Math 450H
First Order Linear ODE Modeling Problems
Prof. Bukiet

1. **Gravity problem**. (Boyce and DiPrima, p. 9, #25) For slowly falling objects the approximation of drag force being proportional to velocity is okay. For faster falling, it is more accurate to assume that the drag force is proportional to the square of velocity. Write a differential equation for the velocity of a falling object of mass m if the drag is proportional to the square of velocity. Determine the limiting velocity.

2. **Escape Velocity**. (Boyce and DiPrima, p. 58) A body of constant mass m is projected away from the earth in a direction perpendicular to the earth's surface with an initial velocity v_0. Assuming that there is no air resistance, but taking into account the variation of the earth's gravitational field with distance, find an expression for the velocity of the ensuing motion. Find the initial velocity that is required to lift the body to a given maximum altitude x_{max} above the earth's surface and find the minimum initial velocity for which the body won't return to the earth's surface.

3. **Torricelli's Principle**. (Boyce and DiPrima, p. 60) If a tank containing a certain liquid has an outlet near the bottom and $h(t)$ is the height of the liquid at time t, Torricelli's principle states that the outflow velocity v at the outlet is equal to the velocity of a particle falling freely (without drag) from the height h. Thus $v = \sqrt{2gh}$ (Show this). The outflow area is smaller than the opening's area. (Let it be $\alpha area$. Derive the ODE describing this situation.

Consider a water tank in the shape of a right circular cylinder, 3m high and having radius of 1m. How long will it take to drain of water if α for water is 0.6.

4. **Population Dynamics**. (Boyce and DiPrima, p. 91, #22) Suppose that a population can be divided into 2 parts: those who have a given disease and can infect others, and those who do not have the disease and are susceptible. Let x be the proportion of susceptible individuals and y be the proportion of infectious individuals: so $x + y = 1$. Assume that the disease spreads by contact between sick and susceptible people and that the rate of spread is proportional to the number of contacts. Assume people are

"well-mixed". Set up the ODEs describing these dynamics, reduce to one equation, find equilibrium points and their stability.

5. **Population Dynamics**. (Boyce and DiPrima, p. 92, #24) Consider the cohort of individuals born in a given year ($t = 0$) and let $n(t)$ be the subset of these individuals surviving t years later. Let $x(t)$ be the number of members of this cohort who have not had smallpox by year t and who are therefore still susceptible. Let β be the rate at which susceptibles contract smallpox, and let ν be the rate at which people who contract smallpox die from the disease. Finally, let $\mu(t)$ be the death rate from all causes other than smallpox. Then $\frac{dx}{dt}$, the rate at which the number of susceptibles declines is given by:

$$\frac{dx}{dt} = -[\beta + \mu(t)]x$$

Also,

$$\frac{dn}{dt} = -\nu\beta x - \mu(t)n$$

Let $z = x/n$ to get $\frac{dz}{dt} = -\beta z(1 - \nu z)$ with $z(0) = 1$. Daniel Bernoulli (1760) estimated that $\nu = \beta = 1/8$. Find the proportion of 20 year olds with small pox. On the basis of this work, Bernoulli figured that innoculating people (eliminating smallpox) would increase life expectancy by about 3 years from 26 years 7 months at the time.