An $M^X/G/1$ Queueing System with Disasters and Repairs under a Multiple Adapted Vacation Policy

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Abstract

We consider a queueing system with batch Poisson arrivals subject to disasters which occur only when the server is busy and clear the system. Following a disaster the server initiates a repair period during which arriving customers accumulate without receiving service. The server operates under a Multiple Adapted Vacation policy. We analyze this system using the supplementary variables technique and obtain the probability generating function of the number of customers in the system in stationarity the fraction of customers who complete service, and the Laplace transform of the system time of a typical customer in stationarity. Finally, we examine a variation of the model in which the system is subject to disasters even when the server is taking a vacation or is under repair.

KEYWORDS: BATCH-ARRIVALS, MULTIPLE ADAPTED VACATIONS, DISASTERS, SUPPLEMENTARY VARIABLES METHOD.

1 Introduction

We study a single server queue with batch Poisson arrivals and general service times. The system suffers random disasters (catastrophes) which, when they occur, instantly remove all customers, including the one in service, from the system. Disasters affect the system only during the time the server is busy serving customers. Immediately after the occurrence of a disaster the server undergoes a repair of random duration while customers arriving during the repair period wait for the server to become available when the repair process is completed. When upon a service completion the system becomes empty the server takes a string of vacations of random length during which any customers that may have arrived wait without receiving

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service. Vacation lengths are i.i.d. random variables and the number of vacations in each string is also random. A string of vacations ends when either customers arrive or when it reaches its full length. In the second case the server remains idle, waiting for customers to arrive.

For a comprehensive survey of queues with vacations see Doshi [10] and Takagi [23]. Baba [7] studied the $M^{[X]}/G/1$ queueing model with multiple vacations while Kella [14] studied a vacation model with control policy. Takagi [23] proposed the concept of variable vacations in which the maximum number of vacations that the server may take is a random variable. Tian and Zhang [24] termed this vacation policy the Multiple Adaptive Vacation policy (MAV). A special case of MAV is the so called randomized vacations policy which corresponds to the case where the maximum number of vacations the server may take at each busy period termination is distributed according to a truncated geometric distribution (see Ke and Huang [12]). Among papers which have treated discrete time queues under a MAV policy we mention Zhang and Tian [28]. The corresponding literature for continuous time queueing systems is limited. Ma et al. [18] studied an $M/G/1$ queue model with MAV and pure decrementing service policy using the embedded Markov chain method. For the same model the probability generating function of the stationary queue length is obtained in [19] by using an embedded Markov chain method and regenerative arguments. A queue with a Batch Markovian Arrival Process (BMAP) under the MAV policy is studied in Banik [4] using matrix-analytic methodology.

Disasters in queueing systems are events that remove all customers from the system. Queues with disasters are natural models for communications or manufacturing systems subject to catastrophic failures which result in the loss of all customers currently in service, or waiting to be served.

Boxma et al. [6] considered an $M/G/1$ queue where disasters occur according to an independent renewal process and studied the model for different disaster rules. Retrial queues with disasters were studied by Artalejo and Gomez-Corral [1], and Shin [21] and queueing networks with disasters by Chao [9]. Discrete time queues with disasters have been analyzed by many authors, see for instance Park et al. [20]. Yechiali [26] studied $M/M/c$ queues with disasters and customers that become impatient during the repair periods following disasters. See also Chakravarthy [8] for a generalization of this model. Kumar et al. [15] derived the transient analysis for the $M/G/1$ queue with disasters. The busy period of such models has been studied by Yashkov and Yashkova [25]. Finally we refer to the papers of Lee et al. [17]

and Yang et al. [27] which include repair times after disasters occurring for discrete and continuous time models respectively. In the literature, disasters are also referred to as queue flushing, mass exodus and catastrophes (see for instance Economou and Gomez [11] and Kyriakidis and Abakuks [16]).

The paper is organized as follows. In section 2 the model is described and in section 3 it is analyzed using the method of supplementary variables as a continuous time Markov process with state space which has a discrete component (number of customers in the system and state of the server, i.e. busy, idle, on vacation, or under repair, and number of vacations taken within a given string of vacations) and a continuous component which is either the elapsed service time of the customer being served or the elapsed vacation or repair time. Balance equations for the stationary distribution of this process are given following the standard supplementary variables methodology and they are solved using generating functions. In section 4 the partial probability generating functions of the stationary number of customers in the system according to the condition of the server (busy, on vacation, or under repair) are given. In section 5 commentary on the results obtained is given and special cases are considered. Additional performance measures such as the distribution of the number of customers in the system at departure epochs is given. Section 6 discuses the structure of the busy periods of the system while section 7 obtains the Laplace transform of the stationary workload and the system time of a "typical customer" arriving in stationarity. Conditional system times given that the typical customer suffers a disaster or does not are also obtained. Finally, in section 8 the system is analyzed under the assumption that the disaster mechanism is also operational when the server is on vacation or under repair.

2 The Model

The system consists of a single server to which customers arrive in batches according to a Poisson process with rate λ and are processed according to the FIFO discipline. Denote the size of nth batch by χ_n, $n = 1, 2, \ldots$. Batches are assumed to be i.i.d. with pgf (probability generating function) $\chi(z) := \sum_{n=1}^{\infty} \mathbb{P}(\chi_1 = n) z^n$. (Without loss of generality we assume that there are no empty batches.) Each customer in the batch is served singly and the service requirements of the customers are assumed to be i.i.d. random variables with common distribution S which will be assumed absolutely continuous with corresponding density

S' and hazard rate function $\mu(x) = \frac{S'(x)}{1-S(x)}$, $x \geq 0$. The Laplace transform of S will be denoted by $\hat{S}(s) := \int_0^\infty e^{-sx} dS(x)$ and the mean, assumed finite, by $m_S := \mathbb{E}S$.

The server follows a MAV policy: At the end of the busy period the server either remains idle (with probability $1 - g_0$) or takes a vacation with probability g_0. If at the end of this vacation no customers have arrived then the server, independently of everything else, takes a new vacation with probability g_1, or remains idle and available to serve the first customer that arrives with probability $1 - g_1$. The process is repeated: Thus the policy is determined by the sequence of probabilities $\{g_k\}$, $k = 0, 1, 2, \ldots$. An alternative way to describe this policy is to assume that the number of vacations to be taken at the end of the ith busy period is a random variable ζ_i with distribution $\mathbb{P}(\zeta_i = k) = f_k$, $k = 0, 1, 2, \ldots$. If we set $F_k = \sum_{l=k}^\infty f_l$ then the probability that the server will take a vacation at the end of a busy period is $g_0 = F_1$ while the probability that he will take a vacation after k vacations have been completed with no arrivals is

$$g_k := \mathbb{P}(\zeta_i \geq k+1 | \zeta_i \geq k) = \frac{F_{k+1}}{F_k}, \quad k = 0, 1, 2, \ldots. \tag{2.1}$$

$\hat{F}(z) := \sum_{k=0}^\infty f_k z^k$ denotes the pgf of the number of potential vacations. Vacations have independent durations with common distribution function U, assumed again to be absolutely continuous with density U' and hazard rate $u(x) = \frac{U'(x)}{1-U(x)}$, $x \geq 0$. The corresponding Laplace transform will be denoted by $\hat{U}(s) = \int_0^\infty e^{-sx} dU(x)$ and the mean by $m_U := \mathbb{E}U < \infty$.

Finally, the system is subject to catastrophic failures which, following the common practice, we shall term disasters. Disasters occur according to a Poisson process with rate δ, independently of all other processes in the system, *provided that the server is busy, serving customers*. When a disaster occurs all customers present, including the one in service, are removed from the system and the server initiates a repair period. Repairs have i.i.d. durations with distribution function R, assumed absolutely continuous, with density function R', hazard rate $r(x) = \frac{R'(x)}{1-R(x)}$, $x \geq 0$, and Laplace transform $\hat{R}(s) = \int_0^\infty e^{-sx} dR(x)$. During a repair period, any customers that may have arrived wait in line. As soon as the server is repaired, if there are customers waiting in line, a new busy period starts immediately, otherwise the server takes a string of vacations following the MAV policy described above.

For any probability distribution F on $[0, \infty)$ with finite mean m we will denote the corresponding equilibrium (or integrated tail) distribution by $F_e(x) := m^{-1} \int_0^x [1 - F(y)]dy$ and will use the fact that its Laplace transform is given by $\hat{F}_e(s) = \frac{1-\hat{F}(s)}{sm}$.

We should point out that the assumption requiring the absolute continuity of the distribution functions of service, repair, and vacations times is necessary in order to provide a Markovian description of the system via the use of supplementary variables which leads to a system of ordinary differential equations for the occupation densities of states. However the final expressions obtained for the probability generating function of the stationary number of customers in the system and other quantities of interest are meaningful even when the aforementioned distribution functions are not absolutely continuous. Since the class of absolutely continuous distribution functions are dense within the space of probability distribution functions one could generalize standard results on the continuity of the dependence of the stationary distribution for the number of customers in the system, the workload etc. and establish rigorously that the results obtained under the absolute continuity assumption hold for arbitrary probability distribution functions.

3 Analysis via Supplementary Variables

Here we derive the steady-state differential-difference equations for the system by treating the elapsed service time, the elapsed repair time, and the elapsed vacation time as supplementary variables. From these we obtain the partial probability generating functions for the number of customers in the system and the state of the sever in stationarity. We will consider the processes

N_t : number of customers in the system at time t

$\mathcal{S}_t$: elapsed service time at time t (if server busy, otherwise 0)

$\mathcal{R}_t$: elapsed repair time at time t (if server on repair, otherwise 0)

$\mathcal{U}_t^j$: elapsed time of the jth vacation at time t (if server on jth vacation otherwise 0)

and the process $\{\xi_t\}$ taking values in the set $\{i, s, r\} \cup \mathcal{V}$ with $\mathcal{V} := \{u_1, u_2, u_3, \ldots\}$. The elements of $\{i, s, r\}$ correspond to the server being *idle*, *serving* customers, or being under *repair* respectively, while the elements of $\{u_1, u_2, \ldots\}$ correspond to the server being on the first, second, etc. vacation within a

string of vacations at the end of a busy cycle. Due to the presence of disasters the system has a regenerative structure with regeneration points the epochs when disasters occur and regenerative cycles of finite expected length (in fact equal to δ^{-1}). Therefore a stationary version of the process exists by virtue of standard results on regenerative processes (e.g. see [2]).

3.1 Balance equations

Suppose the process is stationary under the probability measure $\mathbb{P}$ and define the densities

$$\begin{aligned}
P_0 &:= \mathbb{P}(N_0 = 0, \xi_0 = i), \\
P_n(x) &:= \lim_{h \downarrow 0} \frac{1}{h} \mathbb{P}(N_0 = n, \xi_0 = s;\ x < \mathcal{S}_0 \leq x + h), \\
W_n(x) &:= \lim_{h \downarrow 0} \frac{1}{h} \mathbb{P}(N_0 = n, \xi_0 = r; x < \mathcal{R}_0 \leq x + h), \\
V_{j,n}(x) &:= \lim_{h \downarrow 0} \frac{1}{h} \mathbb{P}(N_0 = n, \xi_t = u_j; x < \mathcal{U}_0^j \leq x + h), \quad j = 1, 2, \ldots.
\end{aligned}$$

The balance equations satisfied in stationarity are

$$\lambda P_0 = \sum_{j=1}^{\infty} (1 - g_j) \int_0^{\infty} V_{j,0}(x) u(x) dx \ + \ (1 - g_0) \left(\int_0^{\infty} P_1(x) \mu(x) dx + \int_0^{\infty} W_0(x) r(x) dx \right) \quad (3.1)$$

$$\frac{d}{dx} P_n(x) + (\lambda + \delta + \mu(x))\, P_n(x) = \lambda \sum_{k=1}^{n-1} \chi_k P_{n-k}(x), \quad x > 0,\ n \geq 1 \quad (3.2)$$

$$\frac{d}{dx} W_0(x) + (\lambda + r(x)) W_0(x) = 0 \quad (3.3)$$

$$\frac{d}{dx} W_n(x) + (\lambda + r(x)) W_n(x) = \lambda \sum_{k=1}^{n} \chi_k W_{n-k}(x), \quad x > 0, n \geq 1 \quad (3.4)$$

$$\frac{d}{dx} V_{j,0}(x) + (\lambda + u(x)) V_{j,0}(x) = 0, \quad j = 1, 2, \ldots \quad (3.5)$$

$$\frac{d}{dx} V_{j,n}(x) + (\lambda + u(x)) V_{j,n}(x) = \lambda \sum_{k=1}^{n} \chi_k V_{j,n-k}(x), \quad x > 0,\ n \geq 1,\ j = 1, 2, \ldots \quad (3.6)$$

The boundary conditions of the above system of differential equations are

$$P_n(0) = \sum_{j=1}^{\infty} \int_0^{\infty} V_{j,n}(x)u(x)dx + \int_0^{\infty} P_{n+1}(x)\mu(x)dx + \int_0^{\infty} W_n(x)r(x)dx + \lambda\chi_n P_0, \quad n \geq 1 \tag{3.7}$$

$$V_{1,0}(0) = g_0 \int_0^{\infty} P_1(x)\mu(x)dx + g_0 \int_0^{\infty} W_0(x)r(x)dx \tag{3.8}$$

$$V_{j,0}(0) = g_{j-1} \int_0^{\infty} V_{j-1,0}(x)u(x)dx, \quad j = 2, 3, \ldots \tag{3.9}$$

$$W_0(0) = \delta \sum_{n=1}^{\infty} \int_0^{\infty} P_n(x)dx \tag{3.10}$$

with normalization condition

$$P_0 + \sum_{n=1}^{\infty} \int_0^{\infty} P_n(x)dx + \sum_{n=0}^{\infty} \left(\int_0^{\infty} W_n(x)dx + \sum_{j=1}^{\infty} \int_0^{\infty} V_{j,n}(x)dx \right) = 1. \tag{3.11}$$

3.2 Solution using partial probability generating functions

Define the partial probability generating functions

$$\begin{aligned}
P(x;z) &:= \lim_{h\downarrow 0} \frac{1}{h} \mathbb{E}[z^{N_0}; \xi_0 = s,\, x < \mathcal{S}_0 \leq x+h] \;=\; \sum_{n=1}^{\infty} z^n P_n(x), \\
W(x;z) &:= \lim_{h\downarrow 0} \frac{1}{h} \mathbb{E}[z^{N_0}; \xi_0 = r,\, x < \mathcal{R}_0 \leq x+h] \;=\; \sum_{n=0}^{\infty} z^n W_n(x), \\
V_j(x;z) &:= \lim_{h\downarrow 0} \frac{1}{h} \mathbb{E}[z^{N_0}; \xi_0 = u_j,\, x < \mathcal{U}_0^j \leq x+h] \;=\; \sum_{n=0}^{\infty} z^n V_{j,n}(x), \quad j = 1, 2, \ldots.
\end{aligned}$$

The partial pgf's for the number of customers in the system in stationarity regardless of the value of the supplementary variables are then given by

$$\begin{aligned}
P(z) &:= \mathbb{E}[z^{N_0}; \xi_0 = s] = \int_0^{\infty} P(x;z)dx, \qquad W(z) := \mathbb{E}[z^{N_0}; \xi_0 = r] = \int_0^{\infty} W(x;z)dx, \\
V_j(z) &:= \mathbb{E}[z^{N_0}; \xi_0 = u_j] = \int_0^{\infty} V_j(x;z)dx, \quad j = 1, 2, \ldots.
\end{aligned} \tag{3.12}$$

Proposition 1. *Let* $\alpha(z) := \lambda(1 - \chi(z))$, $\beta := \hat{U}(\lambda) \in (0,1)$, $C := \frac{1-\hat{F}(\beta)}{1-\beta} \frac{1}{\hat{F}(\beta)}$, *and*

$$K(z) := C\left(1 - \hat{U}(\alpha(z))\right) + 1 - \chi(z) = (1 - \chi(z))\left(1 + \lambda m_U C \hat{U}_e(\alpha(z))\right). \tag{3.13}$$

The partial pgf for the number of customers in the system when the server is busy is given by

$$P(z) \;=\; P(0;z)\, \frac{1 - \hat{S}(\delta + \alpha(z))}{\delta + \alpha(z)} \tag{3.14}$$

where

$$P(0;z) = z\,\frac{\lambda P_0 K(z) - \delta P(1)\hat{R}(\alpha(z))}{\hat{S}(\delta+\alpha(z)) - z}. \tag{3.15}$$

The expression for $P(0;z)$ given in (3.15) depends on the unknown values of P_0 and $P(1)$. These can be determined using Rouché's theorem and the normalization condition (3.11) as we shall see in the sequel.

Proof. From (3.2), (3.4), (3.6), and (3.7) we obtain the linear first order PDE's

$$\begin{aligned}
\frac{\partial}{\partial x}P(x;z) + (\alpha(z)+\delta+\mu(x))P(x;z) &= 0,\\
\frac{\partial}{\partial x}W(x;z) + (\alpha(z)+r(x))W(x;z) &= 0,\\
\frac{\partial}{\partial x}V_j(x;z) + (\alpha(z)+u(x))V_j(x;z) &= 0,
\end{aligned} \tag{3.16}$$

and the equation

$$\begin{aligned}
\sum_{n=1}^{\infty} z^n P_n(0) &= \sum_{j=1}^{\infty}\int_0^{\infty}\sum_{n=1}^{\infty} z^n V_{j,n}(x)u(x)dx + \int_0^{\infty}\sum_{n=1}^{\infty} z^n P_{n+1}(x)\mu(x)dx\\
&\quad + \int_0^{\infty}\sum_{n=1}^{\infty} z^n W_n(x)r(x)dx + \lambda\sum_{n=1}^{\infty}\chi_n z^n P_0
\end{aligned}$$

which, taking into account that $\sum_{n=1}^{\infty} V_{j,n}(x)z^n = V_j(x;z) - V_{j,0}(x)$, $\sum_{n=1}^{\infty} W_n(x)z^n = W(x;z) - W_0(x)$, and $\sum_{n=1}^{\infty} P_{n+1}(x)z^n = z^{-1}P(x;z) - P_1(x)$, gives

$$\begin{aligned}
P(0;z) &= \sum_{j=1}^{\infty}\int_0^{\infty} V_j(x;z)u(x)dx - \sum_{j=1}^{\infty}\int_0^{\infty} V_{j,0}(x)u(x)dx + z^{-1}\int_0^{\infty} P(x;z)\mu(x)dx\\
&\quad - \int_0^{\infty} P_1(x)\mu(x)dx + \int_0^{\infty} W(x;z)r(x)dx - \int_0^{\infty} W_0(x)r(x)dx + \lambda P_0\chi(z).
\end{aligned} \tag{3.17}$$

Solving (3.16) we obtain

$$\begin{aligned}
P(x;z) &= P(0;z)(1-S(x))e^{-(\delta+\alpha(z))x}\\
W(x;z) &= W(0;z)(1-R(x))e^{-\alpha(z)x}\\
V_j(x;z) &= V_j(0;z)(1-U(x))e^{-\alpha(z)x}, \quad j=1,2,\ldots.
\end{aligned} \tag{3.18}$$

The solution of (3.5) is

$$V_{j,0}(x) = V_{j,0}(0)(1-U(x))e^{-\lambda x}, \quad j=1,2,\ldots \tag{3.19}$$

whence we obtain

$$\int_0^\infty V_{j,0}(x)u(x)dx = V_{j,0}(0)\hat{U}(\lambda), \quad j = 1, 2, \ldots \tag{3.20}$$

since $\int_0^\infty (1 - U(x))e^{-\lambda x}u(x)dx = \int_0^\infty e^{-\lambda x}dU(x)$. We thus get

$$V_{j,0}(0) = g_{j-1}\beta V_{j-1,0}(0), \quad j = 2, 3, \ldots. \tag{3.21}$$

Note also that $V_j(0; z) = V_{j,0}(0)$ and $W(0; z) = W_0(0)$ because when the jth vacation begins, or when a repair begins, immediately after a disaster, there are necessarily no customers in the system.

Set $V_{j,0}(0) =: v_j$. Using (2.1), (3.21) gives

$$v_j = \frac{F_j}{F_1}\beta^{j-1}v_1, \quad j = 1, 2, \ldots.$$

Thus

$$\sum_{j=1}^\infty v_j = \sum_{j=1}^\infty \frac{F_j}{F_1}\beta^{j-1}v_1 = \frac{v_1}{F_1}\,\frac{1 - \hat{F}(\beta)}{1 - \beta}. \tag{3.22}$$

From (3.18) and (3.19) we obtain

$$\int_0^\infty V_j(x; z)u(x)dx = \int_0^\infty V_j(0; z)(1 - U(x))e^{-\alpha(z)x}u(x)dx \;=\; v_j\hat{U}(\alpha(z)),\; j = 1, 2, \ldots, \tag{3.23}$$

$$\int_0^\infty V_{j,0}(x)u(x)dx = \int_0^\infty V_{j,0}(0)(1 - U(x))e^{-\lambda x}u(x)dx \;=\; v_j\beta,\; j = 1, 2, \ldots, \tag{3.24}$$

$$\int_0^\infty P(x; z)\mu(x)dx = \int_0^\infty P(0; z)(1 - S(x))e^{-(\delta+\alpha(z))x}\mu(x)dx \;=\; P(0; z)\hat{S}(\delta + \alpha(z)), \tag{3.25}$$

$$\int_0^\infty W(x; z)r(x)dx = \int_0^\infty W(0; z)(1 - R(x))e^{-\alpha(z)x}r(x)dx \;=\; W(0; z)\hat{R}(\alpha(z)). \tag{3.26}$$

v_1 is obtained from (3.1), (3.8), and (3.9):

$$\lambda P_0 = \sum_{j=1}^\infty v_{j+1}\frac{1 - g_j}{g_j} + \frac{1 - g_0}{g_0}v_1.$$

Using (3.21) and the fact that $\frac{1-g_j}{g_j} = \frac{f_j}{F_{j+1}}$ we obtain

$$\lambda P_0 \;=\; \sum_{j=1}^\infty F_1^{-1}f_j\beta^j v_1 + v_1\frac{f_0}{F_1} \;=\; \frac{v_1}{F_1}\left(\hat{F}(\beta) - f_0\right) + v_1\frac{f_0}{F_1} \;=\; \frac{v_1}{F_1}\hat{F}(\beta). \tag{3.27}$$

The following argument clarifies the meaning of (3.27). When a busy period terminates by the departure of a customer leaving the system empty or when a repair period (following a disaster) terminates with no customers in the system, the server takes a string of vacations which may be *empty* (i.e. in fact the server does not take a vacation) with probability $f_0 = 1 - F_1$ or *non-empty* with probability F_1. The rate of non-empty vacation string initiations is v_1 and thus the rate of all vacation string initiations, (including empty strings) is v_1/F_1. Also, $\hat{F}(\beta) := \sum_{j=0}^{\infty} f_j \hat{U}(\lambda)^j$ is the probability that no arrivals occur during a vacation string, *empty or otherwise*. Thus $v_1\hat{F}(\beta)/F_1$ is the rate of entering the idle state of the server whereas λP_0 is the exit rate for the same state. Equation (3.27) expresses the fact that these two should be equal.

Using (3.23)–(3.26) in (3.17) we obtain

$$\begin{aligned} P(0;z) &= \sum_{j=1}^{\infty} v_j \hat{U}(\alpha(z)) - \sum_{j=1}^{\infty} v_j \beta + z^{-1} P(0;z)\hat{S}(\delta + \alpha(z)) + W(0;z)\hat{R}(\alpha(z)) \\ &\quad -v_1 + \sum_{j=1}^{\infty}(1-g_j)v_j\beta + \lambda P_0 \chi(z) - \lambda P_0. \end{aligned}$$

This, after some simplifications, using (3.22) and the fact that $\beta \sum_{j=1}^{\infty} g_j v_j = \sum_{j=2}^{\infty} v_j$, which is a consequence of (3.21), gives

$$\begin{aligned} P(0;z)\left(1 - z^{-1}\hat{S}(\delta + \alpha(z))\right) &= \lambda P_0 \left(\hat{U}(\alpha(z)) - 1\right) C + W(0;z)\hat{R}(\alpha(z)) \\ &\quad + \lambda P_0 \left(\chi(z) - 1\right). \end{aligned} \tag{3.28}$$

Using (3.18) we write (3.10) as

$$W_0(0) = W(0;z) = \delta \sum_{n=1}^{\infty} \int_0^{\infty} P_n(x)dx = \delta \int_0^{\infty} P(x,1)dx = \delta P(1). \tag{3.29}$$

(3.28) and (3.29) establish (3.15). (3.14) follows from (3.12) and (3.18). □

Proposition 2. *Let z_0 be the* unique *zero of the function $h(z) := \hat{S}(\delta + \alpha(z)) - z$ in the open unit disk $|z| < 1$. Then P_0 and $P(1)$ satisfy the relationship*

$$\delta P(1)\hat{R}(\alpha(z_0)) = \lambda P_0 K(z_0). \tag{3.30}$$

Remark 1. *Since $h(0) = \hat{S}(\delta + \lambda) > 0$ and $h(1) = \hat{S}(\delta) - 1 < 0$, z_0 will in fact be real which makes its numerical determination particularly simple.*

Remark 2. *Consider a system with Poisson arrivals (rate λ) in which customers arrive in batches with pgf $\chi(z)$. (This system operates with no vacations, no disasters, and no repairs.) Let Y denote the length the a busy period of this system starting with* a single customer *and denote by $\hat{B}_0(s) := \mathbb{E}[e^{-sY}]$ its Laplace transform. Note that, since we make no assumption regarding the stability of the system without vacations, Y will be a defective random variable with $\mathbb{P}(Y < \infty) < 1$ or, equivalently, $\hat{B}_0(0) < 1$, iff $\lambda m_S > 1$. In any case $\hat{B}_0(s)$ satisfies the equation $\hat{B}_0(s) = \hat{S}(s + \lambda - \lambda\chi(\hat{B}_0(s)))$ or,*

$$\hat{B}_0(s) = \hat{S}(s + \alpha(\hat{B}_0(s))). \tag{3.31}$$

Therefore, the root z_0 in Proposition 2 is precisely $\hat{B}_0(\delta)$.

Proof. The system is stable for all values of the parameters due to the presence of disasters. Hence the power series that defines $P(0; z)$ converges uniformly on the closed unit disk $|z| \leq 1$ and defines an analytic function there. Let $f(z) := -z$ and $g(z) := \hat{S}(\delta + \alpha(z))$ which are both analytic in $|z| \leq 1$. Then

$$|g(z)| \leq \int_0^\infty \mid e^{-(\delta+\alpha(z))x} \mid dS(x) = \int_0^\infty e^{-\delta x}\, e^{-\lambda x \Re(\alpha(z))} dS(x).$$

The real part of $\alpha(z)$ when $|z| = 1$, i.e. $z = e^{i\theta}$, $\theta \in [0, 2\pi)$, is

$$\Re\left(\lambda\left(1 - \sum_{k=1}^\infty \chi_k e^{ik\theta}\right)\right) = \lambda \sum_{k=1}^\infty \chi_k (1 - \cos k\theta) \geq 0, \qquad \theta \in [0, 2\pi)$$

and thus $|g(z)| \leq \int_0^\infty e^{-\delta x} dS(x) < 1$. It follows by Rouché's theorem that $f(z)$ and $f(z) + g(z)$ will have the same number of zeros inside $\mid z \mid < 1$. Since $f(z)$ has only one zero inside this circle, $h(z)$ also has a single zero inside $|z| < 1$, denoted as z_0. The numerator of $P(0; z)$ must therefore also vanish at z_0, otherwise $P(0; z)$ would have a singularity there, and this completes the proof. □

4 Marginal Probability Generating Functions

We are now ready to derive explicit expressions for the marginal pgf's for the number of customers in the system under stationarity according to the state of the server (busy, under repair, or on vacation). From these, the corresponding stationary probabilities can be obtained. To simplify the expressions we will set

$$\gamma := \frac{K(z_0)}{\hat{R}(\alpha(z_0))} = \frac{(1 - \chi(z_0))\left(1 + \lambda m_U C \hat{U}_e(\alpha(z_0))\right)}{\hat{R}(\alpha(z_0))}. \tag{4.1}$$

When there are no repairs, $\gamma = K(z_0)$. If, in addition, there are no vacations then $K(z) = 1 - \chi(z)$ and $\gamma = 1 - \chi(z_0) = 1 - \chi(B_0(\delta))$.

1. The partial pgf of the time stationary probabilities for the system size when the server is working.

$$P(z) = \lambda P_0 z \, \frac{K(z) - \gamma \hat{R}(\alpha(z))}{\hat{S}(\delta + \alpha(z)) - z} \, \frac{1 - \hat{S}(\delta + \alpha(z))}{\delta + \alpha(z)}. \tag{4.2}$$

where P_0 is given by (4.6). Note that $P(1) = \lambda P_0 \gamma / \delta$ is the stationary probability that the server is busy serving customers.

2. The partial pgf of the time stationary probabilities for the system size when the server is under repair. Taking into account (3.12), (3.18), and (3.30) we obtain

$$W(z) \;=\; \lambda P_0 \frac{K(z_0)}{\hat{R}(\alpha(z_0))} \frac{1 - \hat{R}(\alpha(z))}{\alpha(z)} = \lambda P_0 \gamma m_R \hat{R}_e(\alpha(z)). \tag{4.3}$$

Using de l' Hôpital's rule we obtain $W(1) = \lambda P_0 \gamma m_R$. This is the steady state probability that the server is under repair.

3. The partial pgf of the time stationary probabilities for the system size when the server is on the jth vacation.

$$\begin{aligned} V_j(z) \;&=\; \int_0^\infty V_j(x;z)dx \;=\; V_j(0;z) \int_0^\infty (1 - U(x)) e^{-\alpha(z)x} dx \\ &=\; V_j(0;z) \, \frac{1 - \hat{U}(\alpha(z))}{\alpha(z)} \;=\; v_1 \frac{F_j}{F_1} \beta^{j-1} \, m_U \hat{U}_e(\alpha(z)), \quad j = 1, 2, \ldots . \end{aligned}$$

Again, de l' Hôpital's rule gives $V_j(1) = v_j m_U$. Note that

$$V(1) = \sum_{j=1}^{\infty} V_j(1) = m_U \sum_{j=1}^{\infty} v_j = m_U \, \frac{v_1}{F_1} \, \frac{1 - \hat{F}(\beta)}{1 - \beta} = \lambda P_0 m_U C \; . \tag{4.4}$$

This is the probability that the server is on vacation. Also $V(z) := \sum_{j=1}^{\infty} V_j(z)$ is given by

$$V(z) = \frac{v_1}{F_1} \frac{1 - \hat{U}(\alpha(z))}{\alpha(z)} \sum_{j=1}^{\infty} F_j \beta^{j-1} = \lambda P_0 \frac{1 - \hat{F}(\beta)}{(1 - \beta)\hat{F}(\beta)} \frac{1 - \hat{U}(\alpha(z))}{\alpha(z)} = \lambda P_0 m_U C \hat{U}_e(\alpha(z)). \tag{4.5}$$

4. The stationary probability that the server is idle. P_0 can be determined by using the normalization condition $P_0 + P(1) + W(1) + \sum_{j=1}^{\infty} V_j(1) = 1$ which gives

$$P_0 = \left(1 \;+\; \frac{\lambda \gamma}{\delta} \;+\; \lambda \gamma m_R \;+\; \lambda m_U C \right)^{-1} . \tag{4.6}$$

5. The pgf of the number of customers in the system in stationarity. This is given by $\Phi(z) = P_0 + P(z) + W(z) + V(z)$. Thus

$$\begin{aligned}\Phi(z) &= P_0 + zP_0\lambda m_S\left(K(z) - \gamma\hat{R}(\alpha(z))\right)\frac{\hat{S}_e(\delta+\alpha(z))}{\hat{S}(\delta+\alpha(z)) - z} + \lambda\gamma P_0 m_R \hat{R}_e(\alpha(z)) \\ &\quad + \lambda P_0 m_U\, C\, \hat{U}_e(\alpha(z)) \end{aligned} \tag{4.7}$$

or equivalently

$$\Phi(z) = \lambda P_0 \frac{\gamma}{\alpha(z)} + \lambda P_0\left(K(z) - \gamma\hat{R}(\alpha(z))\right)\left(\frac{1}{\alpha(z)} + \frac{z}{\alpha(z)+\delta}\,\frac{1-\hat{S}(\delta+\alpha(z))}{\hat{S}(\delta+\alpha(z)) - z}\right). \tag{4.8}$$

5 Additional Performance Measures and Particular Cases

5.1 Rate of disasters and number of customers removed by them

The rate at which disasters occur is equal to the rate of repair initiations and is given by

$$d := \delta P(1) = W_0(0) = \lambda P_0 \gamma. \tag{5.1}$$

The above is therefore the rate of busy period terminations due to disasters. We next determine the rate of "normal" busy period terminations, i.e. those due to service completions which leave the system empty. This rate is clearly $\int_0^\infty P_1(x)\mu(x)dx$. Equation (3.3) gives $W_0(x) = W_0(0)e^{-\lambda x}(1 - R(x))$ whence we obtain $\int_0^\infty W_0(x)r(x)dx = W_0(0)\hat{R}(\lambda)$ and thus, recalling that $g_0 = F_1$ and $V_{1,0}(0) = v_1$, (3.8) gives

$$\int_0^\infty P_1(x)\mu(x)dx = \frac{v_1}{F_1} - W_0(0)\hat{R}(\lambda) = \frac{v_1}{F_1} - \delta P(1)\hat{R}(\lambda) = \frac{v_1}{F_1} - \lambda P_0\gamma\hat{R}(\lambda). \tag{5.2}$$

Thus, the rate of busy period terminations due to the departure of a customer (as opposed to those terminated by disasters) is given by

$$r_b := \int_0^\infty P_1(x)\mu(x)dx = \lambda P_0\left(\frac{1}{\hat{F}(\beta)} - \gamma\hat{R}(\lambda)\right). \tag{5.3}$$

where, in the above equalities we have also used (5.1) and (3.27). Finally, the rate of initiations of busy periods, b, is given by

$$b = r_b + d = \lambda P_0\left(\frac{1}{\hat{F}(\beta)} - \gamma(1 - \hat{R}(\lambda))\right). \tag{5.4}$$

(Indeed, each busy period always follows a previous one that was either terminated by the departure of a customer, or by the occurrence of a disaster.)

The number of customers removed from the system when a disaster occurs has pgf

$$\Phi_d(z) = P(z)/P(1). \tag{5.5}$$

A rigorous proof of this fact can be obtained using Papangelou's theorem (see [3]). Indeed, the disaster point process has stochastic intensity $\delta \mathbf{1}(\xi_t = s)$ at time t, with respect to the history of the process up to time t. The pgf of the number of customers removed by a disaster is the expectation of z^{N_0-} under the Palm transformation of the stationary probability measure $\mathbb{P}$ with respect to the disaster point process. (In intuitive terms the number of customers in the system just prior to a "typical disaster".) Thus, by Papangelou's theorem, $\Phi_d(z) = \mathbb{E}[z^{N_0-}\mathbf{1}(\xi = s)\delta]/\mathbb{E}[\mathbf{1}(\xi_0 = s)\delta]$, which reduces to the right hand side of (5.5). From (3.14) and (3.15) we have

$$\Phi_d(z) = z\frac{\delta}{\gamma}\frac{K(z) - \gamma\hat{R}(\alpha(z))}{\hat{S}(\delta + \alpha(z)) - z}\,\frac{1 - \hat{S}(\delta + \alpha(z))}{\delta + \alpha(z)}. \tag{5.6}$$

The proportion of customers that complete their service (as opposed to being eliminated from the system as a result of a disaster) is obtained from a ratio of rates argument as follows: The rate of customer departures due to service completion, r, is

$$r = \sum_{n=1}^{\infty}\int_0^{\infty} P_n(x)\mu(x)dx = \int_0^{\infty} P(x;1)\mu(x)dx = P(0;1)\hat{S}(\delta) = \lambda P_0\gamma\frac{\hat{S}(\delta)}{1 - \hat{S}(\delta)}. \tag{5.7}$$

Thus, the fraction of customers who complete their service is

$$\frac{r}{\lambda m_\chi} = P_0\frac{\gamma}{m_\chi}\frac{\hat{S}(\delta)}{1 - \hat{S}(\delta)}. \tag{5.8}$$

The rate of departures due to disasters is $\lambda m_\chi - r = \lambda\left(m_\chi - P_0\gamma\frac{\hat{S}(\delta)}{1-\hat{S}(\delta)}\right)$. Hence, the average number of customers removed from the server by a disaster is

$$\frac{\lambda m_\chi}{d} = \frac{1}{\lambda P_0\gamma}\lambda\left(m_\chi - P_0\gamma\frac{\hat{S}(\delta)}{1 - \hat{S}(\delta)}\right) = \frac{m_\chi}{\gamma P_0} - \frac{\hat{S}(\delta)}{1 - \hat{S}(\delta)}.$$

This can also be obtained of course from (5.6).

5.2 Mean duration of a vacation period

The rate of non-empty vacation string initiations is given by v_1. If we denote by m_V the mean duration of a vacation string then (essentially by Little's law) $v_1 m_V = V(1)$, the stationary probability that the system is on vacation. This, taking into account (4.4) yields

$$m_V = \frac{m_U}{F_1} \frac{1 - \hat{F}(\beta)}{1 - \beta}.$$

(The same result could be also obtained by a renewal theoretical argument.) The above refers to nonempty vacation strings. If we denote by m_V^0 the mean length of a vacation string including empty vacation strings (of mean duration zero) then $m_V^0 = m_U \frac{1-\hat{F}(\beta)}{1-\beta}$.

5.3 The pgf of the system size at a departure epoch

A departing customer will leave behind l customers in the system at a departure epoch if and only if there are $l+1$ customers in the system just before the departure. Thus, if ϕ_l^+ denotes the probability that a departing customer leaves behind l customers in the system from a stochastic intensity argument we obtain

$$\phi_l^+ = D \int_0^\infty \mu(x) P_{l+1}(x) dx$$

where D is a normalizing constant. If

$$\Phi^+(z) := \sum_{l=0}^{\infty} \phi_l^+ z^l$$

denotes the corresponding pgf, taking into account that $\sum_{l=0}^{\infty} z^l P_{l+1}(x) = z^{-1} P(x; z)$, (3.25), and (3.15) we obtain

$$\Phi^+(z) = D \left(K(z) - \gamma \hat{R}(\alpha(z)) \right) \frac{\hat{S}(\delta + \alpha(z))}{\hat{S}(\delta + \alpha(z)) - z} \tag{5.9}$$

with the normalizing constant D determined as $D = \frac{1-\hat{S}(\delta)}{\gamma \hat{S}(\delta)}$. Thus,

$$\Phi^+(z) = \left(\hat{R}(\alpha(z)) - \gamma^{-1} K(z) \right) \frac{\hat{S}(\delta + \alpha(z))}{\hat{S}(\delta)} \frac{1 - \hat{S}(\delta)}{z - \hat{S}(\delta + \alpha(z))} \ . \tag{5.10}$$

Let us now examine what happens when a) repair times are negligible and b) when disasters do not occur. In the first case $\hat{R}(s) \equiv 1$ and hence from (5.10) and (4.1) we obtain

$$\Phi^+(z) = \left(1 - \frac{K(z)}{K(z_0)} \right) \frac{\hat{S}(\delta + \alpha(z))}{\hat{S}(\delta)} \frac{1 - \hat{S}(\delta)}{z - \hat{S}(\delta + \alpha(z))} \ .$$

In an ordinary $M/G/1$ system (with no batches or vacations) $K(z) = 1 - z$ and this gives

$$\Phi^+(z) = \frac{z - z_0}{1 - z_0} \frac{\hat{S}(\delta + \lambda - \lambda z)}{\hat{S}(\delta)} \frac{1 - \hat{S}(\delta)}{z - \hat{S}(\delta + \lambda - \lambda z)} \ .$$

When, in addition, disasters do not occur ($\delta = 0$) then $z_0 = 1$ and it is more convenient to appeal directly to (5.9), (with D determined by the requirement that $\Phi^+(1) = 1$ using de l'Hôpital's rule). Then

$$\Phi^+(z) = DK(z) \frac{\hat{S}(\alpha(z))}{\hat{S}(\alpha(z)) - z}$$

and, taking into account that $K(1) = 0$, $K'(1) = -m_\chi(1 + C\lambda m_U)$, we obtain $D = \frac{1-\lambda m_S m_\chi}{m_\chi(1+C\lambda m_U)}$ and

$$\Phi^+(z) = \frac{1 - \lambda m_S m_\chi}{m_\chi(1 + C\lambda m_U)} \frac{K(z)\hat{S}(\alpha(z))}{\hat{S}(\alpha(z)) - z} \ .$$

5.4 Special Cases

The system without repairs. If the repair period after the occurrence of a disaster has negligible duration then we may set $m_R = 0$ and $\hat{R}(s) \equiv 1$ to obtain

$$\Phi(z) = P_0 + z\lambda P_0 m_S \left(K(z) - K(z_0)\right) \frac{\hat{S}_e(\delta + \alpha(z))}{\hat{S}(\delta + \alpha(z)) - z} + \lambda P_0 m_U \, C \, \hat{U}_e(\alpha(z)) \ . \tag{5.11}$$

The system with no vacations and no repairs. The pgf of the stationary distribution is obtained from (4.7) by setting $\hat{U}(s) = 1$ and $\hat{R}(s) = 1$ for all s (and consequently $m_U = m_R = 0$). Then $K(z) = \alpha(z)$ and (4.7) becomes

$$\Phi(z) = P_0 + zP_0 m_S \left(\alpha(z) - \alpha(z_0)\right) \frac{\hat{S}_e(\delta + \alpha(z))}{\hat{S}(\delta + \alpha(z)) - z} \tag{5.12}$$

where z_0 is given by Proposition 1 and $P_0 = \left(1 + \delta^{-1}\alpha(z_0)\right)^{-1}$. Equivalently, we have the expression

$$\Phi(z) = \frac{\delta}{\delta + \alpha(z)} \left[1 + \frac{(1 - z)\hat{S}(\delta + \alpha(z))}{z - \hat{S}(\delta + \alpha(z))} \frac{\lambda(\chi(z_0) - \chi(z))}{\delta + \alpha(z_0)}\right] . \tag{5.13}$$

The $M/M/1$ queue with disasters (no repairs or vacations). Here we set $\hat{S}(s) = \mu/(\mu + s)$ and $\alpha(z) = \lambda(1 - z)$. Also let $\rho = \lambda/\mu$. In this case $\gamma = 1 - z_0$ and $P_0 = 1 - \rho z_0$. (3.31) becomes in this case the equation

$$\lambda z^2 - (\lambda + \mu + \delta)z + \mu = 0 \tag{5.14}$$

and $z_0 = \frac{\lambda+\mu+\delta-\sqrt{(\lambda+\mu+\delta)^2-4\lambda\mu}}{2\lambda}$. With the appropriate substitutions and simplifications (5.12) gives the stationary number of customers in the system as

$$\Phi(z) = \frac{1 - \rho z_0}{1 - \rho z_0 z}, \tag{5.15}$$

a well known result. The rate of disasters is $\lambda P_0 \gamma = \lambda(1-z_0)(1-\rho z_0) = \delta\rho z_0$. The fraction of customers who complete their service is $(1 - \rho z_0)(1 - z_0)\frac{\frac{\mu}{\mu+\delta}}{1-\frac{\mu}{\mu+\delta}} = (1 - \rho z_0)(1 - z_0)\frac{\mu}{\delta} = z_0$. The average number of customers removed by each disaster is

$$\frac{\lambda(1 - z_0)}{\delta\rho z_0} = \frac{1}{1 - \rho z_0}.$$

Bernoulli Vacations Here $g_i = q$, $i = 0, 1, 2, \ldots$ where $q \in (0, 1)$. Set $p = 1 - q$. Then $\hat{F}(z) = \frac{p}{1-qz}$ and $C = p/q$. In this case C is simply the mean number of vacations taken.

The system with non-terminating vacations. This case also falls under the framework we consider by assuming that the distribution $\{f_j\}$ is defective with $f_j = 0$, $F_j = \mathbb{P}(\zeta \geq j) = 1$, and hence $g_j = 1$, for $j = 1, 2, \ldots$ and $\hat{F}(z) = 0$ for $z \neq 0$. Clearly, in this case the server is never idly waiting for a customer to serve. The rate equations (3.1), (3.8), (3.9) are still valid but with $P_0 = 0$. (3.15) can then be stated as

$$P(0; z) = z\,\frac{(1 - \hat{U}(\alpha(z)))v_1\frac{1}{1-\beta} - \delta P(1)\hat{R}(\alpha(z))}{\hat{S}(\delta + \alpha(z)) - z}.$$

The argument based on Rouché's Theorem gives $\delta P(1) = \frac{1-\hat{U}(\alpha(z_0))}{\hat{R}(\alpha(z_0))}\frac{v_1}{1-\beta}$ and upon substituting in the equation above we obtain

$$Pa(0; z) = z\,\frac{v_1}{1 - \beta}\,\frac{1 - \hat{U}(\alpha(z)) - \frac{\hat{R}(\alpha(z))}{\hat{R}(\alpha(z_0))}(1 - \hat{U}(\alpha(z_0)))}{\hat{S}(\delta + \alpha(z)) - z}.$$

Equations (3.18) are of course still valid. Also,

$$W(z) = \delta P(1)\frac{1 - \hat{R}(\alpha(z))}{\alpha(z)} = \frac{1 - \hat{U}(\alpha(z_0))}{\hat{R}(\alpha(z_0))}\frac{v_1}{1 - \beta}m_R\,\hat{R}_e(\alpha(z))$$

and

$$V(z) = v_1 m_U \frac{1}{1-\beta} \hat{U}_e(\alpha(z)).$$

Setting $\widetilde{\gamma} := \frac{1-\hat{U}(\alpha(z_0))}{\hat{R}(\alpha(z_0))}$, we obtain the following expression for the pgf of the number of customers in the system in stationarity:

$$\Phi(z) = \frac{v_1}{1-\beta}\left[m_S\left(1-\hat{U}(\alpha(z))-\widetilde{\gamma}\hat{R}(\alpha(z))\right)\frac{z\hat{S}_e(\delta+\alpha(z))}{\hat{S}(\delta+\alpha(z))-z}+\widetilde{\gamma}m_R\hat{R}_e(\alpha(z))+m_U\hat{U}_e(\alpha(z))\right].$$

The value of v_1 is easily determined by the requirement that $\Phi(1) = 1$. Thus

$$\frac{v_1}{1-\beta} = \left(\widetilde{\gamma}\delta^{-1} + \gamma m_R + m_U\right)^{-1}.$$

6 The Busy Period

6.1 The pgf of the Number in the System at a Busy Period Initiation Epoch

Denote by ψ_n the probability that the typical busy period in stationarity starts with n customers present and let $\{t_l\}$, $l = 1, 2, \ldots$ be the initiation epochs of the busy periods. The system size process, $\{N_t; t \geq 0\}$, (number of customers in the system) is defined as a process with right–continuous paths a.s.. Then N_{t_l} is the number of customers in the system at the initiation epoch of the lth busy period and

$$\psi_n := \lim_{m\to\infty} \frac{1}{m}\sum_{l=1}^{m} \mathbf{1}(N_{t_l} = n), \quad n = 1, 2, \ldots.$$

Denote by $\Psi(z) := \sum_{n=1}^{\infty} \psi_n z^n$ the corresponding pgf.

Proposition 3. *The number of customers present at the initiation of the typical busy period has pgf given by*

$$\Psi(z) = \frac{\Theta(z) - \Theta(0)}{\Theta(1) - \Theta(0)} \tag{6.1}$$

with $\Theta(z) = K(z) - \gamma\hat{R}(\alpha(z))$.

Proof. Denote by $b_n := \lim_{t\to\infty} \frac{1}{t}\sum_{l=1}^{\infty} \mathbf{1}(t_l \leq t; N_{t_l} = n)$ the rate of busy period initiations that start with n customers present. The rate of all busy period initiations, $b := \lim_{t\to\infty} \frac{1}{t}\sum_{l=1}^{\infty} \mathbf{1}(t_l \leq t) = \sum_{n=1}^{\infty} b_n$ has already been obtained in (5.4). A ratio of rates argument gives

$$\psi_n = \frac{b_n}{b}, \quad n = 1, 2, \ldots,$$

and thus

$$\Psi(z) = \frac{1}{b}\sum_{n=1}^{\infty} b_n z^n \tag{6.2}$$

In view of the analysis of §3

$$b_n \;=\; \lambda P_0 \chi_n + \int_0^\infty W_n(x) r(x) dx + \sum_{j=1}^{\infty}\int_0^\infty V_{j,n}(x) u(x) dx, \quad n = 1, 2, \ldots \tag{6.3}$$

Taking into account (6.3) and (3.7) the infinite sum in (6.2) is equal to

$$\begin{aligned}\sum_{n=1}^{\infty} z^n \left(\lambda P_0 \chi_n + \int_0^\infty W_n(x) r(x) dx + \sum_{j=1}^{\infty}\int_0^\infty V_{j,n}(x) u(x) dx\right) \\ = \sum_{n=1}^{\infty} z^n P_n(0) - \sum_{n=1}^{\infty} z^n \int_0^\infty P_{n+1}(x)\mu(x) dx \\ = \; P(0;z)\left(1 - z^{-1}\hat{S}(\delta + \alpha(z))\right) + \int_0^\infty P_1(x)\mu(x) dx \end{aligned} \tag{6.4}$$

whereas b can be obtained by setting $z = 1$ in (6.4). Hence

$$\Psi(z) = \frac{P(0;z)\left(1 - z^{-1}\hat{S}(\delta + \alpha(z))\right) + \int_0^\infty P_1(x)\mu(x) dx}{P(0;1)\left(1 - \hat{S}(\delta)\right) + \int_0^\infty P_1(x)\mu(x) dx}\,. \tag{6.5}$$

Equation (3.3) gives $W_0(x) = W_0(0)e^{-\lambda x}(1 - R(x))$ whence we obtain $\int_0^\infty W_0(x) r(x) dx = W_0(0)\hat{R}(\lambda)$ and thus, recalling that $g_0 = F_1$, (3.8) gives

$$\int_0^\infty P_1(x)\mu(x) dx = \frac{v_1}{F_1} - W_0(0)\hat{R}(\lambda) = \frac{v_1}{F_1} - \delta P(1)\hat{R}(\lambda) = \frac{v_1}{F_1} - \lambda P_0 \gamma \hat{R}(\lambda), \tag{6.6}$$

where, in the above equalities we have also used (3.29), (3.30), and (4.1). Also, using (3.15), the numerator of (6.5) is written as

$$\sum_{n=1}^{\infty} b_n z^n = P(0;z)\left(1 - z^{-1}\hat{S}(\delta + \alpha(z))\right) = \delta P(1)\hat{R}(\alpha(z)) - \lambda P_0 K(z) = \lambda P_0\left(\gamma\hat{R}(\alpha(z)) - K(z)\right)$$

whereas the denominator of (6.5) becomes (using again (3.15) evaluated at $z = 1$ and (6.6))

$$b = \lambda P_0 \gamma + \frac{v_1}{F_1} - \lambda P_0 \gamma = \lambda P_0\left(\frac{1}{\hat{F}(\beta)} + \gamma(1 - \hat{R}(\lambda)\right).$$

In this last equation we have used the fact that $\frac{v_1}{F_1} = \lambda P_0 \hat{F}(\beta)^{-1}$ which follows from (3.27. Thus, substituting into (6.5), we obtain the pgf of the number present at a busy period initiation epoch as

$$\Psi(z) \;=\; \frac{\gamma\left(\hat{R}(\alpha(z)) - \hat{R}(\lambda)\right) - K(z) + \hat{F}(\beta)^{-1}}{\gamma\left(1 - \hat{R}(\lambda)\right) + \hat{F}(\beta)^{-1}} \;=\; \frac{\Theta(z) - \Theta(0)}{\Theta(1) - \Theta(0)}. \tag{6.7}$$

Here $\Theta(z) = K(z) - \gamma\hat{R}(\alpha(z))$ and $\Theta(0) = \hat{F}(\beta)^{-1}$. □

6.2 The Laplace Transform of the Length of the Typical Busy Period

The expression obtained in the previous subsection for the pgf of the number of customers in the system at the initiation epoch of a busy period allows us to determine easily the Laplace transform of the length of the typical busy period. This will be obtained in terms of the solution of equation (3.31) which gives the Laplace transform of the length of a busy period starting with single customer, in an $M^X/G/1$ queue without disasters. We begin with the following

Remark 3. *Let X, Δ, be independent random variables. If $\hat{F}(s) := \mathbb{E}[e^{-sX}]$ and Δ is exponentially distributed with rate δ then*

$$\mathbb{E}[e^{-s(X\wedge\Delta)}|X<\Delta] = \frac{\hat{F}(s+\delta)}{\hat{F}(\delta)}, \qquad \mathbb{E}[e^{-s(X\wedge\Delta)}|\Delta<X] = \frac{\delta}{\delta+s}\,\frac{1-\hat{F}(s+\delta)}{1-\hat{F}(\delta)},$$
$$\mathbb{E}[e^{-s(X\wedge\Delta)}] = \frac{\delta}{\delta+s} + \frac{s}{\delta+s}\hat{F}(s+\delta).$$

Theorem 4. *The Laplace transform of the length of a typical busy period, $\hat{B}(s)$, in the system we examine is given by*

$$\hat{B}(s) = \frac{\delta}{\delta+s} + \frac{s}{s+\delta}\hat{\Gamma}(s+\delta) \tag{6.8}$$

where $\hat{\Gamma}(s) = \Psi\left(\hat{B}_0(s)\right)$, $\hat{B}_0(s)$ is given by (3.31), and $\Psi(z)$ by (6.7).

Note that the distribution of B is always proper ($\hat{B}(0) = 1$), even when B_0 is defective.

Proof. At an initiation epoch of a typical busy period the distribution of the number of customers present has pgf given by (6.7). Suppose that, at this point the disaster mechanism is "shut off". Let Γ denote the resulting busy period length with corresponding Laplace transform given by

$$\hat{\Gamma}(s) = \sum_{k=1}^{\infty} \hat{B}_0(s)^k\,\psi_k = \Psi(\hat{B}_0(s)).$$

(Γ is a defective random variable if the system without disasters is not stable.) If Δ is an independent exponential random variable with rate δ, the length of the busy period of the actual system, B, has Laplace transform given by $\hat{B}(s) = \mathbb{E}[e^{-s(\Gamma\wedge\Delta)}]$. An appeal to Remark 3 completes the proof. □

6.3 Structure of Busy Periods

We can describe the structure of the sample paths of the system by distinguishing between busy periods (when the server is in state s) and *inactive periods* when the server is in states r, $\mathcal{V}$, or i. Further, we distinguish between busy periods that end normally, i.e. by the departure of a customer, which we shall term of type 0 and busy periods ending by a disaster, which we shall term of type 1. A busy period of type 0 gives rise to an inactive period of type 0, consisting of a (possibly empty) string of vacations, plus an idle period, if there where no arrivals during the vacation string. A busy period of type 1 gives rise to an inactive period of type 1, consisting of a repair period and then, in the absence of arrivals during this repair time, a vacation string and possibly an idle period as in type 0. Thus, each busy period, together with the ensuing inactive period, comprises a busy cycle, either of type 0 or of type 1.

Let ζ_n, $n = 1, 2, \ldots$, be a sequence of random variables taking values in the set $\{0, 1\}$ where $\zeta_n = 0$ if the nth busy cycle is of type 0 and $\zeta_n = 1$ if it is of type 1. We will show that $\{\zeta_n\}$, $n \in \mathbb{N}$ is a time-homogeneous Markov chain and we will obtain its transition probability matrix. Consider first the following

Remark 4. *The pgf of the number of arrivals during a vacation period* conditional on the event that there is at least one arrival *is*

$$\frac{\hat{U}(\lambda(1-\chi(z))) - \hat{U}(\lambda)}{1-\hat{U}(\lambda)}. \tag{6.9}$$

The distribution of the length of the busy period depends on the number of customers at initiation as well as on whether it ends naturally or as the result of a disaster. Let us first focus on the former. If the preceding busy period ended naturally then a string of vacations follows. The probability that during a vacation string (empty or not) there are no arrivals is $\hat{F}(\beta)$ and in this case an idle period intervenes and then the new busy period starts with the arrival of a batch with pdf $\chi(z)$. On the other hand, during a vacation period in which there were arrivals, the pgf of the number of customers that arrived is given by (6.9) Hence, when a busy period ends naturally, i.e. without a disaster occurring, the number of customers at the initiation of the next busy period is a random variable with pgf $\phi_0(z)$ given by

$$\begin{aligned}\phi_0(z) &:= \hat{F}(\beta)\chi(z) + (1-\hat{F}(\beta))\tfrac{\hat{U}(\lambda(1-\chi(z)))-\hat{U}(\lambda)}{1-\hat{U}(\lambda)} = 1-\hat{F}(\beta)\left(1-\chi(z)\right)\Big(1+\lambda m_U \hat{U}_e(\alpha(z))C\Big) \\ &= 1-\hat{F}(\beta)K(z). \end{aligned} \tag{6.10}$$

When a busy period ends as a result of a disaster there follows a repair period. If during this repair period there are no arrivals, an event that happens with probability $\hat{R}(\lambda)$, it will be followed by a a string of vacations and hence the ensuing busy period will start with a number of customers with pgf $\phi_0(z)$. On the other hand, conditional on there being arrivals during the repair period, the pgf of their number is given by $\frac{\hat{R}(\lambda-\lambda\chi(z))-\hat{R}(\lambda)}{1-\hat{R}(\lambda)}$. Thus the ensuing busy period starts with a random number of customers with pgf

$$\phi_1(z) \;=\; \hat{R}(\lambda)\phi_0(z) + \left(1-\hat{R}(\lambda)\right)\frac{\hat{R}(\lambda(1-\chi(z)))-\hat{R}(\lambda)}{1-\hat{R}(\lambda)} \;=\; \hat{R}(\alpha(z)) - \hat{F}(\beta)\hat{R}(\lambda)K(z). \quad (6.11)$$

In the absence of disasters, the Laplace transform of the length of a busy period starting with a random number of customers with pgf ϕ is $\phi(\hat{B}_0(s))$ where, as before, $\hat{B}_0(s)$ denotes the Laplace transform of the length of a busy period starting with a single customer in a system without disasters. Thus a busy period starting with pgf $\phi_0(z)$ ends normally, without a disaster occurring, with probability $\phi_0(B_0(\delta))$. The probability that a busy period ending normally is followed by another busy period ending normally is $\phi_0(B_0(\delta)) = \phi_0(z_0)$ after taking into account Remark 2. Thus

$$\mathbb{P}(\zeta_{n+1}=0|\zeta_n=0) = \phi_0(z_0) = 1-\hat{F}(\beta)K(z_0). \quad (6.12)$$

Arguing similarly, the probability that a busy period ending with a disaster is followed by a busy period ending normally is

$$\phi_1(z_0) = \mathbb{P}(\zeta_{n+1}=0|\zeta_n=1) = \hat{R}(\alpha(z_0))\left(1-\gamma\hat{R}(\lambda)\hat{F}(\beta)\right). \quad (6.13)$$

Consider then a two-state Markov chain $\{\zeta_n\}$, $n\in\mathbb{N}$ with state space $\{0,1\}$ where $\zeta_n=0$ if the nth busy period ends normally and $\zeta_n=1$ if it ends as a result of a disaster. The transition probability matrix of this chain is

$$\begin{bmatrix} \phi_0(z_0) & 1-\phi_0(z_0) \\ \phi_1(z_0) & 1-\phi_1(z_0) \end{bmatrix} = \begin{bmatrix} 1-\hat{F}(\beta)K(z_0) & \hat{F}(\beta)K(z_0) \\ \hat{R}(\alpha(z_0))(1-\gamma\hat{R}(\lambda)\hat{F}(\beta)) & 1-\hat{R}(\alpha(z_0))(1-\gamma\hat{R}(\lambda)\hat{F}(\beta)) \end{bmatrix}.$$

The stationary distribution of the above chain is

$$\pi_0 = \frac{\frac{1}{\hat{F}(\beta)}-\gamma\hat{R}(\lambda)}{\frac{1}{\hat{F}(\beta)}+\gamma(1-\hat{R}(\lambda))} = \frac{r}{b}, \qquad \pi_1 = \frac{\gamma}{\frac{1}{\hat{F}(\beta)}+\gamma(1-\hat{R}(\lambda))} = \frac{d}{b}. \quad (6.14)$$

Of course, comparing with (6.7), we see that $\Psi(z) = \pi_0\phi_0(z) + \pi_1\phi_1(0)$.

We can distinguish four types of busy periods. We denote them by two subscripts, the first (0 or 1) according to whether the preceding busy period terminates normally or as a result of a disaster and the second according to whether the busy period itself terminates normally or as a result of a disaster. The Laplace transforms of their lengths are $\hat{B}_{00}(s) = \frac{\phi_0(\hat{B}_0(s+\delta))}{\phi_0(\hat{B}_0(\delta))}$, $\hat{B}_{10}(s) = \frac{\phi_1(\hat{B}_0(s+\delta))}{\phi_1(\hat{B}_0(\delta))}$, $\hat{B}_{01}(s) = \frac{1-\phi_0(\hat{B}_0(s+\delta))}{1-\phi_0(\hat{B}_0(\delta))}\frac{\delta}{\delta+s}$, and $\hat{B}_{11}(s) = \frac{1-\phi_1(\hat{B}_0(s+\delta))}{1-\phi_1(\hat{B}_0(\delta))}\frac{\delta}{\delta+s}$.

7 Stationary Workload and System Times

Let Ω denote the stationary workload in the system and and T the time a typical customer arriving in stationarity spends in the system regardless of whether he eventually departs as a result of a disaster or upon completing his service. Similarly let T_d denote the system time of a typical customer arriving in stationarity given that his departure is caused by a disaster and T_s that of a typical customer, given that he completes his service. $\hat{T}(s)$, $\hat{T}_d(s)$, and $\hat{T}_s(s)$ denote the corresponding Laplace transforms.

Theorem 5. *The Laplace transform of a typical customer's system time arriving in stationarity and who does not suffer a disaster is given by*

$$\hat{T}_s(s) = \frac{1}{\gamma}\left(\hat{\kappa}(s) - \gamma\hat{R}(s)\right)\frac{\alpha(\hat{S}(s+\delta))}{s-\alpha(\hat{S}(\delta+s))}\frac{1-\hat{S}(\delta)}{1-\hat{S}(s+\delta)}\frac{\hat{S}(\delta+s)}{\hat{S}(\delta)}. \tag{7.1}$$

Similarly, in stationarity, the Laplace transform of a the system time of a customer who is removed by a disaster is

$$\hat{T}_d(s) \;=\; \lambda P_0 \frac{\frac{\gamma}{s} + \frac{\hat{\kappa}(s)-\gamma\hat{R}(s)}{\delta+s}\left(\frac{\delta}{s} - \delta\frac{\chi_e(\hat{S}(s+\delta))\hat{S}(s+\delta)}{s-\alpha(\hat{S}(\delta+s))}\right)}{1 - P_0\gamma\frac{\hat{S}(\delta)}{m_\chi(1-\hat{S}(\delta))}}. \tag{7.2}$$

and finally the Laplace transform of the system time of a customer in stationarity, regardless of whether he completes service or not is

$$\hat{T}(s) \;=\; \lambda P_0\left[\frac{\gamma}{s} + \frac{\hat{\kappa}(s)-\gamma\hat{R}(s)}{\delta+s}\left(\frac{\delta}{s} + s\frac{\chi_e(\hat{S}(s+\delta))\hat{S}(s+\delta)}{s-\alpha(\hat{S}(s+\delta))}\right)\right]. \tag{7.3}$$

In the above P_0 is given by (4.6) and

$$\hat{\kappa}(s) := \frac{s}{\lambda}\left(1 + \lambda m_U C\hat{U}_e(s)\right). \tag{7.4}$$

The Laplace transform of the stationary workload is given by

$$\begin{aligned}\hat{\Omega}(s) &= P_0 \frac{\lambda\gamma + \delta - s}{\alpha(\hat{S}(s)) + \delta - s} \\ &\quad + P_0 \frac{\delta - s}{\alpha(\hat{S}(s)) + \delta - s} \left(\lambda\gamma m_R \hat{R}_e(\alpha(\hat{S}(s))) + \lambda m_U C \hat{U}_e(\alpha(\hat{S}(s))) \right).\end{aligned} \tag{7.5}$$

Proof. Let $\bar{\mathcal{S}}_t$ denote the *residual* service time of the customer in service at time t, when $\xi_t = s$ (i.e. when the server is busy) otherwise set $\bar{\mathcal{S}}_t = 0$. Similarly let $\bar{\mathcal{R}}_t$, and $\bar{\mathcal{U}}_t^j$, $j = 1, 2, \ldots$, denote the *residual life* of the repair process, or vacation time at time t (provided again the server is under repair or on vacation). Also, denote by Q_t the number of customers in the system at time t, *excluding any that may be receiving service at that time*.

Suppose that at time 0 the system is in the stationary regime and a batch containing a tagged customer arrives. We proceed in two steps. *In the first step*, in order to avoid the complications that arise from the possibility of disasters, we imagine that at the arrival instant the disaster mechanism shuts off and disasters no longer occur in the system. We first determine the Laplace transform of the sojourn time of the tagged customer under these circumstances. The system time of the tagged customer consists of four parts, namely:

1) A delay due to the residual service time of the customer that may be in service at time 0, which is equal to $\tau_1 := \mathbf{1}(\xi_0 = s)\bar{\mathcal{S}}_0$.

2) A delay that ensues if the server happens to be on vacation or under repair a time 0. This delay is equal to $\tau_2 := \mathbf{1}(\xi_0 = r)\bar{\mathcal{R}}_0 + \sum_{j=1}^{\infty} \mathbf{1}(\xi_0 = u_j)\bar{\mathcal{U}}_0^j$.

3) The sum of the service requirements of all customers in the system at time 0 *not receiving service*, Q_0. This is the total number of customers if the server is on vacation or under repair, or the number of customers in queue, excluding the one in the server, if the server is busy at time 0.

4) Finally, the sum of the service requirements of all customers preceding the tagged customer in the batch that arrived at time 0, together with the tagged customer's own service requirement. The Laplace transform of this last delay is $\chi_e(\hat{S}(s))\hat{S}(s)$ from a discrete renewal theoretic argument (i.e. a size biasing argument), since we consider not a "typical" batch but a typical customer.

From (3.18) the joint transform of Q_0, τ_1 and τ_2 is

$$\begin{aligned}\mathbb{E}[z^{Q_0}e^{-s_1\tau_1-s_2\tau_2}] &= P_0 + z^{-1}P(0;z)\int_0^\infty (1-S(x))e^{-x(\delta+\alpha(z))}\int_0^\infty e^{-s_1 y}\frac{S'(x+y)}{1-S(x)}dydx \\ &\quad +W_0(0)\int_0^\infty (1-R(x))e^{-x\alpha(z)}\int_0^\infty e^{-s_2 y}\frac{R'(x+y)}{1-R(x)}dydx \\ &\quad +\sum_{j=1}^\infty v_j\int_0^\infty (1-U(x))e^{-x\alpha(z)}\int_0^\infty e^{-s_2 y}\frac{U'(x+y)}{1-R(x)}dydx.\end{aligned}$$

Taking into account (3.15), (3.22), (3.27), and (3.29),

$$\begin{aligned}&\mathbb{E}[z^{Q_0}e^{-s_1\tau_1-s_2\tau_2}] \\ &= P_0 + z^{-1}P(0;z)\tfrac{\hat{S}(s_1)-\hat{S}(\delta+\alpha(z))}{\alpha(z)+\delta-s_1} + W(0;z)\tfrac{\hat{R}(s_2)-\hat{R}(\alpha(z))}{\alpha(z)-s_2} + \textstyle\sum_{j=1}^\infty V_j(0)\tfrac{\hat{U}(s_2)-\hat{U}(\alpha(z))}{\alpha(z)-s_2} \\ &= P_0 + \lambda P_0\tfrac{K(z)-\gamma\hat{R}(\alpha(z))}{\hat{S}(\delta+\alpha(z))-z}\tfrac{\hat{S}(s_1)-\hat{S}(\delta+\alpha(z))}{\alpha(z)+\delta-s_1} + \lambda P_0\gamma\tfrac{\hat{R}(s_2)-\hat{R}(\alpha(z))}{\alpha(z)-s_2} + \lambda P_0 C\tfrac{\hat{U}(s_2)-\hat{U}(\alpha(z))}{\alpha(z)-s_2}. \qquad (7.6)\end{aligned}$$

Let us then denote by T_1 the part of the system time of the tagged customer that would have been subject to disasters had the disaster mechanism not been switched off (namely delays described in 1, 3, and 4 above) and by T_2 the part that would not have been subject to disasters. We can obtain the joint Laplace transform of T_1 and T_2 from (7.6) by replacing z with $\hat{S}(s_1)$ and then, to account for the system time of the customers preceding the tagged customer in the batch, including the tagged customer, by multiplying the result by $\chi_e(\hat{S}(s_1)\hat{S}(s_1)$. We thus obtain

$$\begin{aligned}\mathbb{E}[e^{-s_1T_1-s_2T_2}] &= \mathbb{E}[\hat{S}(s_1)^{Q_0}e^{-s_1\tau_1-s_2\tau_2}]\,\chi_e(\hat{S}(s_1)\hat{S}(s_1) \\ &= \Bigg[P_0 - \lambda P_0\frac{K(\hat{S}(s_1))-\gamma\hat{R}(\alpha(\hat{S}(s_1)))}{\alpha(\hat{S}(s_1))+\delta-s_1} \\ &\quad +\lambda P_0\gamma\frac{\hat{R}(s_2)-\hat{R}(\alpha(\hat{S}(s_1)))}{\alpha(\hat{S}(s_1))-s_2} + \lambda P_0 C\frac{\hat{U}(s_2)-\hat{U}(\alpha(\hat{S}(s_1)))}{\alpha(\hat{S}(s_1))-s_2}\Bigg]\,\chi_e(\hat{S}(s_1)\hat{S}(s_1)\end{aligned}$$

or, after simplifying,

$$\begin{aligned}\mathbb{E}[e^{-s_1T_1-s_2T_2}] &= \frac{\lambda P_0}{\alpha(\hat{S}(s_1))-s_2}\Bigg[\Big(\hat{\kappa}(\alpha(\hat{S}(s_1)))-\gamma\hat{R}(\alpha(\hat{S}(s_1)))\Big)\frac{s_2-s_1+\delta}{\alpha(\hat{S}(s_1))-s_1+\delta} \\ &\quad -\hat{\kappa}(s_2)+\gamma\hat{R}(s_2)\Bigg]\,\chi_e(\hat{S}(s_1)\hat{S}(s_1). \qquad (7.7)\end{aligned}$$

Now comes the *second step*, namely restoring the disaster mechanism after time 0. The tagged customer will manage to obtain service and leave the system provided that $T_1 < \Delta$ where Δ is again an independent exponential random variable with rate δ. Trivially, $\mathbb{E}[e^{-s_1T_1+s_2T_2}|T_1<\Delta] = \frac{\mathbb{E}[e^{-(s_1+\delta)T_1+s_2T_2}]}{\mathbb{E}[e^{-\delta T_1}]}$.

From (7.7) we have

$$\mathbb{E}[e^{-(s_1+\delta)T_1-s_2T_2}] = \frac{\lambda P_0}{\alpha(\hat{S}(s_1+\delta))-s_2}\left[\left(\hat{\kappa}(\alpha(\hat{S}(s_1+\delta)))-\gamma\hat{R}(\alpha(\hat{S}(s_1+\delta)))\right)\frac{s_2-s_1}{\alpha(\hat{S}(s_1+\delta))-s_1}\right.$$
$$\left.-\hat{\kappa}(s_2)+\gamma\hat{R}(s_2)\right]\chi_e(\hat{S}(s_1+\delta)\hat{S}(s_1+\delta). \tag{7.8}$$

Setting $s = s_1 = s_2$ above we obtain

$$\mathbb{E}[e^{-(s+\delta)T_1-sT_2}] = \lambda P_0\frac{\hat{\kappa}(s)-\gamma\hat{R}(s)}{s-\alpha(\hat{S}(s+\delta))}\chi_e(\hat{S}(s+\delta))\hat{S}(s+\delta) \tag{7.9}$$

Setting $s = 0$ in (7.9) we obtain, after some simplifications,

$$\mathbb{E}[e^{-\delta T_1}] = P_0\gamma\frac{\hat{S}(\delta)}{m_\chi(1-\hat{S}(\delta))}. \tag{7.10}$$

(Notice that this is the fraction of customers that complete service as given in (5.8.) Thus, the Laplace transform of a customer who does not suffer a disaster is given by the ratio of the right hand sides of (7.9) and (7.10) thus obtaining (7.1).

In similar fashion, starting with the relation

$$\mathbb{E}[e^{-s_1\Delta-s_2T_2}|T_1>\Delta] = \frac{\mathbb{E}[e^{-s_2T_2}]-\mathbb{E}[e^{-(s_1+\delta)T_1-s_2T_2}]}{1-\mathbb{E}[e^{-\delta T_1}]}\frac{\delta}{\delta+s_1}$$

(where Δ is again an independent exponential random variable with rate δ) and using the fact that $\mathbb{E}[e^{-s_2T_2}] = \frac{\lambda P_0}{s_2}\left(\gamma\frac{s_2+\delta}{\delta}+\hat{\kappa}(s_2)-\gamma\hat{R}(s_2)\right)$ which follows by setting $s_1 = 0$ in (7.7) we see that the Laplace transform of the system time of a customer removed by a disaster is

$$\hat{T}_d(s) = \lambda P_0\frac{\frac{1}{s}\left(\gamma+\frac{\gamma s}{\delta}+\hat{\kappa}(s)-\gamma\hat{R}(s)\right)-\frac{\hat{\kappa}(s)-\gamma\hat{R}(s)}{s-\alpha(\hat{S}(s+\delta))}\chi_e(\hat{S}(s+\delta))\hat{S}(s+\delta)}{1-P_0\gamma\frac{\hat{S}(\delta)}{m_\chi(1-\hat{S}(\delta))}}\frac{\delta}{\delta+s}$$

which, after some rearrangement, gives (7.2).

Finally, the Laplace transform of the typical customer's system time, regardless of whether the customer suffers a disaster or not can be obtained by the fact that

$$\mathbb{E}[e^{-s_1(T_1\wedge\Delta)-s_2T_2}] = \frac{\delta}{\delta+s_1}\mathbb{E}[e^{-s_2T_2}]+\frac{s_1}{\delta+s_1}\mathbb{E}[e^{-(s_1+\delta)T_1-s_2T_2}] \tag{7.11}$$
$$= \frac{\delta}{\delta+s}\frac{\lambda P_0}{s}\left(\gamma\frac{s+\delta}{\delta}+\hat{\kappa}(s)-\gamma\hat{R}(s)\right)+\frac{s}{\delta+s}\lambda P_0\frac{\hat{\kappa}(s)-\gamma\hat{R}(s)}{s-\alpha(\hat{S}(s+\delta))}\chi_e(\hat{S}(s+\delta))\hat{S}(s+\delta)$$

which yields (7.3).

The argument for the stationary workload is similar. Begin with the joint distribution of Q_0 and $\bar{\mathcal{R}}_0$:

$$\begin{aligned}\mathbb{E}[z^{Q_0}e^{-s\bar{\mathcal{R}}_0}] &= P_0 + P_0\lambda m_U C\hat{U}_e(\alpha(z)) + P_0\gamma\lambda m_R\hat{R}_e(\alpha(z)) \\ &\quad + z^{-1}P(0;z)\int_0^\infty e^{-x(\alpha(z)+\delta)}(1-S(x))\int_0^\infty e^{-ys}\frac{S'(x+y)}{1-S(x)}dydx.\end{aligned}$$

Substituting z with $\hat{S}(s)$ in the above we obtain (7.5). □

Remark 5. *When the batches are of size 1, i.e. $\chi(z) \equiv 1$ and $\alpha(z) \equiv \lambda - \lambda z$ then (7.11) can be obtained from (4.8) by setting $z = 1 - \frac{s}{\lambda}$, in agreement with the distributional version of Little's law which holds in this case.*

Special Cases: When there are no repairs and no vacations (an $M^X/G/1$ queue with disasters) (7.5) reduces to

$$\hat{\Omega}(s) = P_0\frac{\alpha(z_0)+\delta-s}{\alpha(\hat{S}(s))+\delta-s}$$

with $P_0 = \frac{\delta}{\delta+\alpha(z_0)}$. When in addition the batch size is always 1, i.e. $\chi(z) = z$, then

$$\hat{\Omega}(s) = \frac{\delta}{\delta+\lambda-\lambda z_0}\,\frac{\delta+\lambda-\lambda z_0-s}{\delta+\lambda-s-\lambda\hat{S}(s)},$$

which is the expression obtained in ([13]). The system time of a customer who completes service in this case is

$$\hat{T}_s(s) = \frac{s-\lambda(1-z_0)}{s-\lambda+\lambda\hat{S}(s+\delta)}\frac{\hat{S}(s+\delta)}{\hat{S}(\delta)}\frac{1-\hat{S}(\delta)}{1-z_0}.$$

8 Disasters Affecting All Aspects of the System's Operation

In this section we consider a variation of the model in which disasters can happen at any time, regardless of the state of the server. This includes periods where the server is idle, on vacation, and even under repair. In all cases, a new repair period starts (which may of course be in turn interrupted by a new disaster). The

balance equations in this case are given by

$$(\delta+\lambda)P_0 = \sum_{j=1}^{\infty}(1-g_j)\int_0^\infty V_{j,0}(x)u(x)dx + (1-g_0)\left(\int_0^\infty P_1(x)\mu(x)dx + \int_0^\infty W_0(x)r(x)dx\right)$$

$$\frac{d}{dx}P_n(x) + (\lambda+\delta+\mu(x))\,P_n(x) = \lambda\sum_{k=1}^{n-1}\chi_k P_{n-k}(x), \quad x>0,\ n\geq 1$$

$$\frac{d}{dx}W_0(x) + (\delta+\lambda+r(x))W_0(x) = 0$$

$$\frac{d}{dx}W_n(x) + (\delta+\lambda+r(x))W_n(x) = \lambda\sum_{k=1}^{n}\chi_k W_{n-k}(x), \quad x>0, n\geq 1$$

$$\frac{d}{dx}V_{j,0}(x) + (\delta+\lambda+u(x))V_{j,0}(x) = 0, \quad j=1,2,\ldots$$

$$\frac{d}{dx}V_{j,n}(x) + (\delta+\lambda+u(x))V_{j,n}(x) = \lambda\sum_{k=1}^{n}\chi_k V_{j,n-k}(x), \quad x>0,\ n\geq 1,\ j=1,2,\ldots$$

The boundary conditions of the above system of differential equations are

$$\begin{aligned}
P_n(0) &= \sum_{j=1}^{\infty}\int_0^\infty V_{j,n}(x)u(x)dx + \int_0^\infty P_{n+1}(x)\mu(x)dx + \int_0^\infty W_n(x)r(x)dx + \lambda\chi_n P_0, \quad n\geq 1\\
V_{1,0}(0) &= g_0\int_0^\infty P_1(x)\mu(x)dx + g_0\int_0^\infty W_0(x)r(x)dx\\
V_{j,0}(0) &= g_{j-1}\int_0^\infty V_{j-1,0}(x)u(x)dx, \quad j=2,3,\ldots\\
W_0(0) &= \delta
\end{aligned}$$

with normalization condition (3.11). With z_0 given by (3.31 set

$$C_\delta := \frac{1}{\hat{F}(\hat{U}(\lambda+\delta))}\,\frac{1-\hat{F}(\hat{U}(\lambda+\delta))}{1-\hat{U}(\lambda+\delta)}, \tag{8.1}$$

$$M_\delta(s) := 1 + C_\delta(\delta+\lambda)m_U\hat{U}_e(s), \tag{8.2}$$

$$\eta := \frac{\hat{R}(\delta+\alpha(z_0))}{(\delta+\alpha(z_0))M_\delta(z_0)} \tag{8.3}$$

$$\Theta_\delta(\delta+\alpha(z)) := \eta(\delta+\alpha(z))M_\delta(\delta+\alpha(z)) - \hat{R}(\delta+\alpha(z)) \tag{8.4}$$

$$\hat{\theta}(s) = \eta s M_\delta(s) - \hat{R}(s). \tag{8.5}$$

With the above definitions, arguing as in section 3.2,

$$P(0;z) = z\,\frac{P_0(\delta+\alpha(z))M_\delta(\delta+\alpha(z)) - \delta\hat{R}(\delta+\alpha(z))}{\hat{S}(\delta+\alpha(z)) - z}. \tag{8.6}$$

P_0 is determined using Rouché's theorem as

$$P_0 = \delta\eta. \tag{8.7}$$

$$P(0;z) = \frac{z\delta}{\hat{S}(\delta+\alpha(z)) - z}\left(\eta(\delta+\alpha(z))M_\delta(\delta+\alpha(z)) - \hat{R}(\delta+\alpha(z))\right) = \frac{z\delta\Theta_\delta(\delta+\alpha(z))}{\hat{S}(\delta+\alpha(z)) - z}. \tag{8.8}$$

8.1 Marginal Probability Generating Functions

1. The partial pgf of the time stationary probabilities for the system size when the server is working.

$$P(z) = \frac{z\delta m_S}{\hat{S}(\delta+\alpha(z)) - z}\Theta_\delta(\delta+\alpha(z))\,\hat{S}_e(\delta+\alpha(z)). \tag{8.9}$$

2. The partial pgf of the time stationary probabilities for the system size when the server is under repair.

$$W(z) \;=\; \delta m_R \hat{R}_e(\delta+\alpha(z)). \tag{8.10}$$

3. The partial pgf of the time stationary probabilities for the system size when the server is on the jth vacation.

$$V_j(z) \;=\; V_j(0;z)\,\frac{1-\hat{U}(\delta+\alpha(z))}{(\delta+\alpha(z))} \;=\; v_1\frac{F_j}{F_1}\beta_\delta^{j-1}\,m_U\hat{U}_e((\delta+\alpha(z))), \quad j=1,2,\ldots.$$

Note that

$$V(1) = \sum_{j=1}^{\infty} V_j(1) = m_U\hat{U}_e(\delta)\sum_{j=1}^{\infty} v_j = m_U\hat{U}_e(\delta)\,\frac{v_1}{F_1}\,\frac{1-\hat{F}(\beta_\delta)}{1-\beta_\delta} = P_0(\delta+\lambda)C_\delta m_U\hat{U}_e(\delta)\;. \tag{8.11}$$

This is the probability that the server is on vacation. Also $V(z) := \sum_{j=1}^{\infty} V_j(z)$ is given by

$$V(z) \;=\; P_0(\delta+\lambda)C_\delta m_U\hat{U}_e(\delta+\alpha(z)). \tag{8.12}$$

4. The pgf of the number of customers in the system in stationarity.

$$\begin{aligned}
\Phi(z) &= P_0 + m_S P(0,z)\hat{S}_e(\delta+\alpha(z)) + (\delta+\lambda)P_0 m_U\hat{U}_e(\delta+\alpha(z))C_\delta + \delta m_R\hat{R}_e(\delta+\alpha(z)) \\
&= \frac{\delta\Theta_\delta(\delta+\alpha(z))}{\delta+\alpha(z)} + \frac{z\delta\Theta_\delta(\delta+\alpha(z))}{\hat{S}(\delta+\alpha(z))-z}\,\frac{1-\hat{S}(\delta+\alpha(z))}{\delta+\alpha(z)} + \frac{\delta}{\delta+\alpha(z)} \\
&= \frac{\delta}{\delta+\alpha(z)}\left(1+\Theta_\delta(\delta+\alpha(z))\frac{(1-z)\hat{S}(\delta+\alpha(z))}{\hat{S}(\delta+\alpha(z))-z}\right)
\end{aligned} \tag{8.13}$$

5. The pgf of the system size at a departure epoch

$$\Phi^+(z) \;=\; D_\delta\Theta_\delta(\delta+\alpha(z))\,\frac{\hat{S}(\delta+\alpha(z))}{\hat{S}(\delta+\alpha(z))-z} \tag{8.14}$$

where $D_\delta := \frac{\hat{S}(\delta)-1}{\hat{S}(\delta)}\,\frac{1}{\Theta_\delta(\delta)}$.

6. The pgf of the system size at a busy period initiation epoch. The ratio of rates argument which led to (6.5) still holds. From (3.3) and (3.8)

$$\int_0^\infty P_1(x)\mu(x)dx = \frac{v_1}{F_1} - \delta\hat{R}(\delta+\lambda).$$

Taking into account (8.6) and using the arguments of section 6.1 the pgf of the number present at a busy period initiation epoch is

$$\begin{aligned}\Psi(z) &= \frac{\delta\left(\hat{R}(\delta+\alpha(z)) - \hat{R}(\delta+\lambda)\right) + P_0\left(\frac{\lambda+\delta}{\hat{F}(\hat{U}(\lambda+\delta))} - (\delta+\alpha(z))M_\delta(\delta+\alpha(z))\right)}{\delta\left(\hat{R}(\delta) - \hat{R}(\delta+\lambda)\right) + P_0\left(\frac{\lambda+\delta}{\hat{F}(\hat{U}(\lambda+\delta))} - \delta M_\delta(\delta)\right)} \\ &= \frac{\Theta_\delta(\delta+\alpha(z)) - \Theta_\delta(\delta+\lambda)}{\Theta_\delta(\delta) - \Theta_\delta(\delta+\lambda)}. \end{aligned} \tag{8.15}$$

with $\Theta_\delta(\delta+\alpha(z))$ given by (8.4). In the above we have taken into account that $M_\delta(\delta+\lambda) = \frac{1}{\hat{F}(\hat{U}(\delta+\lambda))}$ as a consequence of (8.1) and (8.2).

7. Negligible Repair Times. The pgf of the number of customers in stationarity is given in this case by

$$\Phi(z) = \frac{\delta}{\delta+\alpha(z)}\,\frac{z(1-\hat{S}(\delta+\alpha(z)))}{z-\hat{S}(\delta+\alpha(z))} + \frac{\delta}{\delta+\alpha(z_0)}\,\frac{M_\delta(\delta+\alpha(z))}{M_\delta(\delta+\alpha(z_0))}\,\frac{(1-z)\hat{S}(\delta+\alpha(z))}{\hat{S}(\delta+\alpha(z))-z}.$$

8.2 Stationary Workload and System Times

Since here the system is subject to disasters even when the server is on vacation, or under repair, we need not distinguish the delay of a tagged customer into a part that is not subject to disasters and one that is. Within the framework of §7 set $\tau = \bar{\mathcal{S}}_0\mathbf{1}(\xi_0 = s) + \bar{\mathcal{R}}_0\mathbf{1}(\xi_0 = r) + \sum_{j=1}^\infty \bar{\mathcal{U}}_0^j\mathbf{1}(\xi_0 = u_j)$. Then

$$\begin{aligned}\mathbb{E}[z^{Q_0}e^{-s\tau}] &= P_0 + z^{-1}P(0;z)\frac{\hat{S}(s)-\hat{S}(\delta+\alpha(z))}{\delta+\alpha(z)-s} + \delta\frac{\hat{R}(s)-\hat{R}(\delta+\alpha(z))}{\delta+\alpha(z)-s} \\ &\quad +(\delta+\lambda)P_0C_\delta\frac{\hat{U}(s)-\hat{U}(\delta+\alpha(z))}{\delta+\alpha(z)-s}. \end{aligned} \tag{8.16}$$

Taking into account that $P_0 = \delta\eta$ and substituting z by $\hat{S}(s)$ in (8.16) we obtain

$$\begin{aligned}\mathbb{E}[(\hat{S}(s))^{Q_0}e^{-s\tau}] &= P_0 + \delta\frac{\hat{R}(\delta+\alpha(\hat{S}(s))) - \eta(\delta+\alpha(\hat{S}(s)))M_\delta(\hat{S}(s))}{\delta+\alpha(\hat{S}(s))-s} \\ &\quad +\delta\frac{\hat{R}(s)-\hat{R}(\delta+\alpha(\hat{S}(s)))}{\delta+\alpha(\hat{S}(s))-s} + (\delta+\lambda)P_0C_\delta\frac{\hat{U}(s)-\hat{U}(\delta+\alpha(\hat{S}(s)))}{\delta+\alpha(\hat{S}(s))-s} \\ &= \frac{\delta\hat{\theta}(s)}{s-\delta-\alpha(\hat{S}(s))}. \end{aligned}$$

Thus, the system time of a customer arriving in stationarity and switching off the disaster mechanism would be

$$T(s) = \frac{\delta\,\hat{\theta}(s)}{s-\delta-\alpha(\hat{S}(s))}\,\chi_e(\hat{S}(s))\hat{S}(s)$$

and the system time of a customer provided he does not suffer a disaster would be

$$\hat{T}_s(s) \;=\; \frac{\hat{\theta}(s+\delta)}{\hat{\theta}(\delta)}\,\frac{\alpha(\hat{S}(s+\delta))}{\alpha(\hat{S}(s+\delta))-s}\,\frac{\hat{S}(s+\delta)}{\hat{S}(\delta)}.$$

Arguing as in §7 we obtain also the stationary workload

$$\hat{\Omega}(s) = \frac{\delta}{\delta+\alpha(\hat{S}(s))}\left(1-\frac{s\,\Theta(\hat{S}(s))}{\delta+\alpha(\hat{S}(s))-s}\right).$$

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