A Batch Arrival Queue System with Feedback and Unliable Server

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Abstract

A queueing system with batch Poisson arrivals and Bernoulli feedback which is subject to disasters and repairs is studied. We also consider that server takes vacation of arbitrary distribution whenever remains empty and study two different vacation policies, single and multiple. We analyze this system using the supplementary variables technique and we obtain the probability generating function of the stationary queue length distribution and the Laplace transform of the busy period's distribution.

KEYWORDS: BATCH-ARRIVALS, SINGLE AND MULTIPLE VACATIONS, FEEDBACK, SUPPLEMENTARY VARIABLES METHOD.

1 Introduction

We study a single server queue with batch Poisson arrivals and general service times. After completion of the service customer can immediately join the tail of the queue as a feedback customer with Bernoulli probability, otherwise the customer may depart forever from the system. The system suffers random disasters which, when they occur, instantly remove all customers (queue and service) from the system. Disasters affect the system only during the time the server is busy serving customers. Immediately after the occurrence of a disaster the server undergoes for a repair with random duration. We also consider that server takes vacations whenever empties. We investigate two different models refereing the vacation policy which is followed, the multiple and the single vacation policy.
For an analytical study of queues with vacations one may refer to Doshi [6], Takagi [18] and Tian and Zhang [20]. Numerous researchers have studied queueing models under different vacation policies depending on the number of possible consecutive vacations server may take every time that system empties. For example Baba [1] studied the *multiple policy* where the number of possible consecutive vacations is unbounded and there is not an idle state so server either working or is on vacation. Choudhury [5] studied the *single policy* where only one vacation server may take each time that is empty and if returning

from vacation and waiting area is empty then stays idle until first customer arrive. The case where the maximum number of vacations is a random variable was proposed by Takagi [18] and termed as *multiple adaptive vacation* (MAV) policy by Tian and Zhang [20].

Disasters also refereed as catastrophes in queueing systems are random events that, when they occur, remove all customers from the system (queue and service). Boxma et al. [3] considered an $M/G/1$ queue system where disasters occurring at certain random times and studied the model for different disaster rules. Yechiali [23] combined disasters with impatience customers while Chakravarthy [4] gave a generalization for the same model. Repair times after disasters occurring for discrete and continuous time models studied respectively by Lee et al. [12] and Yang et al. [24]. The busy period for queues with disasters was obtained by Yashkov and Yashkova [22]. We also refer Boudali and Economou [2] where they studied the effect of catastrophes on the strategic customer behavior and the paper of Kim and Lee [11] who studied a M/G/1 system with working breakdowns suffering by disasters.

Queueing models with Bernoulli feedback first studied by Takács [16], where customers who completed their services feedback instantaneously to the tail of the queue with probability r or leave the system forever with probability $1-r$. Further studies on the queue length, the total sojourn time and waiting time are provided by Disney et al. [7], [8]. The Bernoulli feedback queues with vacation were studied by Takagi [18], Takine et al. [17], Wortman and Disney [21]. A batch arrival queue system with feedback and optional server vacations under single vacation policy derived by Madan and Al-Rawwash [13]. Based on multiple vacation policy Thangaraj and Vanitha [19] studied a two phase single server queue system with Bernoulli feedbacks.

2 Model description and notations

Consider a single server queue which customers arrive in batches according to a Poisson process with rate λ. Batches are assumed to be i.i.d. with χ_n, $n = 1, 2, \ldots$ the nth batch size and probability generating function $X(z) = \sum_{n=1}^{\infty} \mathbb{P}(\chi_1 = n) z^n$. Service times are assumed to be i.i.d. random variables with common distribution S, corresponding density S' and hazard rate function $\mu(x) = \frac{S'(x)}{1-S(x)}$, $x \geq 0$. The Laplace transform of S will be denoted by $\hat{S}(s) := \int_0^{\infty} e^{-sx} dS(x)$ and the mean, assumed finite, by $m_S := \mathbb{E}S$. As soon as service of a customer is complete, then with probability r he join the tail of the original queue as a feedback customer or else with probability $1-r$ he may leave the system. System is subject to disasters which occur according the Poisson process with rate δ, independently of all other processes in the system only when server is busy. When a disaster occurs all customers present, including the one in service, are removed from the system. Immediately after disaster server initiates a repair period which durations are i.i.d. with distribution function R, density function R', hazard rate $r(x) = \frac{R'(x)}{1-R(x)}$, $x \geq 0$, and Laplace transform $\hat{R}(s) = \int_0^{\infty} e^{-sx} dR(x)$ and the mean is $m_R := \mathbb{E}R < \infty$. During a repair period, any customers that may have arrived wait in line. At the end of the busy period or at the end of the repair period if no customer is present server follows a vacation policy. We study two different models separately each of them with different vacation policy (single and multiple). Under single vacation policy server takes one vacation of random length while returning from the vacation, if he finds no customers waiting, he stays idle until he finds at least one customer waiting for service. Contrary, under multiple vacation when he returns from the vacation, if he finds no customers waiting, he immediately takes another vacation and so on, until he finds at least one customer waiting for service. Vacations for both models have independent durations with common distribution function U, with density U' and hazard rate $u(x) = \frac{U'(x)}{1-U(x)}$, $x \geq 0$. The corresponding Laplace transform will be denoted by

$\hat{U}(s) = \int_0^\infty e^{-sx} dU(x)$ and the mean is $m_U := \mathbb{E}U < \infty$. Various stochastic processes involved in the system are independent of each other.

3 Analysis of the steady–state distribution

In this section we derive the steady-state differential-difference equations for the system by treating the elapsed service time, the elapsed repair time, and the elapsed vacation time as supplementary variables for both models (multiple and single). We will consider the processes

$$\begin{array}{rcl} N_t & : & \text{number of customers in the system at time } t \\ \bar{S}_t & : & \text{elapsed service time at time } t \text{ (if server busy, otherwise 0)} \\ \bar{R}_t & : & \text{elapsed repair time at time } t \text{ (if server on repair, otherwise 0)} \\ \bar{U}(t) & : & \text{elapsed vacation time at time } t \text{ (if server on vacation otherwise 0)} \end{array}$$

and the process $\{\xi_t\}$ taking values in the set $\{i, s, r, v\}$, whose elements correspond to the server being *idle*, *serving* customers, being under *repair*, and being on *vacation* respectively. Also let

$$\begin{array}{rcl} P_0(t) & = & \mathbb{P}(N_t = 0, \xi_t = i), \\ P_n(x,t)dx & = & \mathbb{P}(N_t = n, \xi_t = s;\ x < \bar{S}_t \leq x + dx), \\ W_n(x,t)dx & = & \mathbb{P}(N_t = n, \xi_t = r; x < \bar{R}_t \leq x + dx), \\ V_n(x,t)dx & = & \mathbb{P}(N_t = n, \xi_t = v; x < \bar{U}(t) \leq x + dx). \end{array}$$

Due to the presence of disasters the system has a regenerative structure (with regeneration points the epochs when disasters occur) and therefore a stationary version of the process exists. In steady-state $P_0 = \lim_{t\to\infty} P_0(t);\ P_n(x) = \lim_{t\to\infty} P_n(x,t);\ W_n(x) = \lim_{t\to\infty} W_n(x,t);\ V_n(x) = \lim_{t\to\infty} V_n(x,t)$. Note that due model's nature server in multiple vacation case never stays idle so $P_0(t)$ exist only in single vacation case.

3.1 Multiple Model

The balance equations satisfied by the stationary distribution are

$$\frac{d}{dx}P_n(x) + (\lambda + \delta + \mu(x))\, P_n(x) = \lambda \sum_{k=1}^{n-1} \chi_k P_{n-k}(x), \quad x > 0,\ n \geq 1 \tag{1}$$

$$\frac{d}{dx}W_0(x) + (\lambda + r(x))W_0(x) = 0 \tag{2}$$

$$\frac{d}{dx}W_n(x) + (\lambda + r(x))W_n(x) = \lambda \sum_{k=1}^{n} \chi_k W_{n-k}(x), \quad x > 0, n \geq 1 \tag{3}$$

$$\frac{d}{dx}V_0(x) + (\lambda + u(x))V_0(x) = 0, \quad j = 1, 2, \ldots \tag{4}$$

$$\frac{d}{dx}V_n(x) + (\lambda + u(x))V_n(x) = \lambda \sum_{k=1}^{n} \chi_k V_{n-k}(x), \quad x > 0,\ n \geq 1 \tag{5}$$

The boundary conditions of the above system of differential equations are

$$\begin{aligned}
P_n(0) &= (1-r)\int_0^\infty P_{n+1}(x)\mu(x)dx + r\int_0^\infty P_n(x)\mu(x)dx \\
&\quad + \int_0^\infty V_n(x)u(x)dx + \int_0^\infty W_n(x)r(x)dx \qquad (6)\\
V_0(0) &= (1-r)\int_0^\infty P_1(x)\mu(x)dx + \int_0^\infty V_0(x)u(x)dx + \int_0^\infty W_0(x)r(x)dx \qquad (7)\\
V_n(0) &= 0, \quad n = 1,2,\dots \qquad (8)\\
W_0(0) &= \delta\sum_{n=1}^\infty \int_0^\infty P_n(x)dx, \quad n \geq 1 \qquad (9)
\end{aligned}$$

with normalization condition

$$\sum_{n=1}^\infty \int_0^\infty P_n(x)dx + \sum_{n=0}^\infty \left(\int_0^\infty W_n(x)dx + \int_0^\infty V_n(x)dx\right) = 1. \qquad (10)$$

Define the partial probability generating functions (pgf)

$$P(x;z) := \sum_{n=1}^\infty z^n P_n(x), \quad W(x;z) := \sum_{n=0}^\infty z^n W_n(x), \quad V(x;z) := \sum_{n=0}^\infty z^n V_n(x).$$

The partial pgf's for the number of customers in the system in stationarity regardless of the value of the supplementary variables are then given by

$$P(z) := \int_0^\infty P(x;z)dx, \quad W(z) := \int_0^\infty W(x;z)dx, \quad V(z) := \int_0^\infty V(x;z)dx. \qquad (11)$$

Proposition 1. *The partial pgf for the number of customers in the system when the server is busy under the multiple vacation policy is given by*

$$P(z) \;=\; P(0;z)\,\frac{1-\hat{S}(\delta+\alpha(z))}{\delta+\alpha(z)} \qquad (12)$$

where

$$P(0;z) = z\,\frac{V(0;z)\left(1-(\hat{U}(\alpha(z)))\right) - \delta P(1)\hat{R}(\alpha(z))}{\hat{S}(\delta+\alpha(z))\left((1-r(1-z)\right)-z}.$$

and $\alpha(z) := \lambda(1-\chi(z))$.

Proof. Multiplying (1), (3), (5), and (6) by z^n and adding we obtain the linear first order PDE's

$$\begin{aligned}
\frac{\partial P(x;z)}{\partial x} + (\alpha(z)+\delta+\mu(x))P(x;z) &= 0,\\
\frac{\partial W(x;z)}{\partial x} + (\alpha(z)+r(x))W(x;z) &= 0, \qquad (13)\\
\frac{\partial V(x;z)}{\partial x} + (\alpha(z)+u(x))V(x;z) &= 0,
\end{aligned}$$

and the equation

$$\begin{aligned}\sum_{n=1}^{\infty} z^n P_n(0) &= \int_0^\infty \sum_{n=1}^{\infty} z^n V_n(x)u(x)dx + (1-r)\int_0^\infty \sum_{n=1}^{\infty} z^n P_{n+1}(x)\mu(x)dx \\ &\quad + \int_0^\infty \sum_{n=1}^{\infty} z^n W_n(x)r(x)dx + r\int_0^\infty \sum_{n=1}^{\infty} z^n P_n(x)\mu(x)dx \qquad (14)\end{aligned}$$

which, taking into account that $\sum_{n=1}^{\infty} V_n(x)z^n = V(x;z) - V_0(x)$, $\sum_{n=1}^{\infty} W_n(x)z^n = W(x;z) - W_0(x)$, and $\sum_{n=1}^{\infty} P_{n+1}(x)z^n = z^{-1}P(x;z) - P_1(x)$, gives

$$\begin{aligned}P(0;z) &= z^{-1}(1-r)\int_0^\infty P(x;z)\mu(x)dx - (1-r)\int_0^\infty P_1(x)\mu(x)dx + r\int_0^\infty P(x;z)\mu(x)dx \\ &\quad + \int_0^\infty V(x;z)u(x)dx - \int_0^\infty V_0(x)u(x)dx + \int_0^\infty W(x;z)r(x)dx - \int_0^\infty W_0(x)r(x)dx. \quad (15)\end{aligned}$$

Solving (13) we obtain

$$\begin{aligned}P(x;z) &= P(0;z)(1-S(x))e^{-(\delta+\alpha(z))x} \\ W(x;z) &= W(0;z)(1-R(x))e^{-\alpha(z)x} \qquad (16)\\ V(x;z) &= V(0;z)(1-U(x))e^{-\alpha(z)x}.\end{aligned}$$

From (16) we obtain

$$\begin{aligned}\int_0^\infty P(x;z)\mu(x)dx &= \int_0^\infty P(0;z)(1-S(x))e^{-(\delta+\alpha(z))x}\mu(x)dx = P(0;z)\hat{S}(\delta+\alpha(z)), \quad (17)\\ \int_0^\infty W(x;z)r(x)dx &= \int_0^\infty W(0;z)(1-R(x))e^{-\alpha(z)x}r(x)dx = W(0;z)\hat{R}(\alpha(z)), \quad (18)\\ \int_0^\infty V(x;z)u(x)dx &= \int_0^\infty V(0;z)(1-U(x))e^{-\alpha(z)x}u(x)dx = V(0;z)\hat{U}(\alpha(z)). \quad (19)\end{aligned}$$

Using (17)–(18) in (15) we obtain

$$\begin{aligned}P(0;z) &= V(0;z)\hat{U}(\alpha(z)) + z^{-1}(1-r)P(0;z)\hat{S}(\delta+\alpha(z)) + W(0;z)\hat{R}(\alpha(z)) \\ &\quad -V_0(0) + rP(0;z)\hat{S}(\delta+\alpha(z)).\end{aligned}$$

Note that since no customers are present when vacations or repair period starts then $V(0;z) = V_0(0)$ and $W(0;z) = W_0(0)$. Using (16) we write (9) as

$$W(0;z) = W_0(0) = \delta\sum_{n=1}^{\infty}\int_0^\infty P_n(x)dx = \delta\int_0^\infty P(x,1)dx = \delta P(1) \qquad (20)$$

and hence

$$P(0;z) = z\,\frac{V_0(0)\left(1-\hat{U}(\alpha(z))\right) - \delta P(1)\hat{R}(\alpha(z))}{\hat{S}(\delta+\alpha(z))\left((1-r(1-z)\right) - z}.$$

□

We can prove via Rouche's theorem (proof given in appendix) that denominator of $P(0;z)$ has an unique root in the open unit disk $|z| < 1$. The numerator of $P(0;z)$ must therefore also vanish at z_0 so we can establish a relationship for $V_0(0)$ and $P(1)$ as

$$V_0(0)\left(1-\hat{U}(\alpha(z_0))\right) = \delta P(1)\hat{R}(\alpha(z_0)) \qquad (21)$$

3.2 Single Model

In this subsection we derive the partial pgf for the number of customers when server is busy under the single vacation policy. Analysis is similar as in multiple vacation policy so we give only the sketch of proof. The balance equations satisfied by the stationary distribution are same as in previous model plus the one given below for the idle case.

$$\lambda P_0 = \int_0^\infty V_0(x)u(x)dx \tag{22}$$

Instead boundary conditions and normalization condition are not exactly same as in multiple model. Specifically conditions (8) and (9) holds while relations for $P_n(0)$ and $V_0(0)$ replaced by the followings

$$\begin{aligned} P_n(0) &= (1-r)\int_0^\infty P_{n+1}(x)\mu(x)dx + \int_0^\infty V_n(x)u(x)dx \\ &\quad + r\int_0^\infty P_n(x)\mu(x)dx + \int_0^\infty W_n(x)r(x)dx + \lambda\chi_n P_0 \end{aligned} \tag{23}$$

$$V_0(0) = (1-r)\int_0^\infty P_1(x)\mu(x)dx + \int_0^\infty W_0(x)r(x)dx \tag{24}$$

with normalization condition

$$P_0 + \sum_{n=1}^{\infty}\int_0^\infty P_n(x)dx + \sum_{n=0}^{\infty}\left(\int_0^\infty W_n(x)dx + \int_0^\infty V_n(x)dx\right) = 1. \tag{25}$$

Proposition 2. *The partial pgf for the number of customers in the system when the server is busy under the single vacation policy is given by*

$$P(z) = P(0;z)\,\frac{1-\hat{S}(\delta+\alpha(z))}{\delta+\alpha(z)} \tag{26}$$

where

$$P(0;z) = z\,\frac{V_0(0)\left(\hat{U}(\lambda)\,(1-X(z)) + 1 - \hat{U}(\alpha(z))\right) - \delta P(1)\hat{R}(\alpha(z))}{\hat{S}(\delta+\alpha(z))\,((1-r(1-z)) - z}.$$

and $\alpha(z) := \lambda(1-X(z))$.

Proof. The linear first order PDE'S described by relations (13) are the same for single model, so and their solution given by relations (16). It's obvious that relations (17), (18) and (19) are also valid for single model. From the boundary condition (23) multiplying by z^n and summarize then

$$\begin{aligned} P(0;z) &= z^{-1}(1-r)\int_0^\infty P(x;z)\mu(x)dx - (1-r)\int_0^\infty P_1(x)\mu(x)dx + r\int_0^\infty P(x;z)\mu(x)dx \\ &\quad + \int_0^\infty V(x;z)u(x)dx - \int_0^\infty V_0(x)u(x)dx \\ &\quad + \int_0^\infty W(x;z)r(x)dx - \int_0^\infty W_0(x)r(x)dx + \lambda P_0\chi(z). \end{aligned} \tag{27}$$

Solving (4) yields

$$V_0(x) = V_0(0)(1-U(x))e^{-\lambda x}$$

whence we obtain

$$\lambda P_0 = \int_0^\infty V_0(x)u(x)dx = V_0(0)\hat{U}(\lambda) \tag{28}$$

Using (17)–(18) and (28) in (27) we obtain

$$P(0;z) = z\,\frac{V_0(0)\left(\hat{U}(\lambda)\,(1-X(z)) + 1 - \hat{U}(\alpha(z))\right) - \delta P(1)\hat{R}(\alpha(z))}{\hat{S}(\delta+\alpha(z))\,((1-r(1-z)) - z}.$$

□

Based on same lines as in multiple case we can establish a relationship for $V_0(0)$ and $P(1)$ for the single vacation case as

$$V_0(0)\left(\hat{U}(\lambda)\,(1-X(z_0)) + 1 - \hat{U}(\alpha(z_0))\right) = \delta P(1)\hat{R}(\alpha(z_0)) \tag{29}$$

Note that value z_0 is identical for both models since denominators for both partial pgf's are equal.

4 State Phase Probabilities and Different pgf's

We derive the stationary probabilities for the state of the server (busy, under repair, or on vacation for both models plus the idle state for single model) based on the corresponding marginal pgfs for the number of customers in the system. In order to simplify the expressions from equation (21) we set $\gamma_1 := \frac{\hat{R}(\alpha(z_0))}{1-\hat{U}(\alpha(z_0)}$ for multiple model and respectively $\gamma_2 := \frac{\hat{R}(\alpha(z_0))}{\hat{U}(\lambda)(1-X(z_0))+1-\hat{U}(\alpha(z_0))}$ from equation (29) for single model.

- **Server is working.**
  From proposition 1 for multiple model

$$P(z) = z\delta\,\frac{\gamma_1\left(1-(\hat{U}(\alpha(z)))\right) - \hat{R}(\alpha(z))}{\hat{S}(\delta+\alpha(z))\,((1-r(1-z)) - z}\,\frac{1-\hat{S}(\delta+\alpha(z))}{\delta+\alpha(z)}P(1). \tag{30}$$

  From proposition 2 for single model

$$P(z) = z\delta\,\frac{\gamma_2\left(\hat{U}(\lambda)\,(1-X(z)) + 1 - \hat{U}(\alpha(z))\right) - \hat{R}(\alpha(z))}{\hat{S}(\delta+\alpha(z))\,((1-r(1-z)) - z}\,\frac{1-\hat{S}(\delta+\alpha(z))}{\delta+\alpha(z)}P(1). \tag{31}$$

- **Server is under repair.**
  Taking into account (11), (16), and (20) we obtain

$$W(z) \;=\; \delta\frac{1-\hat{R}(\alpha(z))}{\alpha(z)}P(1). \tag{32}$$

  The probability that the server is under repair using the Hospital rule is $W(1) = P(1)\delta m_R$. Relation (32) is valid for both models using in each case the appropriate value of $P(1)$.

- **Server is on vacation.**
  Taking into account (11), (16) and (21) for multiple model we obtain

$$V(z) = \delta\gamma_1 \frac{1 - \hat{U}(\alpha(z))}{\alpha(z)} P(1). \tag{33}$$

  Taking into account (11), (16) and (29) for single model we obtain

$$V(z) = \delta\gamma_2 \frac{1 - \hat{U}(\alpha(z))}{\alpha(z)} P(1). \tag{34}$$

  For both models the probability that the server is on vacation using the Hospital rule is $V(1) = \delta P(1) m_U \gamma_i$ for $i = 1, 2$ respectively for multiple and single model.

- **Server is idle.**
  Server stays idle only when operates under the single vacation policy. Using relations (28) and (29) then

$$P_0 = \delta\gamma_2 \lambda^{-1} \hat{U}(\lambda) P(1) \tag{35}$$

- **The probability that the server is busy.**
  Multiple Model
  The probability that the server is busy $P(1)$ can be determined by using the normalization condition $P(1) + W(1) + V(1) = 1$ which gives

$$P(1) = (1 \, + \, \gamma_1 \delta m_U \, + \, \delta m_R)^{-1}. \tag{36}$$

  Single Model
  Due the existence of idle probability normalization condition in single model is $P_0 + P(1) + W(1) + V(1) = 1$ from which $P(1)$ is

$$P(1) = \Big(1 \, + \, \gamma_2 \delta m_U \, + \, \delta\gamma_2 \lambda^{-1} \hat{U}(\lambda) \, + \, \delta m_R\Big)^{-1}. \tag{37}$$

- **The pgf of the number of customers in the system in stationarity.**
  We can now derive the pgf of the number of customers in the system in stationarity for both models summing the corresponding marginal pgf's as given above.
  Multiple Model
  Using relations (30), (32) and (33) then $\Phi(z) = P(z) + W(z) + V(z)$ or

$$\begin{aligned} \Phi(z) \;\; = \;\; & \delta P(1) \{ z \frac{\gamma_1 \Big(1 - (\hat{U}(\alpha(z)))\Big) - \hat{R}(\alpha(z))}{\hat{S}(\delta + \alpha(z))\,((1 - r(1-z)) - z} \, \frac{1 - \hat{S}(\delta + \alpha(z))}{\delta + \alpha(z)} \\ & + \gamma_1 \frac{1 - \hat{U}(\alpha(z))}{\alpha(z)} + \frac{1 - \hat{R}(\alpha(z))}{\alpha(z)} \}. \end{aligned} \tag{38}$$

  where $P(1)$ given by relation (36).
  Single Model

Using relations (30), (32) and (34) for the marginal pgf's and relation (35) for the idle probability then $\Phi(z) = P(z) + P(0) + W(z) + V(z)$ or

$$\Phi(z) = \delta P(1)\{z \frac{\gamma_2 \left(\hat{U}(\lambda)(1 - X(z)) + 1 - \hat{U}(\alpha(z))\right) - \hat{R}(\alpha(z))}{\hat{S}(\delta + \alpha(z))((1 - r(1 - z)) - z} \frac{1 - \hat{S}(\delta + \alpha(z))}{\delta + \alpha(z)}$$
$$+\gamma_2 \frac{1 - \hat{U}(\alpha(z))}{\alpha(z)} + \frac{1 - \hat{R}(\alpha(z))}{\alpha(z)} + \gamma_2 \lambda^{-1} \hat{U}(\lambda)\}. \quad (39)$$

where $P(1)$ given by relation (37).

4.1 The pgf of the system size at a departure epoch

A departing customer will leave behind l customers in the system at a departure epoch if and only if there are $l + 1$ customers in the system just before the departure. Thus, if ϕ_l^+ denotes the probability that a departing customer leaves behind l customers in the system from a stochastic intensity argument we obtain

$$\phi_l^+ = C_0(1 - r) \int_0^\infty \mu(x) P_{l+1}(x) dx$$

where C_0 is a normalizing constant. If $\Phi^+(z) := \sum_{l=0}^\infty \phi_l^+ z^l$ denotes the corresponding pgf we obtain

$$\Phi_i^+(z) = C_0(1 - r)\delta P(1) \frac{\gamma_i - \hat{R}(\alpha(z))}{\hat{S}(\delta + \alpha(z))((1 - r(1 - z)) - z} \hat{S}(\delta + \alpha(z)) \quad (40)$$

where $C_0 = \frac{1 - \hat{S}(\delta)}{\delta(1 - r)(\gamma_i - 1)P(1)\hat{S}(\delta)}$. Thus,

$$\Phi_i^+(z) = \frac{1 - \hat{S}(\delta)}{(\gamma_i - 1)\hat{S}(\delta)} \frac{\hat{S}(\delta + \alpha(z))\left(\gamma_i - \hat{R}(\alpha(z))\right)}{\hat{S}(\delta + \alpha(z))((1 - r(1 - z)) - z} \quad (41)$$

for $i = 1, 2$ corresponding to multiple and single model respectively.

4.2 Special Cases

The system without repairs.
We assume that after disaster the repair time is negligible so server takes vacation immediately. In order to obtain the pgf for the number of customers in both models we replace $\hat{R}(s) = 1$ in (38) and (39), so

$$\Phi(z) = \delta P(1)\{z \frac{\gamma_1 \left(1 - \hat{U}(\alpha(z))\right)}{\hat{S}(\delta + \alpha(z))((1 - r(1 - z)) - z} \frac{1 - \hat{S}(\delta + \alpha(z))}{\delta + \alpha(z)}$$
$$+\gamma_1 \frac{1 - \hat{U}(\alpha(z))}{\alpha(z)}\}. \quad (42)$$

$$\Phi(z) = \delta P(1)\{z \frac{\gamma_2 \left(\hat{U}(\lambda)(1 - X(z)) + 1 - \hat{U}(\alpha(z))\right)}{\hat{S}(\delta + \alpha(z))((1 - r(1 - z)) - z} \frac{1 - \hat{S}(\delta + \alpha(z))}{\delta + \alpha(z)}$$
$$+\gamma_2 \frac{1 - \hat{U}(\alpha(z))}{\alpha(z)} + \gamma_2 \lambda^{-1} \hat{U}(\lambda)\}. \quad (43)$$

The system without vacations.

Consider now the queue system as described with Poisson batch arrivals disasters, repairs and Bernoulli feedback without any vacation policy when queue is empty. Then based on single vacation model (since it contains the idle state) we replace $m_U = 0$ and $\hat{U}(\lambda) = 1$ forall s in relations (37) and (39). So the probability for busy server is

$$P(1) = \big(1 \ + \ \delta\gamma_3\lambda^{-1} \ + \ \delta m_R\big)^{-1}. \tag{44}$$

where $\gamma_3 := \frac{\hat{R}(a(z))}{1-\alpha(z_0)}$. The pgf for the number of customers in system is given by

$$\begin{aligned}\Phi(z) \quad &= \quad \delta P(1)\{z\frac{\gamma_3(1-X(z))-\hat{R}(\alpha(z))}{\hat{S}(\delta+\alpha(z))\,((1-r(1-z))-z}\,\frac{1-\hat{S}(\delta+\alpha(z))}{\delta+\alpha(z)}\} \\ &\quad +\frac{1-\hat{R}(\alpha(z))}{\alpha(z)}+\gamma_3\lambda^{-1}\}. \end{aligned} \tag{45}$$

5 Analysis of a Busy Period

5.1 The pgf of the System Size at a Busy Period Initiation Epoch

Denote by ψ_n the probability that the typical busy period in stationarity starts with n customers present and let $\{t_l\}$, $l = 0, 1, 2, \ldots$ be the initiation epochs of the busy periods. The N_{t_l} is the number of customers in the system at the initiation epoch of the lth busy period and ψ_n the steady-state probability that an arbitrary customer finds n customers in the system at a busy period initiation epoch. The probabilities where system begins repair or vacation given respectively by

$$\begin{aligned}\mathbb{P}[\text{server stopped by disaster}] \quad &= \quad \frac{W_0(0)}{W_0(0)+V_0(0)} \\ \mathbb{P}[\text{server stopped by vacation}] \quad &= \quad \frac{V_0(0)}{W_0(0)+V_0(0)}\end{aligned}$$

where $W_0(0)$ given by relation (20) with the appropriate value of $P(1)$ for each model while $V_0(0)$ given by equation(21) for multiple and equation (29) for single model respectively.

$$\begin{aligned}W_0(0) \quad &= \quad \delta P(1) \\ V_0(0) \quad &= \quad \delta\gamma_i P(1) \qquad\qquad for \ \ i=1,2\end{aligned}$$

Conditioning on the number of customers which arrive during the vacation (repair) we obtain for multiple vacation model the probability of ψ_n as

$$\psi_n = \frac{W_0(0)}{W_0(0)+V_0(0)}\sum_{k=1}^{n} b_k\chi_n^{(k)*} \ + \ \frac{V_0(0)}{W_0(0)+V_0(0)}\sum_{k=1}^{n}\alpha_k\chi_n^{(k)*}$$

Similarly for single vacation model ψ_n probability equals

$$\psi_n \quad = \quad \frac{W_0(0)}{W_0(0)+V_0(0)}\sum_{k=1}^{n} b_k\chi_n^{(k)*} \ + \ \frac{V_0(0)}{W_0(0)+V_0(0)}\left(\hat{U}(\lambda)\chi_n+\Big(1-\hat{U}(\lambda)\Big)\sum_{k=1}^{n}\alpha_k\chi_n^{(k)*}\right)$$

where $\chi_n^{(k)*}$ is the k-fold convolution of χ_n and $\alpha_k = P[k$ batches arrive during a vacation time$]$ while $b_k = P[k$ batches arrive during a repair time$]$. Note that $\alpha_0 = \hat{U}(\lambda)$. Now multiplying by appropriate powers of z the previous expressions for probabilities and then summing we obtain the corresponding pgf's for both models.

$$\Psi(z) = \frac{W_0(0)}{W_0(0)+V_0(0)}\{\hat{R}(\alpha(z)) - b_0\} + \frac{V_0(0)}{W_0(0)+V_0(0)}\{\hat{U}(\alpha(z)) - a_0\} \qquad Multiple \tag{46}$$

$$\begin{aligned}\Psi(z) &= \frac{V_0(0)}{W_0(0)+V_0(0)}\left(\hat{U}(\lambda)X(z) + \left(1-\hat{U}(\lambda)\right)\{\hat{U}(\alpha(z)) - a_0\}\right) \\ &\quad + \frac{W_0(0)}{W_0(0)+V_0(0)}\{\hat{R}(\alpha(z)) - b_0\} \qquad Single\end{aligned} \tag{47}$$

5.2 The Laplace Transform of Busy Period

We consider an $M^{[X]}/G/1$ queue system with Bernoulli feedback without disasters or vacations where the number of services that a tagged customer receives is geometric with pgf $R(z) = \frac{1-r}{1-zr}$ with r denoted as the feedback probability. In order to derive the duration of busy period which produced by the total service of a tagged customer we decompose this time to single services. If we denote as T the duration of busy period produced by customer's single service and using the well known methodology which developed by Takács we obtain an integral equation for this duration.

Proposition 3. *The LST of the busy period duration which produced by a single service of a tagged customer for $M^{[X]}/G/1$ queue system with Bernoulli feedback given by the unique solution of the equation*

$$\hat{T}(s) = \hat{S}(s + \lambda - \lambda X(R(\hat{T}(s)))). \tag{48}$$

Proof. Firstly we define as L the duration of busy period produced by a single batch of customers. Following the Takás methodology we condition on two events–namely, on the duration of the service time u of the initiating customer and on the number A of batch arrivals during this service time. Given that $u = x$ and $A = n$ then

$$T = x + L_1 + ... + L_n.$$

Where L_i is the busy period generated by the i batch of customers. Since the L_i's are i.i.d. as L with corresponding denoted as LST $\hat{G}(s)$ and are also independent of x, we have

$$E\{e^{-sT}|s = x, A = n\} = E\{e^{-s(x+L_1+...+L_n)}\} = E\{e^{-sx}\}E\{e^{-s(L_1+...+L_n)}\} = e^{-sx}[\hat{G}(s)]^n.$$

We remove now the conditions on s and A. We have

$$E\{e^{-sT}|s = x\} = \sum_{n=0}^{\infty} e^{-sx}[\hat{G}(s)]^n \frac{e^{-\lambda x}(\lambda x)^n}{n!} = e^{-(s+\lambda-\lambda\hat{G}(s))x}.$$

Finally

$$E\{e^{-sT}\} = \int_0^{\infty} E\{e^{-sT}|s = x\}dS(x) = \int_0^{\infty} e^{-(s+\lambda-\lambda\hat{G}(s))x}dS(x).$$

Thus

$$\hat{T}(s) = \hat{S}(s + \lambda - \lambda\hat{G}(s)). \tag{49}$$

Given that each batch size distributed according the probability a_k the LST for the $\hat{G}(s)$ is

$$\hat{G}(s) = \sum_{k=1}^{\infty} a_k [\hat{Z}(s)]^k = X(\hat{Z}(s)), \tag{50}$$

where $\hat{Z}(s)$ is the LST of the busy period which generated by a single customer total service. Note that this duration not depend on the service discipline but on the random number of services which a customer totally receives and distributed by the geometric distribution with probability r, so

$$\hat{Z}(s) = \sum_{k=1}^{\infty} r_k [\hat{T}(s)]^k = R(\hat{T}(s)). \tag{51}$$

From relations (51), (50) and (49) we obtain the integral equation (48). $\square$

The LST of the duration of busy period of a single's customer total service is $R(\hat{T}(s))$ where $\hat{T}(s)$ given by relation (48).

Theorem 4. *The LST of the length of a busy period, $\hat{B}(s)$, in our system is given by*

$$\hat{B}(s) = \frac{\delta}{\delta + s} + \frac{s}{s + \delta}\hat{\Gamma}(s + \delta)$$

where

$$\hat{\Gamma}(s) = \Psi\left(R(\hat{T}(s))\right),$$

$\hat{T}(s)$ is given by (48), and $\Psi(z)$ by (46) or (47) depending on the model.

Proof. We may distinguish two types of busy periods, type 1 that terminate because the system empties after the departure of a customer, and type 2 that terminate because of a disaster. In order to derive type's 1 busy period duration we use result from previous proposition. At an initiation epoch of a typical busy period the distribution of the number of customers present has pgf given by (46). Let Γ denote the resulting busy period with corresponding LST given by

$$\hat{\Gamma}(s) = \sum_{k=1}^{\infty} \left(R(\hat{T}(s))\right)^k \mathbb{P}\left(k \text{ customers waiting at initiation}\right) = \Psi\left(R(\hat{T}(s))\right).$$

For type's 2 busy period duration let D an independent exponential r.v. with rate δ which corresponds to disaster event. Let Q_1 be the probability that a busy period ends by a service completion and $Q_2 = 1 - Q_1$ the probability that it ends by a disaster. Then

$$Q_1 = \mathbb{P}(\Gamma < D) = \mathbb{E}e^{-\delta\Gamma} = \hat{\Gamma}(\delta)$$

Denote also by $\hat{B}_i(s)$ the LST of a type i busy period, $i = 1, 2$.and thus

$$\begin{aligned} \hat{B}_1(s) &= \mathbb{E}[e^{-s(\Gamma\wedge D)} \mid \Gamma < D] = \frac{\hat{\Gamma}(s+\delta)}{\hat{\Gamma}(\delta)}, \\ \hat{B}_2(s) &= \mathbb{E}[e^{-s(\Gamma\wedge D)} \mid \Gamma > D] = \frac{1 - \hat{\Gamma}(s+\delta)}{1 - \hat{\Gamma}(\delta)} \frac{\delta}{s+\delta}. \end{aligned}$$

Finally the length of the busy period of the actual system has LST $\hat{B}(s)$ given by

$$\begin{aligned}\hat{B}(s) &= \mathbb{E}[e^{-s(\Gamma\wedge D)}] \\ &= Q_1\mathbb{E}[e^{-s(\Gamma\wedge D)} \mid \Gamma < D] + Q_2\mathbb{E}[e^{-s(\Gamma\wedge D)} \mid \Gamma > D] \\ &= \frac{\delta}{\delta+s} + \frac{s}{s+\delta}\hat{\Gamma}(s+\delta).\end{aligned}$$

□

6 Appendix

Remark: Since $h(0) = \hat{S}(\delta+\lambda)(1-r) > 0$ and $h(1) = \hat{S}(\delta) - 1 < 0$, z_0 will in fact be real which makes its numerical determination particularly simple.

Proof. The system is stable for all values of the parameters due to the presence of disasters. Hence the power series that defines $P(0;z)$ converges uniformly on the closed unit disk $|z| \leq 1$ and defines an analytic function there. Let $f(z) := -z$ and $g(z) := \hat{S}(\delta + \alpha(z))\left((1 - r(1-z)\right)$ which are both analytic in $|z| \leq 1$. Then

$$|g(z)| \leq \int_0^\infty \mid e^{-(\delta+\alpha(z))x} \mid dS(x) = \int_0^\infty e^{-\delta x}\, e^{-\lambda x \Re(\alpha(z))} dS(x).$$

The real part of $\alpha(z)$ when $|z| = 1$, i.e. $z = e^{i\theta}$, $\theta \in [0, 2\pi)$, is

$$\Re\left(\lambda\left(1 - \textstyle\sum_{k=1}^\infty \chi_k e^{ik\theta}\right)\right) = \lambda \textstyle\sum_{k=1}^\infty \chi_k(1-\cos k\theta) \geq 0, \qquad \theta \in [0, 2\pi)$$

and thus $|g(z)| \leq \int_0^\infty e^{-\delta x} dS(x) < 1$. It follows by Rouché's theorem that $f(z)$ and $f(z) + g(z)$ will have the same number of zeros inside $\mid z \mid < 1$. Since $f(z)$ has only one zero inside this circle, $h(z)$ also has a single zero inside $|z| < 1$, denoted as z_0. The numerator of $P(0;z)$ must therefore also vanish at z_0 otherwise $P(0;z)$ would have a singularity there and this completes the proof. □

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