

Bifurcation Analysis of Gilet's Walking Droplet Model

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Abstract

Bouncing droplets on a vibrating fluid bath can exhibit wave-particle behavior, such as being propelled by the waves they generate. These droplets seem to walk across the bath, and thus are dubbed *walkers*. These walkers can exhibit exotic dynamical behavior which give strong indications of chaos, but many of the interesting dynamical properties have yet to be proven. In recent years discrete dynamical models have been derived and studied numerically. We prove the existence of a Neimark–Sacker bifurcation for a variety of eigenmode shapes of the Faraday wave field from one such model. Then we reproduce numerical simulations and produce new numerical simulations and apply our theorem to the test functions used for that model in addition to new test functions. Evidence of chaos is shown by numerically studying a global bifurcation.

Keywords: hydrodynamic quantum analogs, bouncing droplets, chaos, bifurcations

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Pilot wave theory is based upon the idea that particle trajectories drive the statistics seen in quantum mechanics. In the early days of quantum mechanics it was seen as a promising explanation of the statistics arising in experiments. However, in later years the theory was abandoned and the idea that the statistics is inherent in quantum phenomena was generally accepted. In more recent years, fluid dynamic experiments with droplets bouncing on a vibrating bath were observed to exhibit quantum-like behavior. This effectively revived the theory in the form of hydrodynamic quantum analogs. While the (integro-differential) models developed for these systems agree well with the experiments, they are quite complex and difficult to analyze. In recent years simpler (discrete dynamical) models have been developed. The numerical simulations of these models show evidence of exotic dynamics such as period doubling bifurcations, Neimark–Sacker bifurcations, and even chaos. We study one such model and prove the existence of Neimark–Sacker bifurcations and accurately predict the existence of dynamics that were overlooked in the original numerical experiments. We also study the possibility of an exotic global bifurcation through numerical experiments. Finally, our numerical experiments show evidence of quantum-like behavior.

1 Introduction

Pilot wave theory in quantum mechanics, specifically the works of de Broglie [1] and later Bohm [2], showed great potential until the 1960s when most physicists abandoned it and instead subscribed to the idea that the statistics is inherent in quantum phenomena. However, in recent years the idea of particle trajectories driving the statistics seen in quantum experiments has made a resurgence due to the study of quantum analogs in hydrodynamic pilot wave theory through the experimental works of Couder et al. [3, 4, 5, 6, 7, 8, 9] and the theoretical works of Bush et al. [10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

In 2005 Couder and collaborators observed droplets moving across a vibrating fluid bath by interacting with its own waves. They later observed quantum-like behavior by reproducing single particle diffraction experiments [3]. In 2010, Bush and collaborators developed hydrodynamic models for a bouncing droplet experiencing propulsion from its own waves. They reproduced some of Couder’s experiments numerically and performed other experiments. One such experiment trapped the particle in a circular corral [16, 17, 18] and the statistics of the dynamics of the particle were qualitatively similar to electrons trapped in a magnetic field. This was then repeated in the laboratory, which led to a surprisingly close agreement between theory and experiment.

While the models Bush and his collaborators developed agree well with experiments, the equations are quite difficult to study analytically. The complexity of the equations naturally created interest in developing realistic simplified mathematical models exhibiting important dynamical features of the original, while being easier to analyze. Several such reduced dynamical models were developed and two of them, devised by Shirokoff [20] and Gilet [21], showing considerable promise are actually planar discrete dynamical system models. Shirokoff [20] developed a model in which he derived a map for the motion of a particle in a square cavity. In this model, using numerics, he dis-

covered cascading period doubling bifurcations indicative of chaos. Gilet [21] included the amplitude of the waves in his model (1) for the straight line motion of a particle. He observed, again by numerical means, evidence of Neimark–Sacker bifurcations and chaotic dynamics.

$$\begin{aligned} w_{n+1} &= \mu[w_n + \Psi(x_n)], \\ x_{n+1} &= x_n - Cw_n\Psi'(x_n); \end{aligned} \tag{1}$$

In this model w is the amplitude of the waves, x is the position of the walker, $C \in [0, 1]$ is a constant representing wave-particle coupling, $\mu \in [0, 1]$ is the damping factor, and $\Psi \in \mathbb{R}$ is a single eigenmode of the Faraday wave field.

The remainder of this paper is organized as follows: In section 2 we prove some basic dynamical properties of (1). We analyze the map further in section 3 and formulate the conditions for it to have a Neimark–Sacker bifurcation. In our analysis, we vary μ while treating C as a constant, and find the value of μ for which the eigenvalues become complex conjugates. Then, we apply the genericity conditions for a Neimark–Sacker bifurcation to determine conditions on Ψ which make the system a generic 2-D map. In section 4 we apply our theorem for the conditions on Ψ to test functions that Gilet studied numerically. We end by discussing the possibility of a homoclinic-like bifurcation leading to chaotic dynamics in section 5.

2 Basic properties of the map

In this section we reproduce much of the calculation done in [21]. First, define $F : \mathbb{R}^2 \mapsto \mathbb{R}^2$ as,

$$\begin{bmatrix} w_{n+1} \\ x_{n+1} \end{bmatrix} = F(w_n, x_n) = \begin{bmatrix} \mu[w_n + \Psi(x_n)] \\ x_n - Cw_n\Psi'(x_n) \end{bmatrix} \tag{2}$$

We find the fixed points of (1) by solving,

$$\begin{aligned} w_* &= \mu[w_* + \Psi(x_*)] \\ x_* &= x_* - Cw_*\Psi'(x_*). \end{aligned}$$

Notice that this gives us two families of fixed points:

$$(0, x_*) \text{ such that } \Psi(x_*) = 0; \mu \neq 0 \text{ and} \tag{3}$$

$$(\hat{w}_*, \hat{x}_*) \text{ such that } \hat{w}_* = \frac{\mu}{1-\mu}\Psi(\hat{x}_*), \Psi'(\hat{x}_*) = 0; C \neq 0, \mu \neq 1. \tag{4}$$

The derivative matrix for F is,

$$DF(w, x) = \begin{pmatrix} \mu & \mu\Psi'(x) \\ -C\Psi'(x) & 1 - Cw\Psi''(x) \end{pmatrix}. \tag{5}$$

2.1 Properties of F in neighborhoods of the second family

We first notice that for fixed points (4), $\mu \in [0, 1)$ because if $\mu = 1$ this fixed point does not exist. Substituting the fixed points (4) into the derivative matrix (5), we obtain

$$DF(\hat{w}_*, \hat{x}_*) = \begin{pmatrix} \mu & 0 \\ 0 & 1 - \frac{\mu C}{1-\mu} \Psi(\hat{x}_*) \Psi''(\hat{x}_*) \end{pmatrix}.$$

Therefore, the eigenvalues are

$$\lambda_1 = \mu, \lambda_2 = \mu, 1 - \frac{\mu C}{1-\mu} \Psi(\hat{x}_*) \Psi''(\hat{x}_*)$$

Observe that since $\mu \in [0, 1)$, if $\Psi(\hat{x}_*) \Psi''(\hat{x}_*) < 0$ the fixed point is a saddle. However, if $\Psi(\hat{x}_*) \Psi''(\hat{x}_*) > 0$, we always have $\lambda_2 < 1$, so we must find a condition on μ for which $\lambda_2 = -1$. For this subsection we call this condition $\hat{\mu}$,

$$\hat{\mu} = \frac{1}{1 + C \Psi(\hat{x}_*) \Psi''(\hat{x}_*) / 2}.$$

Then, for $\mu < \hat{\mu}$ this fixed point is a sink and for $\mu > \hat{\mu}$ this fixed point is a saddle. If $\mu = \hat{\mu}$, the fixed point becomes nonhyperbolic.

When the fixed point is a saddle, it is easy to see that the linear stable and unstable manifolds are,

$$W^s = \{(w, x) : x = \hat{x}_*\}, \quad (6)$$

$$W^u = \{(w, x) : w = \hat{w}_* \frac{\mu}{1-\mu} \Psi(\hat{x}_*)\}. \quad (7)$$

2.2 Properties of F in neighborhoods of the first family

Since we can translate any fixed point (3) to the origin, without loss of generality, we assume the fixed point is $(w_*, x_*) = (0, 0)$ with the relevant conditions on Ψ' . Substituting the fixed point (3) into the derivative matrix (5), we obtain

$$DF(0, 0) = \begin{pmatrix} \mu & \mu \Psi'(0) \\ -C \Psi'(0) & 1 \end{pmatrix}. \quad (8)$$

The characteristic polynomial is

$$(\mu - \lambda)(1 - \lambda) + C\mu \Psi'(0)^2 = 0 \Rightarrow \lambda^2 - (1 + \mu)\lambda + \mu(1 + C\Psi'(0)^2) = 0,$$

and the eigenvalues are,

$$\lambda = \frac{1}{2}(1 + \mu) \pm \frac{i}{2} \sqrt{4\mu(1 + C\Psi'(0)^2) - (1 + \mu)^2}. \quad (9)$$

Notice that $\lambda \in \mathbb{C} \setminus \mathbb{R}$ if $4\mu(1 + C\Psi'(0)^2) - (1 + \mu)^2 > 0$, i.e.

$$2C\Psi'(0)^2 - 2\sqrt{C\Psi'(0)^2(C\Psi'(0)^2 + 1)} + 1 < \mu < 2C\Psi'(0)^2 + 2\sqrt{C\Psi'(0)^2(C\Psi'(0)^2 + 1)} + 1 \quad (10)$$

Now, if $|\lambda| < 1$, we get a stable focus, and when $|\lambda| > 1$ we get an unstable focus. This indicates that there may be a Neimark–Sacker bifurcation at the fixed point when $|\lambda|$ passes through unity, which occurs when μ goes from $\mu \leq \mu_* := 1/(1 + C\Psi'(0)^2)$ to $\mu > \mu_*$.

3 Neimark–Sacker bifurcation

In a continuous two-dimensional system (assuming genericity), a Hopf bifurcation occurs when a stable fixed point changes stability to become unstable as we vary the parameter forwards and either a stable limit cycle is generated about that unstable fixed point by varying the parameter forward (supercritical Hopf) or an unstable limit cycle is created about the stable fixed point by varying the parameter backwards (subcritical Hopf). Similarly, for a discrete two-dimensional map (again assuming genericity), a Neimark–Sacker bifurcation occurs in the same manner, except instead of a limit cycle an invariant closed Jordan curve (a topological circle) is born.

We consider the genericity conditions outlined in [22]. That is, the map must be locally conjugate near the fixed point to a specific normal form, there are no strong resonances, and the first Lyapunov coefficient must be nonzero. The Lyapunov coefficient also determines if the bifurcation is supercritical (if positive) or subcritical (if negative).

In [21], Gilet conjectured a supercritical Neimark–Sacker bifurcation occurs (assuming the map is generic) at the fixed points (3). He also observed evidence of this in the iterates of the map for chosen test functions. Here we prove the map (2) is generic and a Neimark–Sacker bifurcation occurs as we vary the parameter μ at the fixed point (r.f. [22, 23, 24]). We also show the map allows for both supercritical and subcritical Neimark–Sacker bifurcations.

Theorem 1. *The map (2) is generic about some fixed point (w_*, x_*) if the eigenmode satisfies the following property,*

$$\begin{aligned} \hat{d} = & \Psi'''(x_*)\Psi'(x_*)(1 + C\Psi'(x_*)^2)(1 + 2C\Psi'(x_*)^2)(4 + 3C\Psi'(x_*)^2) \\ & + 2\Psi''(x_*)^2 \{5 + C\Psi'(x_*)^2 [1 - C\Psi'(x_*)^2 (31 + 21C\Psi'(x_*)^2)]\} \neq 0. \end{aligned} \quad (11)$$

and a Neimark–Sacker bifurcation occurs at the fixed points (3) when

$$\mu = \mu_* = \frac{1}{1 + C\Psi'(x_*)^2}. \quad (12)$$

Furthermore, if $\hat{d} < 0$, the map undergoes a supercritical Neimark–Sacker bifurcation, and if $\hat{d} > 0$, the map undergoes a subcritical Neimark–Sacker bifurcation in a neighborhood of the fixed point.

Proof. As done in Section 2.2, without loss of generality we translate the fixed point to $(w_*, x_*) = (0, 0)$. We showed in Section 2.2 that the pair of eigenvalues λ are complex conjugates if

$$2C\Psi'(0)^2 - 2\sqrt{C\Psi'(0)^2(C\Psi'(0)^2 + 1)} + 1 < \mu < 2C\Psi'(0)^2 + 2\sqrt{C\Psi'(0)^2(C\Psi'(0)^2 + 1)} + 1 \quad (13)$$

and $|\lambda| = 1$ when

$$\mu = \mu_* = \frac{1}{1 + C\Psi'(0)^2}.$$

This shows that a Neimark–Sacker bifurcation occurs at the fixed point if the map is generic.

Next we show the map is generic (c.f. [22, 23, 24]) via three conditions (C.1), (C.2), and (C.3),

(C.1) We show that $r'(\mu_*) \neq 0$.

Notice, since $r(\mu_*) = 1$,

$$\left. \frac{d}{d\mu} (r(\mu)^2) \right|_{\mu=\mu_*} = 2r(\mu_*)r'(\mu_*) = 2r'(\mu_*),$$

so if $\left. \frac{d}{d\mu} (r(\mu)^2) \right|_{\mu=\mu_*} \neq 0$, $r'(\mu_*) \neq 0$. Then, since $C > 0$ and $\Psi'(0) \in \mathbb{R}$,

$$\left. \frac{d}{d\mu} (r(\mu)^2) \right|_{\mu=\mu_*} = (1 + C\Psi'(0)^2) \neq 0. \quad (14)$$

This shows that the transversality condition is satisfied.

(C.2) We show the arguments of the eigenvalues satisfy the first set of nondegeneracy conditions for a Neimark–Sacker bifurcation.

Let $\theta_* = \tan^{-1} A$, where

$$A = \frac{\sqrt{4\mu_*(1 + C\Psi'(0)^2) - (1 + \mu_*)^2}}{1 + \mu_*} = \frac{\sqrt{4 - (1 + \mu_*)^2}}{1 + \mu_*} \quad (15)$$

Observe that $\theta_* = 0$ if $A = 0$, $\theta_* = \pm\pi$ if $A = 0$, $\theta_* = \pm 2\pi/3$ if $A = \pm\sqrt{3}$, and $\theta_* \rightarrow \pm\pi/2$ as $A \rightarrow \pm\infty$. Since A is clearly positive and bounded, this rules out each case except $A = \sqrt{3}$. In order to get $\sqrt{3}$ we need $1 + \mu = 1$; however, since μ is positive this is not possible.

Thus the first nondegeneracy condition is satisfied.

(C.3) We compute the normal form for the Neimark–Sacker bifurcation and derive the conditions for which the second nondegeneracy condition is satisfied.

We compute the eigenvectors: $(DF)q = \lambda q$ and $(DF)^T p = \bar{\lambda} p$,

$$q = \begin{pmatrix} \mu_* \Psi'(0) \\ \lambda - \mu_* \end{pmatrix} \text{ and } p = \begin{pmatrix} \bar{\lambda} - 1 \\ \mu_* \Psi'(0) \end{pmatrix} \quad (16)$$

For the normalization, we take

$$\langle p, q \rangle = \bar{p} \cdot q = i\mu_* \Psi'(0) \sqrt{4 - (1 + \mu_*)^2}. \quad (17)$$

Now we change the variables to

$$\begin{pmatrix} w \\ x \end{pmatrix} = zq + \bar{z}\bar{q} = \begin{bmatrix} (z + \bar{z})\mu_*\Psi'(0) \\ (\lambda - \mu_*)z + (\bar{\lambda} - \mu_*)\bar{z} \end{bmatrix} \quad (18)$$

Substituting this into F yields

$$F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}; \text{ where} \quad (19)$$

$$F_1 = (z + \bar{z})\mu_*^2\Psi'(0) + \mu_*\Psi((\lambda - \mu_*)z + (\bar{\lambda} - \mu_*)\bar{z}),$$

$$F_2 = (\lambda - \mu_*)z + (\bar{\lambda} - \mu_*)\bar{z} - C(z + \bar{z})\mu_*\Psi'(0)\Psi'((\lambda - \mu_*)z + (\bar{\lambda} - \mu_*)\bar{z}).$$

Now we take the inner product

$$\begin{aligned} \langle p, F \rangle = & (z + \bar{z})\mu_*^2(\lambda - 1)\Psi'(0) + \mu_*(\lambda - 1)\Psi((\lambda - \mu_*)z + (\bar{\lambda} - \mu_*)\bar{z}) + (\lambda - \mu_*)\mu_*\Psi'(0)z \\ & + (\bar{\lambda} - \mu_*)\mu_*\Psi'(0)\bar{z} - C\mu_*^2\Psi'(0)^2(z + \bar{z})\Psi'((\lambda - \mu_*)z + (\bar{\lambda} - \mu_*)\bar{z}). \end{aligned} \quad (20)$$

Finally, to get the normal form we divide this through by $\langle p, q \rangle$ and take the Taylor series,

$$\begin{aligned} H = & \left[(z + \bar{z})\mu_*^2(\lambda - 1)\Psi'(0) + \mu_*(\lambda - 1) \sum_{j+k \geq 1} \frac{1}{j!k!} \partial_{z^j \bar{z}^k} \Psi(0) z^j \bar{z}^k + (\lambda - \mu_*)\mu_*\Psi'(0)z \right. \\ & \left. + (\bar{\lambda} - \mu_*)\mu_*\Psi'(0)\bar{z} - C\mu_*^2\Psi'(0)^2(z + \bar{z}) \left(\Psi'(0) + \sum_{j+k \geq 1} \frac{1}{j!k!} \partial_{z^j \bar{z}^k} \Psi'(0) z^j \bar{z}^k \right) \right] \\ & / \left(i\mu_*\Psi'(0)\sqrt{4 - (1 + \mu_*^2)} \right), \end{aligned} \quad (21)$$

where $\partial_{z^j \bar{z}^k} \Psi(0) := \partial_{z^j \bar{z}^k} \Psi((\lambda - \mu_*)z + (\bar{\lambda} - \mu_*)\bar{z})|_{(z, \bar{z})=(0,0)}$, and similarly for $\partial_{z^j \bar{z}^k} \Psi'(0)$.

By matching linear terms, it is easy to show the normal form can be written as,

$$\begin{aligned} H = & \lambda z + \left[(\lambda - 1) \sum_{j+k \geq 2} \frac{1}{j!k!} \partial_{z^j \bar{z}^k} \Psi(0) z^j \bar{z}^k - C\mu_*\Psi'(0)^2(z + \bar{z}) \sum_{j+k \geq 1} \frac{1}{j!k!} \partial_{z^j \bar{z}^k} \Psi'(0) z^j \bar{z}^k \right] \\ & / \left(i\Psi'(0)\sqrt{4 - (1 + \mu_*^2)} \right). \end{aligned} \quad (22)$$

We are ready now to compute the nondegeneracy condition required to satisfy the final genericity condition. From [22, 24], we have the formula,

$$d(0) = \operatorname{Re} \left(\frac{\bar{\lambda} g_{21}}{2} \right) + \operatorname{Re} \left(\frac{\bar{\lambda}(\bar{\lambda} - 2)}{2(\lambda - 1)} g_{20} g_{11} \right) - \frac{1}{2} |g_{11}|^2 - \frac{1}{4} |g_{02}|^2. \quad (23)$$

where $g_{jk}/(j!k!)$ is the coefficient of the $z^j \bar{z}^k$ term. We compute the relevant terms

$$\begin{aligned} g_{20} &= \frac{(\lambda - 1)(\lambda - \mu_*)^2 \Psi''(0) - 2C\mu_* \Psi'(0)^2 (\lambda - \mu_*) \Psi'(0)}{i\sqrt{4 - (1 + \mu_*)^2} \Psi'(0)} \\ g_{02} &= \frac{(\lambda - 1)(\bar{\lambda} - \mu_*)^2 \Psi''(0) - 2C\mu_* \Psi'(0)^2 (\bar{\lambda} - \mu_*) \Psi'(0)}{i\sqrt{4 - (1 + \mu_*)^2} \Psi'(0)} \\ g_{11} &= \frac{(\lambda - 1)(\lambda - \mu_*)(\bar{\lambda} - \mu_*) \Psi''(0)}{i\sqrt{4 - (1 + \mu_*)^2} \Psi'(0)} - \frac{C\mu_* \Psi'(0)^2 (\lambda - \mu_* + \bar{\lambda} - \mu_*) \Psi'(0)}{i\sqrt{4 - (1 + \mu_*)^2} \Psi'(0)} \\ g_{21} &= \frac{(\lambda - 1)(\lambda - \mu_*)(\bar{\lambda} - \mu_*) \Psi'''(0)}{i\sqrt{4 - (1 + \mu_*)^2} \Psi'(0)} - \frac{C\mu_* \Psi'(0)^2 ((\lambda - \mu_*)^2 + 2(\lambda - \mu_*)(\bar{\lambda} - \mu_*)) \Psi'''(0)}{i\sqrt{4 - (1 + \mu_*)^2} \Psi'(0)} \end{aligned}$$

If we factor out certain terms this formula greatly simplifies to,

$$d(0) = \frac{\Psi'''(0)}{2\Psi'(0)} \frac{\operatorname{Re}(-i\bar{\lambda}\hat{g}_{21})}{\sqrt{4 - (1 + \mu_*)^2}} - \frac{\Psi''(0)^2 \operatorname{Re} \left(\frac{\bar{\lambda}(\bar{\lambda} - 2)}{2(\lambda - 1)} \hat{g}_{20} \hat{g}_{11} \right)}{\Psi'(0)^2 (4 - (1 + \mu_*)^2)} - \frac{\Psi''(0)^2 \left(\frac{1}{2} |\hat{g}_{11}|^2 + \frac{1}{4} |\hat{g}_{02}|^2 \right)}{\Psi'(0)^2 (4 - (1 + \mu_*)^2)} \quad (24)$$

where

$$\hat{g}_{20} = (\lambda - \mu_*) [\lambda^2 + \mu_* - 2C\Psi'(0)^2 \mu_* - \lambda(1 + \mu_*)] \quad (25)$$

$$\hat{g}_{02} = (\bar{\lambda} - \mu_*) [1 + \mu_* - 2C\Psi'(0)^2 \mu_* - \lambda\mu_* - \bar{\lambda}] \quad (26)$$

$$\begin{aligned} \hat{g}_{11} &= -1 + \lambda + \lambda\mu_* - C\Psi'(0)^2 \lambda\mu_* - \lambda^2 \mu_* + \lambda\mu_* - C\Psi'(0)^2 \bar{\lambda}\mu_* \\ &\quad - \mu_* - \mu_*^2 + 2C\Psi'(0)^2 \mu_*^2 + \lambda\mu_*^2 \end{aligned} \quad (27)$$

$$\begin{aligned} \hat{g}_{21} &= (\mu_* - \lambda) [1 - \lambda - \lambda\mu_* + C\Psi'(0)^2 \lambda\mu_* + \lambda^2 \mu_* - \bar{\lambda}\mu_* + 2C\Psi'(0)^2 \bar{\lambda}\mu_* \\ &\quad + \mu_* + \mu_*^2 - 3C\Psi'(0)^2 \mu_*^2 - \lambda\mu_*^2]. \end{aligned} \quad (28)$$

Then, substituting in for λ and μ_* gives

$$\begin{aligned} d(0) &= C^2 \Psi'(0)^2 \left[\Psi'''(0) \Psi'(0) (1 + C\Psi'(0)^2) (1 + 2C\Psi'(0)^2) (4 + 3C\Psi'(0)^2) \right. \\ &\quad \left. + 2\Psi''(0)^2 \{ 5 + C\Psi'(0)^2 [1C\Psi'(0)^2 (31 + 21C\Psi'(0)^2)] \} \right] \\ &\quad / [4(1 + C\Psi'(0)^2)^4 (4 + 3C\Psi'(0)^2)]. \end{aligned} \quad (29)$$

Since

$$\frac{C^2\Psi'(0)^2}{4(1+C\Psi'(0)^2)^4(4+3C\Psi'(0)^2)} > 0,$$

we need only be concerned with,

$$\begin{aligned} \hat{d} := & \Psi'''(x_*)\Psi'(x_*)(1+C\Psi'(x_*)^2)(1+2C\Psi'(x_*)^2)(4+3C\Psi'(x_*)^2) \\ & + 2\Psi''(x_*)^2 \{5+C\Psi'(x_*)^2 [1-C\Psi'(x_*)^2(31+21C\Psi'(x_*)^2)]\} \end{aligned} \quad (30)$$

Consequently, the Neimark–Sacker criteria imply that the bifurcation occurs if $\hat{d} \neq 0$, and it is supercritical for $\hat{d} < 0$ and subcritical for $\hat{d} > 0$.

This shows that for certain properties of Ψ the second nondegeneracy condition is satisfied, thereby completing the proof. □

4 Application of results to test functions

In [21], Gilet uses the test functions,

$$\Psi(x, \beta) = \frac{1}{\sqrt{\pi}} \cos \beta \sin 3x + \frac{1}{\sqrt{\pi}} \sin \beta \sin 5x \quad (31)$$

where β is a fixed parameter that can be changed in order to tweak the shape of the eigenmode. He shows numerical constructions of the iterate-space for $\beta = \pi/3$ and $\beta = \pi/6$, and studies the statistics of the iterates for other values of β . We too choose to study the eigenmode for $\beta = \pi/3$, for which the map exhibits supercritical Neimark–Sacker bifurcations at various fixed points, but we also study $\beta = 5\pi/6$, for which the map exhibits a subcritical Neimark–Sacker bifurcation at the origin.

It should be noted that for the sake of plotting, we consider the map (2) to be written as

$$\begin{bmatrix} x_{n+1} \\ w_{n+1} \end{bmatrix} = F(x_n, w_n) = \begin{bmatrix} x_n - Cw_n\Psi'(x_n) \\ \mu[w_n + \Psi(x_n)] \end{bmatrix} \quad (32)$$

4.1 Supercritical Neimark–Sacker bifurcations for $\beta = \pi/3$

For $\beta = \pi/3$, we now consider the map (32) on the domain $x \in [0, \pi/2]$. On this domain, the map has three fixed points $(x_*, 0)$ about which a Neimark–Sacker bifurcations occurs; one of which is the origin with the other two being the following:

$$\begin{aligned} x_* &= \tan^{-1} \left(\sqrt{\frac{15 + \sqrt{3} - \sqrt{6(8 - \sqrt{3})}}{9 - \sqrt{3} + \sqrt{6(8 - \sqrt{3})}}} \right) \approx 0.7269 \\ x_* &= \tan^{-1} \left(\sqrt{\frac{15 + \sqrt{3} + \sqrt{6(8 - \sqrt{3})}}{9 - \sqrt{3} - \sqrt{6(8 - \sqrt{3})}}} \right) \approx 1.3515 \end{aligned}$$

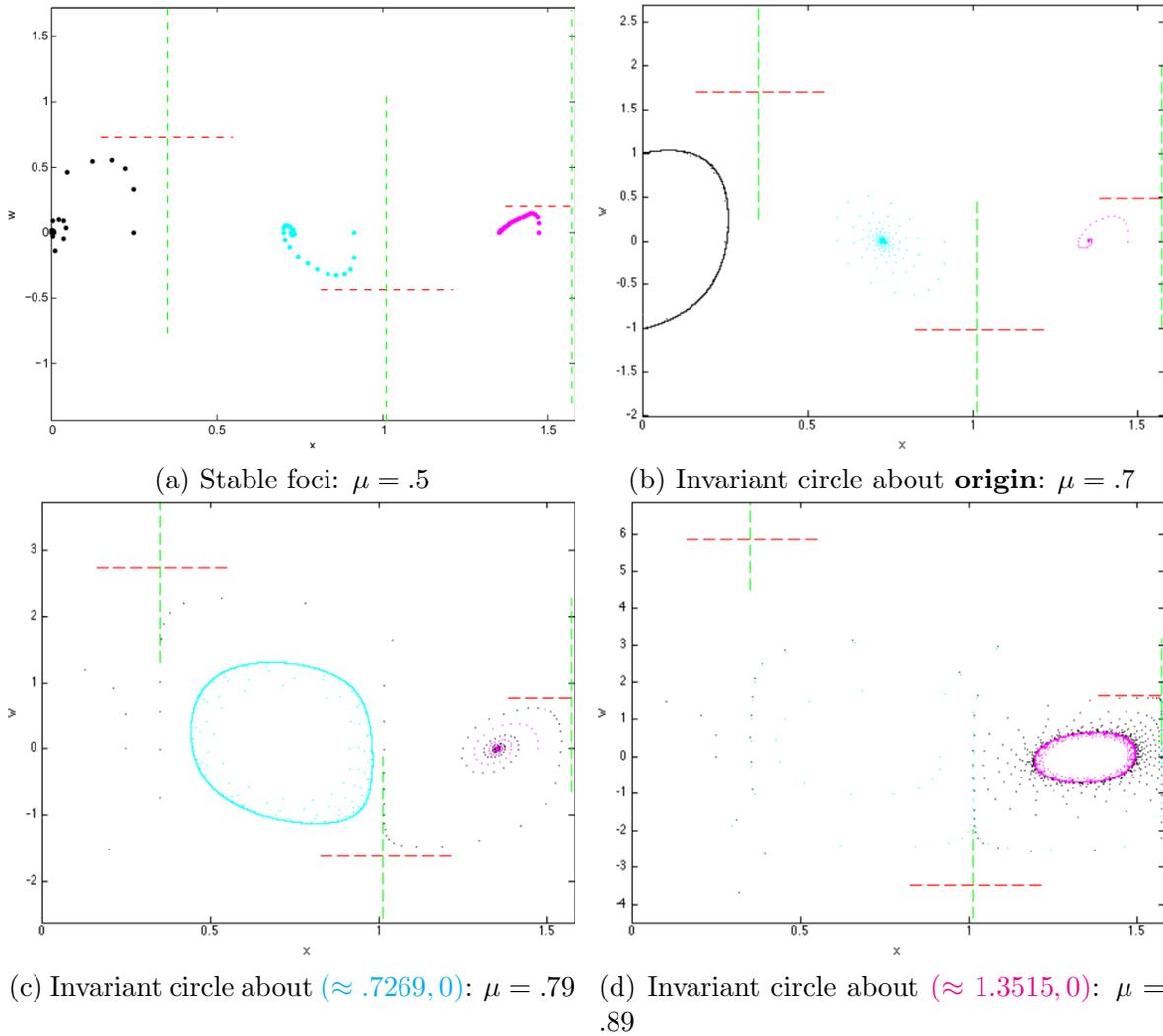


Figure 1: For each plot the **green** lines represent the linear **stable manifold** and the **red** lines represent the linear **unstable manifold** at the respective saddle fixed points. The **black** markers represent iterates originating from a neighborhood of the **origin**, the **cyan** markers represent iterates originating from a neighborhood of $(\approx .7269, 0)$, and the **magenta** markers represent the iterates originating from a neighborhood of $(\approx 1.3515, 0)$.

At the fixed points a supercritical Neimark–Sacker bifurcation occurs for $\mu_* = 0.64894, 0.742027, 0.879451$, respectively, for which $\hat{d} = -4076.61, -1747.52, -410.779$. We also have three other fixed points: $(\approx 0.3490, 0)$, $(\approx 1.0128, 0)$, $(\pi/2, 0)$, which are saddles and therefore not subject to Neimark–Sacker bifurcation. Let us vary the parameter μ from $\mu = 0.5$ to $\mu = 0.89$. We illustrate the progression of the bifurcations in the Fig. 1. In Fig. 1a, the relevant fixed points are all stable foci. Next, in Fig. 1b, we have passed the critical value for the origin and a stable invariant circle is now visible. Finally, in Fig. 1c and 1d, the critical values for the next two fixed points, respectively, are passed. We notice in Fig. 1c and 1d that the preceding fixed point(s), respectively, are now unstable focus(foci). This shows that each fixed point undergoes a supercritical Neimark–Sacker bifurcation in accordance with Theorem 1.

In addition to Neimark–Sacker bifurcation, we notice a curious phenomenon. As the saddle fixed points move away from the Neimark–Sacker fixed points the trajectories from the neighborhood of one Neimark–Sacker fixed point crosses into that of another. Furthermore, as the invariant circle increases in radius it seems to collide with on of its neighboring saddles just before the onset of chaos. We analyze this further in Section 5.

4.2 Subcritical Neimark–Sacker bifurcations for $\beta = 5\pi/6$

For $\beta = 5\pi/6$ lets once again consider the map (32), however lets simplify matters and restrict our domain to a neighborhood of the origin. Here lies a subcritical Neimark–Sacker bifurcation at $\mu_* = 0.999847$ where $\hat{d} = 4.88756$. Since this is subcritical, lets vary our parameter μ backwards from $\mu = 0.9999$ to $\mu = .999$. This is illustrated in Fig. 2. First the origin is an unstable focus, then as μ passes μ_* backwards the origin becomes a stable focus, however if an initial point is taken further out the iterates diverge, which indicates an unstable invariant circle. In Fig. 2b, we represent this unstable invariant circle by the black dashed curve.

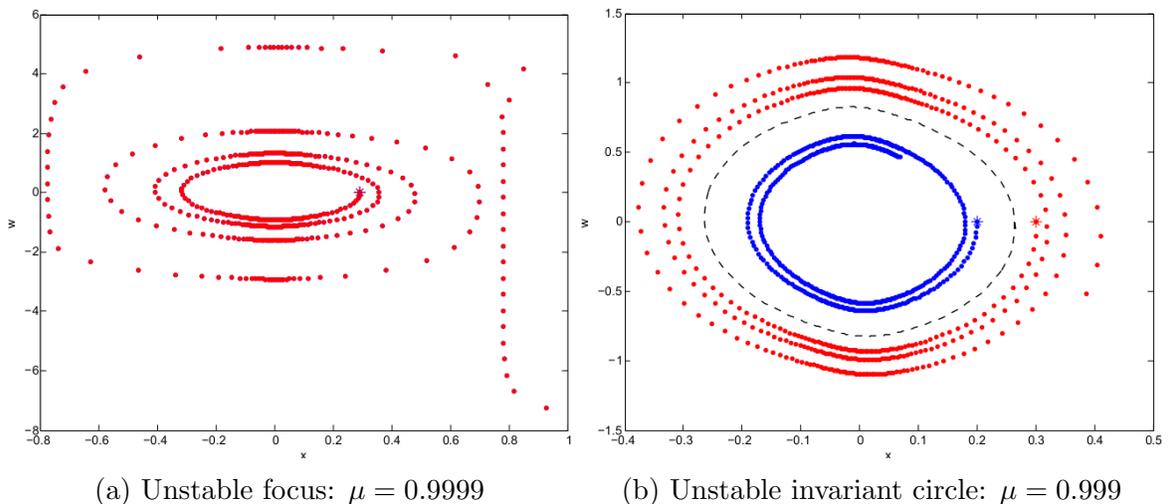


Figure 2: A star denotes the initial point. In the second plot the blue iterates have an initial point inside the invariant circle, and the red iterates have an initial point outside the invariant circle. Finally, the black dashed curve represents the unstable invariant circle.

5 Evidence of global bifurcations leading to chaos

In Fig. 1d we observe the appearance of quite exotic dynamics. If we continue to vary μ we find more exotic, chaotic-like, dynamics. We illustrate this in Fig. 3 where we see a scatter of iterates in a seemingly random manner. Moreover, we observe as μ is varied, the iterates remain within a compact set. Here the iterates seem to satisfy the transitivity and sensitivity conditions for chaos. Proving these conditions is the object of panned future research.

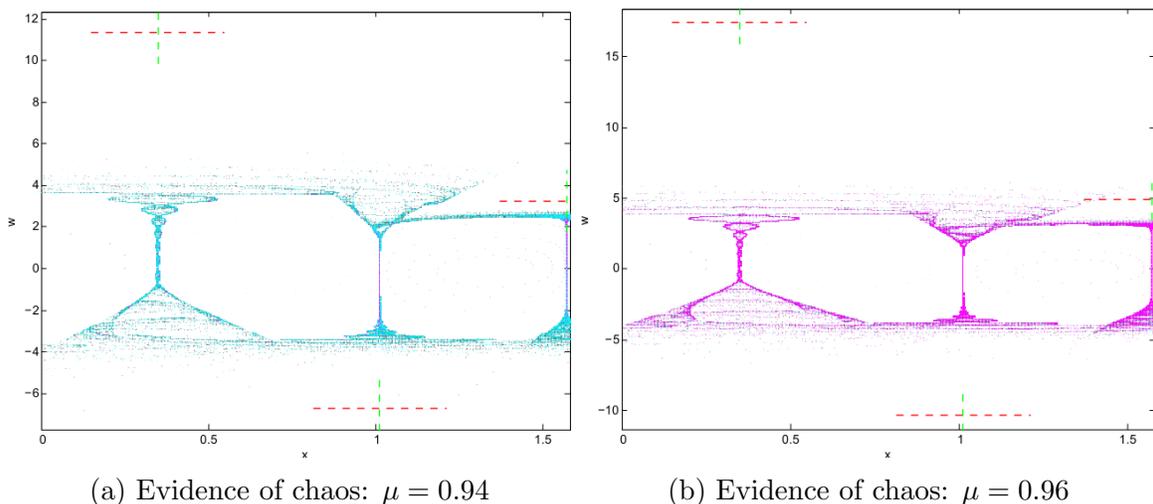


Figure 3: Exotic, chaotic-like, trajectories. For each plot the green lines represent the linear **stable manifold** and the red lines represent the linear **unstable manifold** at the respective saddle fixed points. The **black** markers represent iterates originating from a neighborhood of the **origin**, the **cyan** markers represent iterates originating from a neighborhood of $(\approx .7269, 0)$, and the **magenta** markers represent the iterates originating from a neighborhood of $(\approx 1.3515, 0)$.

Now let us vary μ near the onset of chaos to study the bifurcation. This is illustrated in Fig. 4. First (Fig. 4a) we observe our invariant circle for $\mu = 0.913$ as before, then as we increase to $\mu = 0.914$ the invariant circle starts to disintegrate (Fig. 4b). Finally, in Fig. 4c, for $\mu = 0.915$ we see the iterates leaving the neighborhood of the Neimark–Sacker fixed point and perhaps nested invariant circles (both stable and unstable). This provides evidence of an exotic (perhaps new) bifurcation as the invariant circle collides with the stable manifold of the saddle and perhaps even a cascading Neimark–Sacker bifurcation after collision.

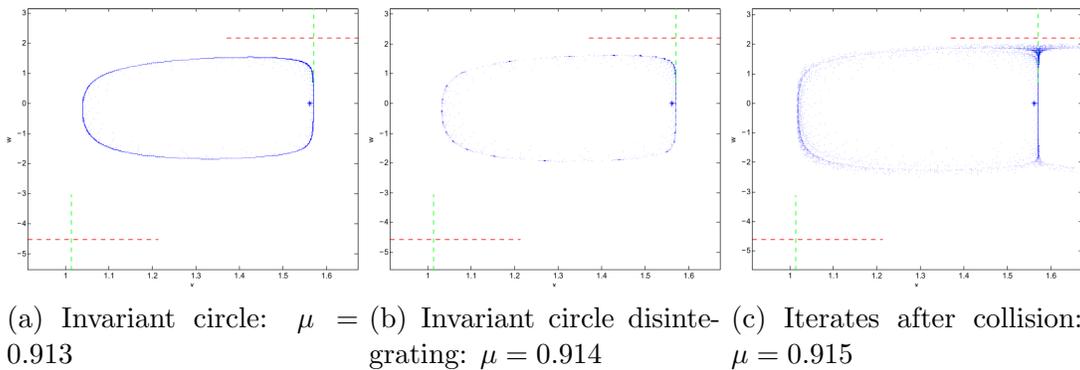


Figure 4: The green lines represent the linear stable manifold and the red lines represent the linear unstable manifold at the respective saddle fixed points. The blue markers are the iterates.

5.1 Potentiality of quantum-like behavior

We notice in our figures that iterates prefer to intersect the linear stable manifolds. These droplets already exhibit wave-particle duality and a host of other quantum phenomena. If we do not restrict the region of study, we observe that the iterates seem to prefer certain positions as shown in Fig. 5. Further analysis is certainly necessary and planned in the near future.

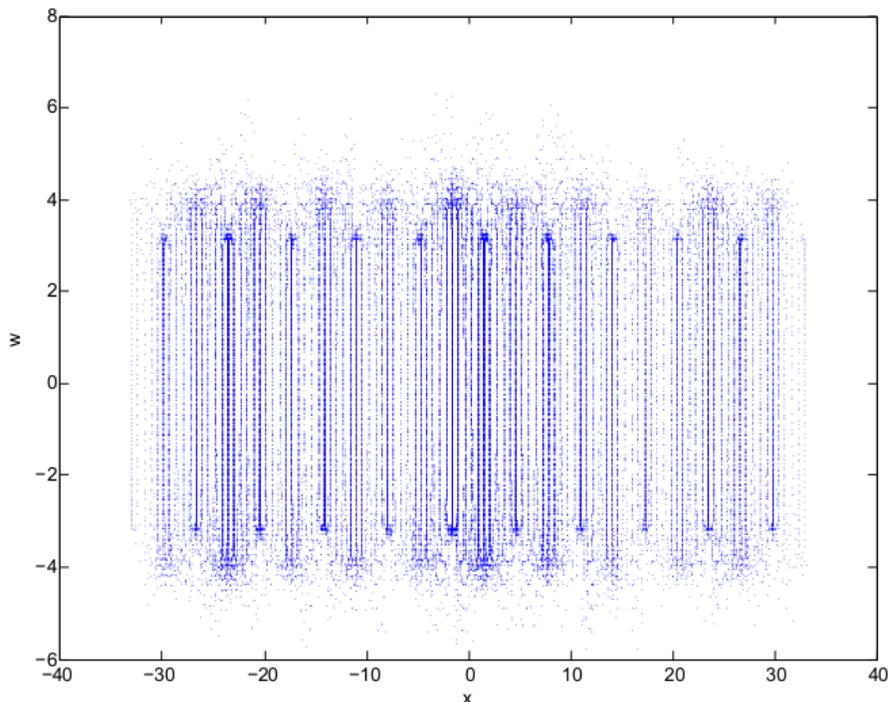


Figure 5: Iterates of the map (32) generated from four initial points chosen near the origin.

6 Conclusion

Walkers have been known to exhibit rich exotic dynamics in addition to strong connections with quantum mechanics. The investigations of Bush and collaborators [10, 8, 11, 12, 13, 14, 15, 16, 17, 18, 19] have shown evidence of exotic dynamics, such as Hopf bifurcations and chaos, in their models of walkers. However, the equations have generally been too complex to prove the existence of certain bifurcations or chaos. In recent years, models involving discrete dynamical systems have been studied by Shirokoff [20] and Gilet [21]. Maps are certainly easier to analyze than flows in many respects. This opened up opportunities to rigorously verify certain heretofore unproven properties. Proving these properties is not only important in its own respect, but may also provide insight into the phenomena observed for walkers and for their continuous models.

We have proved the existence of Neimark–Sacker bifurcations for Gilet’s model (1), and applied our theorem to test functions to show complete agreement between the theory and the numerics. Using numerics, we also observed evidence for more exotic phenomena, which seem to lead to chaos. One such curious phenomenon is a homoclinic-like bifurcation when the invariant circle collides with the stable manifold of a neighboring saddle fixed point. This also appears to give rise to other invariant circles leading to cascading Neimark–Sacker bifurcations wherein the invariant circles change stability and give birth to twin stable invariant circles.

Finally, while not the focus of this work, Gilet, as did we, observed quantum-like phenomena for the position of the iterates. We notice that after the onset of chaos the iterates prefer certain positions, namely along the stable manifolds of each saddle fixed point. In [21], Gilet studies the statistics that develop in these quantum-like orbits. He writes, “Future work could include the search for wave-particle coupling dynamics that yields $\text{PDF}(x) \sim |\Psi(x)|^2$ as observed in quantum mechanics.” We agree that this would be a fruitful endeavor and perhaps studying the dynamics of the system further will provide insight into the right choice of wave-particle coupling.

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References

- [1] L. de Broglie. *Ondes et mouvements*. Gauthier-Villars, Paris, France, 1st edition, 1926.
- [2] D. Bohm and J.P. Vigier. Model of the causal interpretation of quantum theory in terms of a fluid with irregular fluctuations. *Phys. Rev.*, 96:208, 1954.
- [3] Y. Couder and E. Fort. Single-particle diffraction and interference at a macroscopic scale. *Phys. Rev. Lett.*, 97(154101), 2006.
- [4] A. Eddi, E. Fort, F. Moisy, and Y. Couder. Unpredictable tunneling of a classical wave-particle association. *Phys. Rev. Lett.*, 102(240401), 2009.
- [5] E. Fort, A. Eddi, A. Boudaoud, J. Moukhtar, and Y. Couder. Path-memory induced quantization of classical orbits. *Proc. Nat. Acad. Sci.*, 107(17515), 2010.
- [6] A. Eddi, E. Sultan, J. Moukhtar, E. Fort, M. Rossi, and Y. Couder. Information stored in faraday waves: the origin of path memory. *J. Fluid Mech.*, 674:433–463, 2011.
- [7] A. Eddi, J. Moukhtar, S. Perrard, E. Fort, and Y. Couder. Level splitting at macroscopic scale. *Phys. Rev. Lett.*, 108(264503), 2012.
- [8] D.M. Harris, J. Moukhtar, E. Fort, Y. Couder, and J.W.M. Bush. Wavelike statistics from pilot-wave dynamics in a circular corral. *Phys. Rev. E*, 88(011001):1–5, 2013.
- [9] S. Perrard, M. Labousse, M. Miskin, E. Fort, and Y. Couder. Self-organization into quantized eigenstates of a classical wave-driven particle. *Nature Comm.*, 5(3219), 2014.
- [10] D.M. Harris and J.W.M. Bush. The pilot-wave dynamics of walking droplets. *Phys. Fluids*, 25(091112):1–2, 2013.
- [11] J. Molacek and J.W.M. Bush. Droplets bouncing on a vibrating bath. *J. Fluid Mech.*, 727:582–611, 2013.
- [12] J. Molacek and J.W.M. Bush. Droplets walking on a vibrating bath: towards a hydrodynamic pilot-wave theory. *J. Fluid Mech.*, 727:612–647, 2013.
- [13] A. Oza, R.R. Rosales, and J.W.M. Bush. A trajectory equation for walking droplets: hydrodynamic pilot-wave theory. *J. Fluid Mech.*, 737:552–570, 2013.
- [14] O. Wind-Willassen, J. Molacek, D.M. Harris, and J.W.M. Bush. Exotic states of bouncing and walking droplets. *Phys. Fluids*, 25(082002):1–11, 2013.
- [15] J.W.M. Bush, A. Oza, and J. Molacek. The wave-induced added mass of walking droplets. *J. Fluid Mech.*, 755(R7), 2014.
- [16] D.M. Harris and J.W.M. Bush. Droplets walking in a rotating frame: from quantized orbits to multimodal statistics. *J. Fluid Mech.*, 739:444–464, 2014.

- [17] A. Oza, D.M. Harris, R.R. Rosales, and J.W.M. Bush. Pilot-wave dynamics in a rotating frame: on the emergence of orbital quantization. *J. Fluid Mech.*, 744:404–429, 2014.
- [18] A. Oza, O. Wind-Willassen, D.M. Harris, R.R. Rosales, and J.W.M. Bush. Pilot-wave dynamics in a rotating frame: Exotic orbits. *Phys. Fluids*, 26(082101), 2014.
- [19] J.W.M. Bush. Pilot-wave hydrodynamics. *Ann. Rev. Fluid Mech.*, 49:269–292, 2015.
- [20] D. Shirokoff. Bouncing droplets on a billiard table. *Chaos*, 23(013115), 2013.
- [21] T. Gilet. Dynamics and statistics of wave-particle interaction in a confined geometry. *Phys. Rev. E*, 90(052917), 2014.
- [22] Y.A. Kuznetsov. *Elements of Applied Bifurcation Theory*, volume 112. Springer-Verlag, New York, NY, 3rd edition, 1995.
- [23] Ju.I. Neimark. On some cases of periodic motions depending on parameters. *Dokl. Akad. Nauk SSSR*, 129:736–739, 1959.
- [24] R. Sacker. On invariant surfaces and bifurcation of periodic solutions of ordinary differential equations. *Report IMM-NYU*, 333:1–62, 1964.
- [25] S. Perrard, M. Labousse, E. Fort, and Y. Couder. Chaos driven by interfering memory. *Phys. Rev. Lett.*, 2014.
- [26] M. Labousse and S. Perrard. Non hamiltonian features of a classical pilot-wave dynamics. *Phys. Rev. E*, 2014.
- [27] S. Protiere, A. Boudaoud, and Y. Couder. Particle-wave association on a fluid interface. *J. Fluid Mech.*, (554):85–108, 2006.