Math 337 – Fall 2004 Final Examination Preparation

Instructions. *These problems are provided to assist you to prepare for the final examination. These problems are intended to supplement your review of the homework, lectures, reading and midterm examinations. Problems marked with an asterisk (*) are among the more difficult that might be expected on the final. Those with two asterisks (**) are especially challenging.*

Problem 1. Suppose

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Compute each expression below if possible:

$$(a)\ AB \qquad (b)\ BA \qquad (c)\ A^2 + B^2 \qquad (d)\ A^t \qquad (e)\ B^tB \qquad (f)\ BB^t$$

$$(g)\ A\mathbf{x} \qquad (h)\ \frac{\mathbf{v}}{B} \qquad (i)\ \mathbf{x}^t\mathbf{x} \qquad (j)\ \mathbf{x}\mathbf{x}^t \qquad (k)\ \mathbf{x}^tA\mathbf{x} \qquad (\ell)\ \mathbf{x}\cdot\mathbf{u} \qquad (m)\ \mathbf{u}\cdot\mathbf{v}$$

$$(n)\ ||\mathbf{x}||^2 + ||\mathbf{u}||^2 \ - \ ||\mathbf{v}||^2 \qquad (o)\ \frac{\mathbf{x}}{\mathbf{v}} \qquad (p)\ 3\mathbf{x} - 2\mathbf{u} \qquad (q)\ 2\mathbf{v} + A \qquad (r)\ \mathbf{v}^tB$$

Problem 2. Suppose A is a 3×5 matrix. For each part below, give the matrix B such that BA is the matrix C described.

(a) C is obtained from A by adding 3 times the first row to the third row.
(b) C is obtained from A by subtracting the third row from the first row.
(c) C is the 2×5 matrix having only zeros as components.
(d) C is obtained from A by multiplying its second row by 7.
(e) C is obtained from A by exchanging its second and third rows.

Problem 3. Find the LU factorization of the following matrices:

$$(a)\ \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad (b)\ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \qquad (c)\ \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad (d)\ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$$

$$(e)\ \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \qquad (f)\ \begin{bmatrix} a & 0 & 0 & 0 \\ 2 & a & 0 & 0 \\ 2 & 2 & a & 0 \\ 2 & 2 & 2 & a \end{bmatrix}$$

Problem 4. Find the general solution of the following systems of equations:

$$(a)\ \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad (b)\ \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$(c)\ \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} \qquad (d)\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Problem 5. For each of the following matrices, find its inverse matrix if possible.

$$(a)\ \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \qquad (b)\ \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \qquad (c)\ \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad (d)\ \begin{bmatrix} 1 & 1 & 1 \\ a & 0 & -a \\ a^2 & 0 & a^2 \end{bmatrix}$$

Problem 6. For each of the following matrices, find a basis for the four fundamental subspaces. Also, find a basis for the fundamental subspaces for each matrix in problem 2.

$$(a)\ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad (b)\ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad (c)\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad (d)\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Problem 7*. Determine if the set of vectors S is a basis for the null space of the matrix A where

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right\} \qquad A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Would S be a basis if it contained additionally the vector $(1,0,0,0)$? Why or why not?

Problem 8.** Consider the two-dimensional subspace V of $\mathbf{R}^3$ having the orthonormal basis $\{\mathbf{a}_1, \mathbf{a}_2\}$. Let P be the matrix that implements projection into V; that is, $P\mathbf{x}$ is the point in V closest to $\mathbf{x}$. Suppose $\mathbf{u}$ and $\mathbf{w}$ are independent vectors in $\mathbf{R}^3$ with $P\mathbf{u}$ and $P\mathbf{w}$ dependent. Show there are scalar values α and β such that $\alpha\mathbf{u} + \beta\mathbf{w} \neq \mathbf{0}$ and $\alpha\mathbf{u} + \beta\mathbf{w}$ is in the left null space of the matrix having $\mathbf{a}_1$ and $\mathbf{a}_2$ as columns.

Problem 9. Suppose that $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is an orthonormal basis for $\mathbf{R}^n$. Suppose $\mathbf{x} \in \mathbf{R}^n$ and

$$\mathbf{x} \;=\; \alpha_1 \mathbf{a}_1 \;+\; \alpha_2 \mathbf{a}_2 \;+\; \ldots \;+\; \alpha_n \mathbf{a}_n$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are scalar values. Clearly showing your work, derive a formula for α_i in terms of $\mathbf{x}$ and as many of $\mathbf{a}_1$, $\mathbf{a}_2$, $\ldots$, $\mathbf{a}_n$ as are needed.

Problem 10*. Suppose that $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are independent vectors in $\mathbf{R}^3$. Moreover, assume that $\mathbf{x}$ and $\mathbf{y}$ are orthonormal (i.e. $||\mathbf{x}|| = 1$, $||\mathbf{y}|| = 1$ and $\mathbf{x} \cdot \mathbf{y} = 0$). Find the linear combination of $\mathbf{x}$ and $\mathbf{y}$ that is closest to $\mathbf{z}$.

Problem 11. Suppose that ϵ is a real number. Find the straight line best fitting the data $\{(-1,-1),\ (0,\epsilon),\ (1,1)\}$ in the least squares sense. That is, find the slope α and intercept β so that the line $b = \alpha t + \beta$ is the least squares best fit for the data points $(t_1, b_1) = (-1,-1)$, $(t_2, b_2) = (0, \epsilon)$ and $(t_3, b_3) = (1, 1)$.

Problem 12. In a certain experiment it is believed that the variables x and y are related by the formula $y = a + b \sin x$ where a and b are constants. The following measurements were made: $(x_1, y_1) = (0, 1)$, $(x_2, y_2) = (0, \frac{1}{2})$, $(x_3, y_3) = (0, \frac{3}{2})$, $(x_4, y_4) = (\pi/2, 0)$. Find the least squares estimates of a and b.

Problem 13. Find the 3×3 matrix P that projects $\mathbf{x} \in \mathbf{R}^3$ into the plane containing the vectors $(1, 1, 1)$ and $(1, -1, 1)$. Find the eigenvalues of P.

Problem 14. Let $A = [1\ 1\ 1\ 1]$. Find an orthonormal basis for the null space of A.

Problem 15*. Give the QR factorization of

$$A \;=\; \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}.$$

Problem 16. Find the determinants of the following matrices:

$$(a)\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad (b)\ \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad (c)\ \begin{bmatrix} 13 & 22 & 17 & 13 \\ 13 & 26 & 18 & 13 \\ 13 & 56 & 77 & 13 \\ 13 & 32 & 78 & 13 \end{bmatrix} \qquad (d)\ \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}^{18}$$

$$(e)\ \begin{bmatrix} 1 & 1 & 1 \\ a & 0 & -a \\ a^2 & 0 & a^2 \end{bmatrix} \qquad (f)\ \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \qquad (g)\ \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 4 & 1 \end{bmatrix} \qquad (h)\ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} [1 \quad 3 \quad -1]$$

Problem 17. For each matrix below find all eigenvalues and as many independent eigenvectors as possible. Diagonalize the matrix using an orthogonal matrix when the matrix is symmetric.

$$(a)\ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad (b)\ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \qquad (c)\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad (d)\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$(e)\ \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} \qquad (f)\ \begin{bmatrix} a & a^2 \\ a^2 & a^3 \end{bmatrix} \qquad (g)\ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \end{bmatrix} \qquad (h)\ \begin{bmatrix} 1 & 0 & a \\ 0 & 2 & 0 \\ a & 0 & 1 \end{bmatrix}$$

Problem 18*. Suppose that $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4, \mathbf{q}_5\}$ is an orthonormal basis (of column vectors) for $\mathbf{R}^5$. If possible, find a basis of eigenvectors for $\mathbf{R}^5$ along with the corresponding eigenvalues for the matrix A given by

$$A \;=\; 3\mathbf{q}_1\mathbf{q}_1^t \;+\; 2\mathbf{q}_2\mathbf{q}_2^t \;+\; 4\mathbf{q}_3\mathbf{q}_3^t.$$

If possible, diagonalize A.

Problem 19*. Consider the sequence $x_0 = 0$, $x_1 = 1$, $x_2 = \frac{3}{2}$, $x_3 = \frac{7}{4}$, $x_4 = \frac{15}{8} \ldots$ which satisfies the relation

$$x_{n+2} = = \frac{3}{2}x_{n+1} \;-\; \frac{1}{2}x_n.$$

(a) Find the matrix A such that

$$\begin{bmatrix} x_{n+2} \\ x_{n+1} \end{bmatrix} \;=\; A \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}.$$

(b) Give a formula for (x_{n+1}, x_n) in terms of powers of A and $(x_1, x_0) = (1, 0)$.
(c) Find the eigenvalues and eigenvectors of A.
(d) Diagonalize A.
(e) Find a explicit formula for x_n.
(f) Which of the following values best approximates x_{314}: 1, 2, π, 432 or $8.315 \cdot 10^{12}$.

Problem 20. Which of the following transformations of $\mathbf{R}^2$ are linear?

$(a)\ T(\mathbf{v}) = (v_1, v_2)$ $\qquad (b)\ T(\mathbf{v}) = (|v_1|, 0)$ $\qquad (c)\ T(\mathbf{v}) = (3v_1, v_2 - 3v_1)$ $\qquad (d)\ T(\mathbf{v}) = (0, 0)$

Problem 21. If possible find matrices satisfying the description.
(a) The matrix transforms $(1, 0)$ into $(2, 4)$ and transforms $(0, 1)$ to $(1, 2)$.
(b) The matrix transforms $(2, 4)$ to $(1, 0)$ and $(1, 2)$ to $(0, 1)$.
(c) The matrix transforms $(2, 4)$ to $(2, 0)$ and $(1, 2)$ to $(1, 0)$.

Partial Solutions

Note well: *These solutions are not complete. There are intended only to indicate whether you have the right approach and solution. On the examination you will be expected to provide complete solutions to problems to receive credit.*

Solution 1.

(a) undefined $\quad (b) \begin{bmatrix} 2 & 1 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} \quad (c)$ undefined $\quad (d) \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad (e) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (f) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$(g) \begin{bmatrix} 5 \\ 4 \end{bmatrix} \quad (h)$ undefined $\quad (i)\ [5] \quad (j) \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \quad (k)\ [14] \quad (\ell)\ 0 \quad (m)$ undefined

$(n)\ -4 \quad (o)$ undefined $\quad (p) \begin{bmatrix} 4 \\ 7 \end{bmatrix} \quad (q)$ undefined $\quad (r) \begin{bmatrix} 1 & 3 \end{bmatrix}$

Solution 2.

$$(a) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad (b) \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (c) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (d) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (e) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Solution 3.

$$(a) \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} \quad (b) \begin{bmatrix} 1 & 0 \\ \frac{b}{a} & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & a - \frac{b^2}{a} \end{bmatrix} \quad (c) \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$$

$$(d) \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{bmatrix} \quad (e) \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{4} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{4}{5} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{6} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{6}{7} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{6}{5} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{6}{5} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{8}{7} \end{bmatrix}$$

$$(f) \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{a} & 1 & 0 & 0 \\ \frac{2}{a} & \frac{2}{a} & 1 & 0 \\ \frac{2}{a} & \frac{2}{a} & \frac{2}{a} & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix}$$

Solution 4.

(a) no solution $\quad (b) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$(c)\ \begin{bmatrix}0\\1\\0\\0\end{bmatrix}+z\begin{bmatrix}1\\-2\\1\\0\end{bmatrix}+w\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\qquad (d)\ \begin{bmatrix}0\\-1\\2\\0\\1\end{bmatrix}+x\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}+u\begin{bmatrix}0\\-1\\-1\\1\\0\end{bmatrix}$$

Solution 5.

$$(a)\ \begin{bmatrix}3&-4\\-2&3\end{bmatrix}\qquad (b)\ \begin{bmatrix}\cos x&-\sin x\\ \sin x&\cos x\end{bmatrix}\qquad (c)\ \frac{1}{4}\begin{bmatrix}3&2&1\\2&4&2\\1&2&3\end{bmatrix}\qquad (d)\ \frac{1}{2a^2}\begin{bmatrix}0&a&1\\2a^2&0&-2\\0&-a&1\end{bmatrix}$$

Solution 6. Bases for null space, column space, row space and left null space in that order:

$$(a)\ \left\{\begin{bmatrix}-1\\1\end{bmatrix}\right\},\quad \left\{\begin{bmatrix}1\\1\end{bmatrix}\right\},\quad \left\{\begin{bmatrix}1\\1\end{bmatrix}\right\},\quad \left\{\begin{bmatrix}-1\\1\end{bmatrix}\right\}$$

$$(b)\ \left\{\begin{bmatrix}-1\\1\end{bmatrix}\right\},\quad \left\{\begin{bmatrix}1\\0\end{bmatrix}\right\},\quad \left\{\begin{bmatrix}1\\1\end{bmatrix}\right\},\quad \left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$$

$$(c)\ \left\{\begin{bmatrix}1\\0\end{bmatrix}\right\},\quad \left\{\begin{bmatrix}1\\0\end{bmatrix}\right\},\quad \left\{\begin{bmatrix}0\\1\end{bmatrix}\right\},\quad \left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$$

$$(d)\ \phi,\quad \left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\},\quad \left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\},\quad \phi$$

Solution 7. One basis for $\mathcal{N}(A)$ is $\{(1,0,0,0),(0,1,0,0)\}$. Since the vectors in S are linearly independent, it is enough to show that S spans $\mathcal{N}(A)$. That is, we need to show that for every value of x and y we can find α and β so that

$$\alpha\begin{bmatrix}1\\1\\0\\0\end{bmatrix}+\beta\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}=\begin{bmatrix}x\\y\\0\\0\end{bmatrix}.$$

This equation is a system of four equations with two unknowns. Elimination shows there is a unique solution: $\alpha=(x+y)/2$ and $\beta=(x-y)/2$.

Solution 8. Since $P\mathbf{u}$ and $P\mathbf{w}$ are dependent, there are numbers α and β, not both zero, such that $\alpha P\mathbf{u}+\beta P\mathbf{w}=\mathbf{0}$. Hence, $P(\alpha\mathbf{u}+\beta\mathbf{w})=0$ and $\alpha\mathbf{u}+\beta\mathbf{w}$ is in the null space of P. If $A=[\mathbf{a}_1\ \mathbf{a}_2]$ then we have $P=A(A^tA)^{-1}A^t=AA^t$ since $A^tA=I$. Thus, $\alpha\mathbf{u}+\beta\mathbf{w}$ is in the null space of AA^t. But A^t and AA^t share the same null space, so $\alpha\mathbf{u}+\beta\mathbf{w}$ is in the null space of A^t. Therefore, $\alpha\mathbf{u}+\beta\mathbf{w}$ is in the left null space of A.

Solution 9. $\alpha_i=\mathbf{a}_i\cdot\mathbf{x}$.

Solution 10. $(\mathbf{z}\cdot\mathbf{x})\mathbf{x}+(\mathbf{z}\cdot\mathbf{y})\mathbf{y}$

Solution 11. $b = t + \epsilon/3$

Solution 12. $a = 1$, $b = -1$

Solution 13.

$$P = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \qquad \lambda_1 = 1,\ \lambda_2 = 1,\ \lambda_3 = 0 \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Solution 14. A basis for the null space of A is $\{(-1,1,0,0),(-1,0,1,0),(-1,0,0,1)\}$. Gram-Schmidt gives

$$\left\{ \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{12}}{12} \\ -\frac{\sqrt{12}}{12} \\ -\frac{\sqrt{12}}{12} \\ \frac{\sqrt{12}}{4} \end{bmatrix} \right\}$$

Solution 15.

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 0 & 2\sqrt{2} \end{bmatrix}$$

Solution 16.

$(a)\ -1 \qquad (b)\ 24 \qquad (c)\ 0 \qquad (d)\ 1 \qquad (e)\ -2a^3 \qquad (f)\ 1 \qquad (g)\ 6 \qquad (h)\ 0$

Solution 17. Note: Answers are not unique.

$$(a) \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix} \qquad (b) \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$(c) \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \qquad (d) \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(e)\ 2, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad (f) \begin{bmatrix} (a^2+1)^{-1/2} & a(a^2+1)^{-1/2} \\ a(a^2+1)^{-1/2} & -(a^2+1)^{-1/2} \end{bmatrix} \begin{bmatrix} a+a^3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (a^2+1)^{-1/2} & a(a^2+1)^{-1/2} \\ a(a^2+1)^{-1/2} & -(a^2+1)^{-1/2} \end{bmatrix}$$

$$(g) \begin{bmatrix} 1 & -3 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/6 & 1/2 & -1/6 \\ -1/3 & 0 & 1/3 \\ -1/6 & -1/2 & 7/6 \end{bmatrix}$$

$$(h) \begin{bmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & -\sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1+a & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1-a \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & -\sqrt{2}/2 \end{bmatrix}$$

Solution 18.

$$A = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 & \mathbf{q}_4 & \mathbf{q}_5 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 & \mathbf{q}_4 & \mathbf{q}_5 \end{bmatrix}^T$$

Solution 19.

$(a)\ A = \begin{bmatrix} 3/2 & -1/2 \\ 1 & 0 \end{bmatrix} \qquad (b) \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = A^n \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} \qquad (d) \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$(e)\ 2 - 2^{1-n} \qquad (f)\ 2$

Solution 20. (a) linear, (b) not linear, (c) linear, (d) linear

Solution 21.

$(a) \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \qquad (b)$ no such matrix $\qquad (c) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2\alpha & -\alpha \\ 2\beta & -\beta \end{bmatrix}$