotonicity with respect to z implies that

$$\theta_n(r_n, z) \le \frac{\pi}{2} \quad \text{for } z \ge z_n.$$

Setting

$$A_n = \int_{-\infty}^{z_n} dz \int_0^{R_n} \frac{re^{c_{R_n}z}}{2}\left((\theta_n)_r^2 + \frac{\sin^2\theta_n}{r^2} - (\theta_+')^2 - \frac{\sin^2\theta_+}{r^2}\right) dr,$$

it follows from Lemma 2.1, applied in the intervals $(0, r_n)$ and (r_n, R_n) for $z > z_n$, that

$$0 = \Phi_{c_{R_n},R_n}(h_n) \ge \Gamma_{c_{R_n},R_n}(h_n) + A_n = 1 + A_n.$$

Hence $A_n \le -1$. On the other hand, using again Lemma 2.1,

$$A_n \ge 2\int_{-\infty}^{z_n} e^{c_{R_n}z}\frac{1 - b^2R_n^2}{1 + b^2R_n^2} \ge -2\int_{-\infty}^{z_n} e^{c_{R_n}z}dz \to 0 \quad \text{as } n \to \infty,$$

where we have used that $z_n \to -\infty$ and c_{R_n} is uniformly bounded (by Lemma 4.1). Hence we have found a contradiction.

It remains to prove (21). Let $a > b$ and $\rho > 0$ be such that $a\rho < 1$, i.e. $2\arctan(a\rho) < \frac{\pi}{2}$. It follows from the proof of Lemma 3.6 that the convergence of $h_{c_R,R}$ to 0 in $C([\rho, R])$ as $z \to \infty$ is uniform with respect to R. Hence, setting $\theta_{c_R,R} = h_{c_R,R} + \theta_+$ there exists $z_\rho \in \mathbb{R}$ such that for all $R > \frac{1}{b}$

$$\theta_{c_R,R} < 2\arctan(ar) \quad \text{in } [\rho, R] \times (z_\rho, \infty).$$

We claim that there exists $z_+^* > z_\rho$ such that (21) holds with $r^* = \rho$. Arguing by contradiction we suppose that there exists $z_n \to \infty$, $r_n \in (0, \rho)$ and $R_n > \frac{1}{b}$ such that, setting $\theta_n = \theta_{c_{R_n},R_n}$,

$$\theta_n(r_n, z_n) = \frac{\pi}{2}.$$

Hence

$$\theta_n(r_n, z) \geq \frac{\pi}{2} \quad \text{if } z \leq z_n.$$

Applying, for $z_\rho < z < z_n$, Lemma 2.1 to the intervals $(0, r_n)$, (r_n, ρ) and (ρ, R_n), we find that

$$\int_{z_\rho}^{z_n} e^{c_{R_n} z} dz \int_0^{R_n} \frac{r}{2} \left((\theta_n)_r^2 + \frac{\sin^2\theta_n}{r^2} - \left(\theta_+'\right)^2 - \frac{\sin^2\theta_+}{r^2} \right) dr$$

$$\geq \int_{z_\rho}^{z_n} 2 \left(\frac{1-(a\rho)^2}{1+(a\rho)^2} \right) e^{c_{R_n} z} dz \to \infty \quad \text{as } z_n \to \infty,$$

since c_{R_n} is uniformly bounded. Hence $0 = \Phi_{c_{R_n},R_n}(h_{c_{R_n},R_n}) \to \infty$ as $n \to \infty$ and we have found a contradiction. ☐

For any $M > 0$ we define L_M^+ as the Hilbert space formed by all the functions f which are measurable on $\mathbb{R}^+ \times (-\infty, M)$ and for which

$$\|f\|_{M,+} := \int_{-\infty}^{M} dz \int_0^\infty r e^{c_R z} |f|^2 dr < \infty\,,$$

with the natural scalar product. Similarly we define the Hilbert space L_∞^+ with the norm

$$\|f\|_{\infty,+} := \int_{\mathbb{R}} dz \int_0^\infty r e^{c_R z} |f|^2 dr$$

In what follows we shall denote by h_R the function

$$h_R(r,z) = \begin{cases} h_{c_R,R}(r,z) & \text{if } r \leq R \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.3 *For any $R > \frac{1}{b}$ and $M > 0$ we have*

1. $\|\frac{\partial h_R}{\partial z}\|_{\infty,+} \leq \sqrt{2}$

2. $\|h_R\|_{\infty,+} \le \frac{\sqrt{8}}{c_R}$
3. $\|\frac{\partial h_R}{\partial r}\|_{M,+} \le \mathcal{Q}$, $\|\frac{\sin(h_R)}{r}\|_{M,+} \le \mathcal{Q}'$

where $\mathcal{Q}$ *and* $\mathcal{Q}'$ *are constants depending only on* b, c_R *and* M.

Proof It is sufficient to prove the estimates for the functions $h_{c_R,R}$. Given any $R > 1/b$, we can repeat for $h_{c_R,R}$ the same arguments used in the proof of Proposition 2.6 to obtain the estimates for a generic element h_n of a minimizing sequence. Since $\Phi_{c_R,R}(h_{c_R,R}) = \mathcal{I}_{c_R,R} = 0$, the constants $\mathcal{Q}$ and $\mathcal{Q}'$ only depend on b, c_R and M. □

Theorem 4.4 *There exist* $z_\infty \in \mathbb{R}$ *and a function* $h_\infty \in C^2((0,\infty)\times\mathbb{R}) \cap C^0([0,\infty)\times\mathbb{R}\setminus\{(0,z_\infty)\})$ *such that:*
(i) h_∞ *solves the differential equation*

$$h_{zz} + c_\infty h_z + h_{rr} + \frac{h_r}{r} - \frac{\sin(2h+2\theta_+) - \sin(2\theta_+)}{2r^2} = 0 \quad \text{in } (0,\infty)\times\mathbb{R} \quad (22)$$

(ii) $h_\infty(r,z) \to 0$ *as* $z \to +\infty$ *and* $h_\infty(r,z) \to \pi - \theta_+$ *as* $z \to -\infty$ *uniformly with respect to* r;
(iii) $h_\infty(0,z) = \pi$ *if* $z < z_\infty$, $h_\infty(0,z) = 0$ *if* $z > z_\infty$;
(iv) h_∞ *is strictly decreasing with respect to* z *in* $\mathbb{R}^+ \times \mathbb{R}$;
(v) h_∞ *is real analytic in* $[0,\infty)\times\mathbb{R}\setminus\{(0,z_\infty)\}$.

Proof Thanks to the Lemma 4.1 and to the uniform bounds $0 \le h_R \le \pi$, through Schauder estimates we can get to say that for any $\rho > 0$ there exists a constant $K = K(\rho,b)$ such that for all $R > 1/b$ $\|h_R\|_{C^4([\rho,R]\times\mathbb{R})} \le K$. By using the previous estimate and Proposition 4.3 together with Lemma 4.1, we deduce the existence of a sequence $R_n \to \infty$ and of a function $h \in C^2(\mathbb{R}^+\times\mathbb{R})$ such that

A1) $h_{R_n} \to h$ in $C^2([\rho,\rho']\times\mathbb{R})$ for any $0 < \rho < \rho'$;
A2) $h, h_z \in L_{\infty,+}$ and $h_r, \sin(h)r^{-1} \in L_{M,+}$ for any $M > 0$;
A3) $0 \le h \le \pi - \theta_+$;
A4) $h_z \le 0$.

From (A1) follows that h is a solution of (22). From (A2) follows (see [26]) that for a.e. $z \in \mathbb{R}$ there exists $h(0,z) = \lim_{r\to 0} h(r,z) = k(z)\pi$ with $k(z) \in \mathbb{Z}$. In view of (A3) and (A4) only one of the following cases can occur:
(A) $h(0,z) = 0$ for all $z \in \mathbb{R}$;
(B) $h(0,z) = \pi$ for all $z \in \mathbb{R}$;
(C) there exists $\zeta \in \mathbb{R}$ such that $h(0,z) = 0$ for $z > \zeta$, $h(0,z) = \pi$ for $z < \zeta$.
Since both (A) and (B) are excluded by Lemma 4.2, we conclude that (C) must occur.

The monotonicity of h with respect to z implies that $h \in C^0([0,\infty)\times\mathbb{R}\setminus\{(0,\zeta)\})$. By reasoning as in the proof of Proposition 6.3 in [2], it easily follows that $\theta := h + \theta_+$ is real analytic on the set $[0,\infty)\times\mathbb{R}\setminus\{(0,\zeta)\}$. So, the same is true for h.

By the strong maximum principle, $h_z < 0$ and $0 < h < \pi - \theta_+$ in the set $\mathbb{R}^+\times\mathbb{R}$.

It follows from (A1) and Proposition 3.9 that $h(r,z) \to 0$ as $z \to +\infty$ and $h(r,z) \to \pi - \theta_+$ as $z \to -\infty$ uniformly in $[\rho,\rho']$, for any $0 < \rho < \rho'$. Then, (A3) and (A4) imply that in both cases the convergence is actually uniform with respect to $r \in [0, \infty)$. Setting $z_\infty = M_0$ and $h_\infty = h$ the proof is complete. $\square$

One easily checks that c_∞ and $\theta_\infty(r, z) := h_\infty(r, z + z_\infty) + \theta_+(r)$ satisfy Theorem 1.2.

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