Detection of complex singularities in two and three dimensions

In this project, Alex Virodov and Nan Maung have developed improved methods for the numerical detection of singular surfaces in two and three dimensions. The work is in collaboration with graduate student Kamyar Malakuti and supervised by Michael Siegel.

Singularities in a 1D function $f(z)$ of a complex variable z can be numerically detected through the asymptotics of the Fourier components $\hat{f}_k$. If

$$\hat{f}_k \approx ck^{-\alpha}\exp(-ikz_0) \tag{0.1}$$

for large k, then

$$f \approx c_\alpha (z - z_0)^{\alpha+1} \tag{0.2}$$

for z near the singularity z_0, where c_α is a constant that depends on c and α. However, until now this procedure has not been generalized to higher dimensions.

1 Detection of singularity surface in 3D

We consider a function $f(x, y, z)$ that is periodic in all three variables. Let $\mathbf{x_0} = (x_0, y_0, z_0)$ be the point on the complex singularity surface of f that is closest to real (physical) space. We assume the singularity surface is paraboloidal near $\mathbf{x_0}$, and after a rotation of variables the surface can be described as $\xi = 0$ with

$$\xi = x' - \mathbf{A}\cdot\mathbf{Y}' + i\mathbf{Y}'\cdot\mathbf{M}\mathbf{Y}' \tag{1.3}$$

$$x' = x - x_0, \qquad \mathbf{Y}' = \begin{pmatrix} y - y_0 \\ z - z_0 \end{pmatrix} \tag{1.4}$$

for $\mathbf{A}$ a real vector and $\mathbf{M}$ a self-adjoint and positive definite 2×2 matrix. This says that there is a singularity at $\mathbf{x_0}$, and that as (y, z) (considered as real parameters) varies away from (y_0, z_0) the imaginary part of the singularity position grows quadratically in the negative direction. The real part of the singularity position is given by $\mathbf{A}\cdot\mathbf{Y}'$ and can vary linearly with y and z.

As part of this grant we have developed a direct fitting procedure based on analysis of the full 3D Fourier spectrum. Consider a singularity in a function f of the form $f \approx f_0\xi^{\alpha+1}$ with ξ given by (1.3). We find that the asymptotic decay of the 3D Fourier coefficients is given by

$$\hat{f}_{\mathbf{k}} \sim c_\alpha k^{\alpha-1}[\det(M)]^{-\frac{1}{2}} e^{-i\mathbf{k}\cdot\mathbf{x_0}} e^{-\frac{1}{4k}(k\mathbf{A}+\mathbf{L})^T\mathbf{M}^{-1}(k\mathbf{A}+\mathbf{L})} \tag{1.5}$$

where $\mathbf{k} = (k, l, m)$ is the wavenumber and $\mathbf{L} = (l, m)^T$. The parameters to be determined by the fit to $\hat{f}_{\mathbf{k}}$ are f_0, α, x_0, y_0, z_0, two components of the vector $\mathbf{A}$, and three distinct components of the self-adjoint matrix $\mathbf{M}$. We have recently developed a fitting procedure that uses the values of $\hat{f}_{\mathbf{k}}$ at 10 distinct $\mathbf{k}$ points to fit these 10 parameters.

The fitting procedure has been validated by computing the (known) singular surface for synthetic data in two and three dimensions. Figure 2 presents an example of the fit for synthetic data of the form

$$f(x, y, z) = 1 - \epsilon_1 e^{ix} + i\epsilon_2 \sin y + i\epsilon_3 \sin z + \epsilon_4 \sin^2(y/2) + \epsilon_5 \sin^2(z/2) \tag{1.6}$$

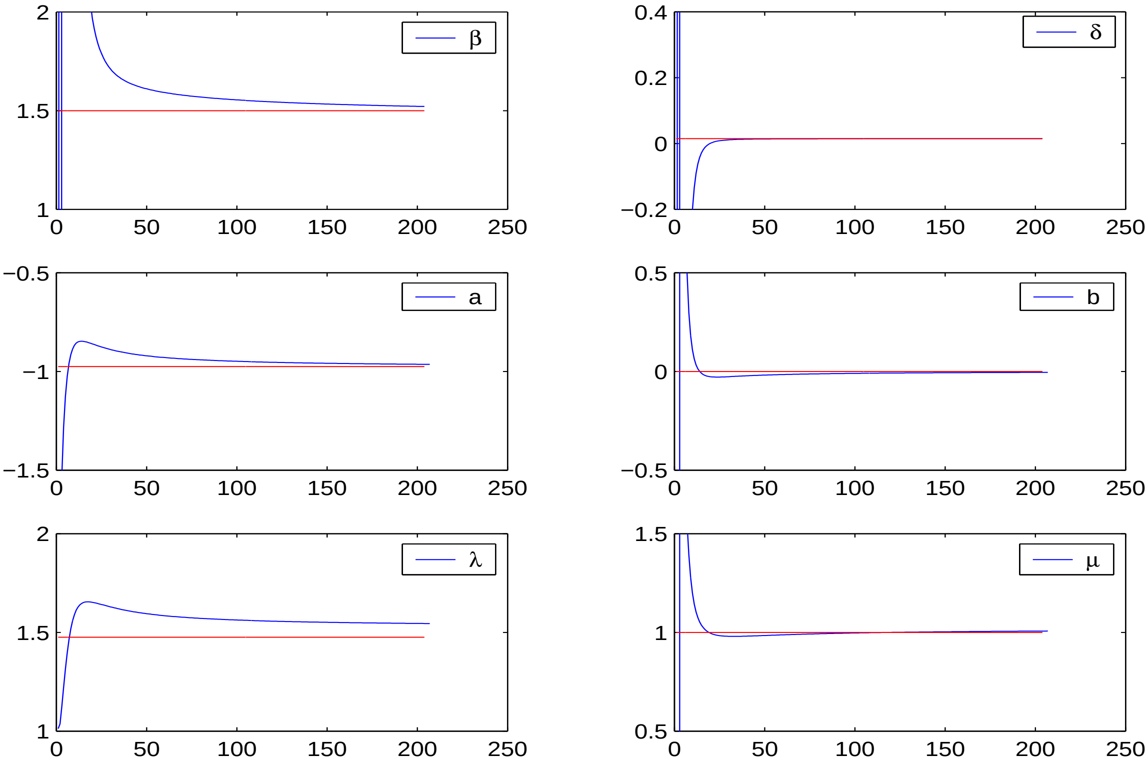

Figure 1: Fits for the asymptotics of the Fourier coefficients $\hat{f}_k$ in the 3D transform. The fits are compared to theoretical values, shown as a horizontal line.

where $\epsilon = 0.985$, $\epsilon_2 = -0.975$, $\epsilon_3 = 0$, and $\epsilon_4 = \epsilon_5 = 4$. For this data, the matrix M in (1.3) is diagonal with elements denoted by $M_{11} = \lambda$ and $M_{22} = \mu$. The vector A in (1.3) is denoted $A = (a, b)$. We also denote $\delta = -Im(x_0)$ and $\beta = \alpha + 1$. A careful Taylor's analysis of (1.6) shows that

$$\delta = -\ln \epsilon_1, \quad \lambda = \frac{\epsilon_4}{4} + \frac{\epsilon_2^2}{4}, \quad \mu = \frac{\epsilon_4}{4} + \frac{\epsilon_2^2}{4} \tag{1.7}$$

$$a = \epsilon_2, \quad b = \epsilon_3. \tag{1.8}$$

The figure shows the computed fits versus the theoretical values (i.e., the right hand sides in (1.7)). The agreement is very good.