Math 222 **Examination #1** October 3, 2001

---

1. Find all points (x_0, y_0) in the plane where the fundamental existence and uniqueness theorem guarantees that the following initial value problem has a unique solution (15):

$$y' = \left(x^2 + y^2 - 1\right)^{1/3}, \quad y(x_0) = y_0.$$

2. Solve the initial value problem (15 pts.)

$$\frac{dy}{dx} - xy = x, \quad y(0) = 0.$$

3. A tank initially contains 10 liters of pure water. A brine (salt water) mixture with salt concentration 1 gram/liter flows into the tank at a rate of 1 liter/min., and a well-stirred mixture exits the tank at a rate of 2 liters/min. Find the amount of salt in the tank after 5 min. (20 points)

4. Suppose that in the general population model (15 pts.)

$$\frac{dP}{dt} = (\beta(t) - \delta(t))\, P, \quad P(0) = 1$$

the birth rate $\beta(t) = 2P$ and the death rate $\delta(t) = P$. Show that $P(t) \to \infty$ in finite time.

5. Use both the Euler and improved Euler methods with step size $h = 0.1$ to find approximate solutions of (20 pts)

$$y' = y(y+1), \quad y(0) = 1$$

at $x = 0.1$. Compare these approximations with the exact solution.

6. Verify that $y_1 = x^{1/2}$ and $y_2 = x^{-1}$ are solutions of (15 pts.)

$$2x^2y'' + 3xy' - y = 0$$

for $x > 0$. Prove that y_1 and y_2 are linearly independent for $x > 0$ and find a general solution.

1. Find a general solution of (15)

$$y''' - y'' + y' - y = 0.$$

2. Find a particular solution of

$$y'' - 2y' + y = e^x$$

using both the method of undetermined coefficients and variation of parameters. (20)

3. A mechanical system is modeled by the initial value problem

$$m\ddot{x} + c\dot{x} + kx = 0, \quad x(0) = 0, \dot{x}(0) = 1,$$

where $m = 1, c = 2$ and $k = 2$. Find the solution in the form $x(t) = A(t)\cos(\omega t - \alpha)$. (15)

4. Find the Laplace transform of the function (15)

$$f(t) = \begin{cases} 1, & 0 < t < 1 \\ e^{1-t}, & t \geq 1 \end{cases}.$$

5. Solve the following initial value problem using the Laplace transform (20)

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 1, \quad y(0) = \frac{dy}{dt}(0) = 0.$$

6. (a) Use the convolution theorem to find the inverse Laplace transform of $\frac{1}{(s-1)(s-2)}$. (8)

   (b) Use the formula for the inverse Laplace transform of $F'(s)$ to find the inverse Laplace transform of $\frac{1}{(s-1)^2}$. (7)

1. Find a power series solution of (15 pts.)

$$y' = 2xy, \quad y(0) = 1.$$

2. Find the first two nonzero terms of two linearly independent power series solutions of

$$\left(1 - x^2\right) \frac{d^2y}{dx^2} + y = 0,$$

and explain why these series must have a radius of convergence ≥ 1.(20 pts.)

3. Use elimination to find the general solution of the system (15 pts.)

$$\begin{aligned} x' &= x + y + e^{3t} \\ y' &= 3y \end{aligned}$$

4. Use the eigenvalue method to solve (15 pts.)

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

5. Find the general solution of (20 pts.)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

6. Compute the matrix exponential e^{At} (or more properly e^{tA}), where (15 pts.)

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

***Hint**: *You can check your answer by using elimination to find a fundamental matrix for the system*

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Math 222 **Final Exam** **19 December 2001**

(All problems worth 10 points)

---

1. Find the solution of the following population model and show that $P(t) \to 2$ as $t \to \infty$:

$$\frac{dP}{dt} = (1-P)(P-2), \quad P(0) = 3/2.$$

2. Find the correct form of, but do not evaluate, a particular solution of

$$y''' - y'' - y' + y = xe^x.$$

3. Find a particular solution of

$$y'' + y = \frac{1}{\sin x}.$$

4. Compute the first two nonzero terms of two linearly independent power series solutions of

$$y'' + x^2 y' + xy = 0,$$

and explain why both of the series converge for all x.

5. Use the Laplace transform to solve

$$\ddot{y} - y = \sin t, \quad y(0) = \dot{y}(0) = 0,$$

and identify the solution as a convolution.

6. Determine the inverse Laplace transform of

$$\frac{2s^2 - 2s + 5}{s^4 - 2s^3 + 5s^2}.$$

7. Use the eigenvalue method to find a general solution of

$$\mathbf{x}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$

8. Use the method of elimination or the definition of the exponential matrix to determine a fundamental matrix for

$$\dot{\mathbf{x}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \mathbf{x}.$$

9. Consider the function f defined on the interval $[0, \pi]$ by $f(t) = \pi$ when $0 \le t < \pi/2$ and $f(t) = 0$ when $\pi/2 \le t < \pi$.

   (a) Sketch the graphs of the odd and even 2π - periodic extensions of f.

   (b) Compute the Fourier series of the odd extension, f_O, of f and give a complete description, with explanations, of the convergence of this series on $[0, \pi]$.

10. Compute a Fourier series expansion of $g(t) := \int_0^t f(\tau)d\tau$, where f is as in problem 9, either directly from 9(b) or by any other method. Describe the convergence of the series on $[0, \pi]$, and use either this series or the one obtained in 9(b) to find a series expansion for π. Explain your answers.

---

1. If a particle has acceleration $a(t) = 12$ m/sec^2 along a straight line, has initial velocity $v_0 = 24$ m/sec and initial position $x_0 = 48$ m, find its position $x = x(t)$ as a function of time. (15 pts.)

2. Answer each of the following questions for the initial value problem: (5 pts. apiece = 15 pts.)

$$\frac{dy}{dx} = \frac{y^{1/3}}{x-3}, \quad y(x_0) = y_0.$$

(a) Find all points (x_0, y_0) in the x, y-plane where the fundamental existence and uniqueness theorem does not guarantee a unique solution.

(b) Find all points in the plane where the fundamental existence and uniqueness theorem does not even guarantee a solution.

(c) Find the solution $y = y(x)$ for any initial point for which there is a unique solution.

3. Solve the initial value problem (15 pts.)

$$\frac{1}{5}y' + y = 4e^{-x}, \quad y(0) = 2.$$

4. A full 200 gallon tank has 50 lbs of salt initially mixed with water. Pure water is pumped into the tank at a rate of 4 gallons/min., while 6 gallons/min. of the thoroughly mixed contents exit the tank. Set up an ODE initial value model that describes the number of pounds of salt in the tank at any time. Solve this problem and explain why this model is valid only for a limited time. (20 pts.)

5. Solve (20 pts.)

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 12y = 0, \quad y(0) = 1, \frac{dy}{dt}(0) = -1.$$

6. Consider the following numerical method for solving the initial value problem (15 pts.)

$$y' = f(x, y), \quad y(x_0) = y_0.$$

Let h be the (constant) step size and define

$$\kappa_1 = f(x_n, y_n), \quad u_{n+1} = y_n + \frac{2}{3}h\kappa_1, \quad \kappa_2 = f\left(x_n + \frac{2}{3}h, u_{n+1}\right),$$

and

$$y_{n+1} = y_n + \frac{h}{4}(\kappa_1 + 3\kappa_2).$$

Perform one step of this method for the case $y' = 6x + 12y, \quad y(+4) = -2,\ h = 3$. Which of the methods studied in class is the above method most like? What would you expect the order of the approximation error for this scheme to be?

Math 222 **Examination #2** **April 10, 2002**

---

1. Use the method of undetermined coefficients to find the correct form of a particular solution of each of the following. Do not evaluate the coeffficients. (15 pts.)

$$(a)\ y'' + y = x + \sin x \qquad\qquad (b)\ y''' + 3y'' + 3y' + y = xe^{-x}$$

2. Find a general solution of (20 pts.)

$$4\frac{d^2y}{dt^2} + y = 2\sec\left(t/2\right), \quad -\pi < t < \pi.$$

3. Find the first two nonzero terms of a pair of linearly independent power series solutions about $x = 0$ for (20 pts.)

$$\left(2 + x^2\right)y'' + xy' + y = 0.$$

(i) Explain why the radius of convergence of these series is not less than $\sqrt{2}$.
(ii) Verify that these solutions are linearly independent.

4. Find the Laplace transform of the function defined as (15 pts.)

$$f(t) = \begin{cases} 0, & 0 < t < 1 \\ (t-1)e^{-t}, & t \geq 1 \end{cases}.$$

5. Solve the following initial value problem using the Laplace transform. You may leave the solution in integral form. (15 pts.)

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = \sin 2t, \quad y(0) = \frac{dy}{dt}(0) = 0.$$

6. Use the method of elimination to solve the (vector) initial value problem (15 pts.)

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\mathbf{x}, \quad x(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Math 222 **FINAL EXAM** **May 10, 2002**

(all problems worth 10 points)

---

1. A tank is filled with 500 gallons of pure water. Brine containing 2 lbs./gallon of salt is pumped into the tank at a rate of 5 gallons/minute. The well-mixed solution exits the tank at a rate of 10 gallons/minute. How long does it take for the tank to empty?

2. Find a particular solution, involving undetermined coefficients that needn't be evaluated, of

$$y^{(4)} + 2y'' + y = x \sin x.$$

3. Find a general solution of

$$y'' + y = \sec x \tan x.$$

4. Solve the following initial value problem using the Laplace transform:

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} = e^t \cos t, \quad y(0) = \frac{dy}{dt}(0) = 0.$$

5. A mass of 0.5 kilograms is hung from a (linear) spring with constant $k = 50$ newtons/meter, and a vertical force $F = 75 \sin 5t$ is applied. The mass is initially at rest and in the equilibrium position. Use the Laplace transform to find the position of the mass at any time.

6. Find the first two nonzero terms of two linearly independent power series solutions of

$$y'' - 2xy' = 8y = 0,$$

and determine the radius of convergence of each series.

7. Use power series to solve

$$\left(x^2 + 1\right) y'' + 2xy' - 2y = 0, \quad y(0) = 0,\ y'(0) = 1.$$

8. Find a general solution of

$$\mathbf{x}' = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix} \mathbf{x} + \begin{bmatrix} t \\ 1 \end{bmatrix}.$$

9. Compute e^{At}, where $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$.

10. Solve the following boundary value problem using Fourier series:

$$\frac{d^2x}{dt^2} + 4x = 4, \quad x(0) = x(2) = 0.$$

Math 222 **Examination #1** October 2, 2002

---

1. Find two different solutions for the initial value problem

$$y' = \frac{9}{2}x^2 y^{1/3}, \quad y(0) = 0,$$

and explain why this does not contradict the fundamental existence and uniqueness theorem. (15 pts)

2. Solve the initial value problem (15 pts)

$$\frac{dx}{dt} = \frac{t(1+x^2)}{x}, \quad x(0) = 1.$$

3. Show that e^x and xe^x are linearly independent solutions of (15 pts)

$$y'' - 2y' + y = 0.$$

4. A 30-gallon capacity tank initially contains 15 gallons of salt water containing 6 pounds of salt. Salt water containing 1 pound of salt per gallon is pumped into the top of the tank at a rate of 2 gallons per minute, while a well-mixed solution exits the bottom of the tank at a rate of 1 gallon per minute. How much salt is in the tank when it is full? (20 pts)

5. On the planet Krypton, New-el's 2^{nd} Law of Motion states the rate of change of momentum of a body is equal to the negative of its velocity. Assuming a flat Krypton with no atmospheric resistance, determine the maximum altitude reached by a ball of (constant) mass one Jor-el after being thrown straight up with an initial velocity of 10 Kal-els/sec. (15 pts)

6. Use the Euler method with constant step size $h = 0.1$ to find approximations y_1 and y_2 (rounding answers to the nearest hundredth) for the solution $y(x)$ of the initial value problem

$$\frac{dy}{dx} = xy(y+1), \quad y(0) = 1$$

at $x = 0.1$ and $x = 0.2$, respectively. Compare the approximations with the exact solution. (20 pts)

Math 222 **Examination #2** November 6, 2002

1. Find the amplitude of oscillation of the steady state solution of (15 pts)

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = \cos t.$$

2. Find the general solution of (15 pts)

$$y''' - y'' - y' + y = e^x.$$

3. Solve the following initial value problem (15 pts)

$$y'' + 3y' + 2y = \frac{1}{1+e^x}, \quad y(0) = y'(0) = 0.$$

4. Find the Laplace transform of the function h defined as (15 pts)

$$h(t) := \int_0^t f(\tau)g\,(t-\tau)\,d\tau,$$

where

$$f(t) := \begin{cases} 1-t, & 0 < t < 1 \\ 0, & t \geq 1 \end{cases} \quad \text{and } g(t) := t,\ t \geq 0.$$

5. Solve the following initial value problem using the Laplace transform: (20 pts)

$$\ddot{x} + 2\dot{x} + 2x = 0, \quad x(0) = \dot{x}(0) = 1. \quad \left(\dot{x} := \frac{dx}{dt}\right)$$

6. For the differential equation (20 pts)

$$\left(x^2 + 2x + 2\right)y'' + y = 0$$

(a) Find the first two nonzero terms of a pair of linearly independent power series solutions (about $x = 0$).

(b) Verify that the solutions in (a) are linearly independent.

(c) Find a lower bound for the radii of convergence of the solutions in (a).

Math 222 **Final Examination** Dec. 18, 2002

(All problems worth 10 points)

---

1. Solve the initial value problem

$$\frac{dy}{dx} = y^2 - 3y + 2, \quad y(0) = 0.$$

2. A tank initially contains 10 grams of arsenic dissolved in 10 liters of water. Pure water flows into the tank at a rate of 2 liters/minute and the well-mixed solution flows out of the tank at the same rate. Show that the amount of arsenic in the tank $\to 0$ as $t \to \infty$.

3. Find a particular solution of

$$y''' + 3y'' + 3y' + y = e^{-x}.$$

4. The unforced, damped oscillation of a mass on a linear spring is modeled by the equation

$$\frac{d^2x}{dt^2} + c\frac{dx}{dt} + 4x = 0, \quad x(0) = 1, \frac{dx}{dt}(0) = 0.$$

Determine the value of c such that the motion is critically damped, and find the solution in this case.

5. Compute the inverse Laplace transform (which may be left in integral form) of

$$\frac{s+1}{s^4 + 2s^3 + 2s^2}.$$

6. Use the Laplace transform method to solve the initial value problem

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = e^t, \quad x(0) = 0, \frac{dx}{dt}(0) = 1.$$

7. Find the first two nonzero terms of two linearly independent power series solutions of

$$y'' + (1+x)y = 0,$$

and explain why each of these series has an infinite radius of convergence.

8. Use variation of parameters to find a particular solution of

$$\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ t \end{bmatrix}.$$

9. Find the fundamental matrix that equals the identity a $t = 0$ (i.e. the exponential matrix) for

$$\mathbf{x}' = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 5 \end{bmatrix} \mathbf{x}.$$

10. (a) Find the Fourier series for the even periodic extension (of period 2) of the function $f(t) := 1 - t$ on $[0, 1]$. (b) Without computing the series, explain why the odd periodic extension of f has a Fourier series that converges at $t = 0$ to a different value than the series in (a).