Consider the nonlinear system

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\left(\alpha^2 + 16\right) & -2\alpha \end{bmatrix} \mathbf{x},$$

where $\mathbf{x} = (x, y)^T$.

(a) Find the exact solution of the system satisfying

$$\mathbf{x}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

for any (x_0, y_0) in the *phase plane* (= the x, y - plane) with $x_0, y_0 \geq 0$. Describe the difference in trajectories for (i) $\alpha < 0$, (ii) $\alpha = 0$, and (iii) $\alpha > 0$.

(b) Find approximate solutions of the system on $0 \leq t \leq 4$ using both the Euler and Runge-Kutta methods for $\alpha = -1$, and $\alpha = 1$ with initial conditions $x_0 = 1,\ y_0 = 0$ and $x_0 = 2,\ y_0 = 0$ using time steps $h = 0.4$ and $h = 0.1$ (16 approximate solutions altogether) .

(c) Plot the approximate trajectories in the phase plane for the results in (b) for the Runge-Kutta method with $\alpha = -1$, $x_0 = 2,\ y_0 = 0,\ h = 0.1$ (3 plots altogether).

(d) What problem do you think you might encounter using the Euler method when $|\alpha|$ is very small? Test this by plotting the approximate trajectory for $\alpha = -0.01,\ x_0 = 1,\ y_0 = 0,\ h = 0.4$ using the Euler method.