Potential theory for initial-boundary value problems of unsteady Stokes flow in two dimensions

Shidong Jiang

1 Introduction

Integral equations have been of great theoretical importance for analyzing boundary value problems. There is a large amount of literature devoted to the classical potential theory and its applications on solving the boundary value problems of elliptic partial differential equations (see, for example, [5, 20, 23, 25, 26, 31, 32, 36, 37, 40]). For elliptic problems, integral equations have been coupled with finite element methods in numerical computation and the resulting boundary element methods have been very popular in engineering science (see, for example, [2, 3, 24]).

For time-dependent problems, integral equation methods are less successful in numerical computation. The primary reason is that the direct implementation of integral equation methods for time-dependent problems is computationally expensive as compared with finite difference or finite element methods. Indeed, the discretizations of integral equations usually lead to dense linear systems. And for time dependent problems, the layer potentials involve integration in both space and time, which makes the evaluation of layer potentials and time marching extremely expensive.

The invention of the Fast Multipole Method (FMM) (see, for example, [16, 17, 4]) has dramatically changed the landscape of the field of scientific computing. Tremendous progress has been made in designing fast and accurate numerical algorithms using FMM and its descendents to solve integral equations for various problems in electromagnetics, elasticity, and fluid mechanics (see, for example, [6, 14, 12, 13, 33, 45, 46, 47]). The numerical tools for solving the heat equation using integral equations have also been developed recently (see, for example, [18, 19, 28, 15, 43, 44]). Hyperbolic potentials have also been applied to study time-dependent problems for scattering problems in electromagnetics (see, for example, [30]). When the hurdle of computational cost has been overcome, integral equation methods offer several advantages as compared with standard finite difference and/or finite element methods. First, problems of complex geometry can be dealt with more easily. Second, the artificial boundary conditions are avoided for exterior problems. Third, the dimension of the problem is reduced by one for certain problems. Fourth, the influences of the initial data, nonhomo-

geneous term, and boundary data are clearly separated in integral equation formulations, and this is physically more attractive for many problems.

In this paper, we study the integral equation formulations for the time-dependent linearized viscous incompressible flow (also called the unsteady Stokes flow or linearized Navier-Stokes flow):

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \mu \Delta \mathbf{u}, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{2}$$

We first derive Green's formula for the unsteady Stokes flow with nonhomogeneous terms in a bounded domain. We then show that the pure initial value problem can be solved using the initial potential; the nonhomogeneous problem can be solved using the volume potential. Next, we derive the jump relation of the double layer potential and apply it to derive a boundary integral equation for the Dirichlet problem. Finally, we derive the jump relation of the single layer potential and apply it to derive a boundary integral equation for the Neumann problem. Since the problem is linear, the integral equation formulations for general initial-boundary value problems of the unsteady Stokes flow are readily available. Here we observe that many numerical tools for solving integral equations for the heat equation [18, 19, 28, 15, 43, 44] can be applied to solve integral equations for the unsteady Stokes flow. Thus, fast numerical algorithms can be readily developed to solve integral equations associated with the unsteady Stokes flow.

Remark 1.1. *In [38], single and double layer potentials have been used to show the existence and uniqueness of the boundary value problems for the unsteady Stokes flow in Lipschitz domains. Our definition of the double layer potential is different from that of [38]. We have symmetrized the double layer potential to make it physically meaningful and more parallel to the steady Stokes case (see, for example, [37]). As compared with [38], we offer elementary but more comprehensive and systematic study of integral equation methods for the unsteady Stokes flow, which hopefully will be more easily accessible to physicists and engineers and more useful for numerical computation.*

Remark 1.2. *As pointed out by one of referees to this paper, Odquist [34] has proposed the layer potential formulation for unsteady Stokes flow in 1932 where he aimed at the proof of the solvability of the Stokes problem. Subsequently, Leray [27] constructed the potential theory for 2-dimensional convex*

domains based on another kernel and proved the above-mentioned existence theorem in 1934.

The paper is organized as follows. First, we present some estimates about the singularities of the Green's function. We then derive Green's formula for the unsteady Stokes flow in Section 3. In Section 4, we study the pure initial value problem of the unsteady Stokes flow. In Section 5, we study the nonhomogeneous problem using the volume potential. In Section 6, we study the Dirichlet problem using double layer potentials. In Section 7, we study the Neumann problem using single layer potentials. Finally, we conclude this paper with a short discussion.

2 Analytical preliminary

2.1 Notation

We use x, y to denote points in $\mathbb{R}^2$, i.e., $x = (x_1, x_2)$, $y = (y_1, y_2)$. We use t and τ to denote time variables. A bounded domain in $\mathbb{R}^2$ is denoted by D, its boundary is denoted by S. S is assumed to be a Lyapunov curve. The outward unit normal vector on S is denoted by $n = (n_1, n_2)$. Vectors are generally boldfaced. The Einstein summation convention will be used.

2.2 Fundamental solutions for the unsteady Stokes flow - the unsteady Stokeslet

The fundamental solution for the unsteady Stokes flow has been derived by several researchers (see, for example, [35, 39, 8, 9, 10, 22, 42, 21]). The unsteady Stokeslet G_{ij} $(i, j = 1, 2)$ for the velocity field (also called Oseen's tensor in literature since Oseen [35] was the first one to define it) and the associated pressurelet p_i $(i = 1, 2)$ satisfy the following six equations

$$\rho \frac{\partial G_{ij}}{\partial t} = -\frac{\partial p_j}{\partial x_i} + \mu \nabla^2 G_{ij} + \delta_{ij} \delta(\mathbf{x}) \delta(t), \quad i, j = 1, 2, \tag{3}$$

$$\frac{\partial G_{ij}}{\partial x_i} = 0, \quad j = 1, 2. \tag{4}$$

The explicit expressions of G_{ij} and p_i are given by the following formulae (see, for example, [22, 42, 21])

$$G_{ij}(x, t) = -\frac{1}{2\pi\rho}(\delta_{ij} - \frac{2x_i x_j}{r^2}) \frac{1 - e^{-r^2/4\nu t}}{r^2} + \frac{1}{4\pi\rho}(\delta_{ij} - \frac{x_i x_j}{r^2}) \frac{e^{-r^2/4\nu t}}{\nu t}, \tag{5}$$

$$p_i(x,t) = \frac{1}{2\pi}\frac{x_i}{r^2}\delta(t), \tag{6}$$

where $r = |x| = \sqrt{x_1^2 + x_2^2}$ and $\nu = \mu/\rho$ is the kinematic viscosity.

2.3 Properties of the unsteady Stokeslet

We now consider the case where the source is at a general point y and time τ. We write $G_{ij}(x,t;y,\tau)$ to denote the unsteady Stokeslet in this case. Due to translation invariance, we actually have $G_{ij}(x,t;y,\tau) = G_{ij}(x-y,t-\tau)$ and similarly $p_i(x,t;y,\tau) = p_i(x-y,t-\tau)$.

We first point out some connections between the unsteady Stokeslet and fundamental solutions of other related PDEs. The Laplace kernel G_l satisfies the equation $-\Delta G_l = \delta(x-y)$ and is given by the formula $G_l(x,y) = \frac{1}{2\pi}\ln\frac{1}{|x-y|}$. The heat kernel G_h satisfies the equation $\rho\frac{\partial G_h}{\partial t} = \mu\Delta G_h + \delta(x-y)\delta(t-\tau)$ and is given by the formula $G_h(x,t;y,\tau) = \frac{1}{4\pi\rho}\frac{1}{\nu(t-\tau)}e^{-\frac{|x-y|^2}{4\nu(t-\tau)}}$, where $\nu = \mu/\rho$ is the kinematic viscosity.

Straightforward computation shows that

$$G_{11} + G_{22} = G_h, \quad \frac{\partial G_{ij}}{\partial t} = \delta_{ij}\frac{\partial G_h}{\partial t} - \nu\frac{\partial^2 G_h}{\partial x_i \partial x_j}, \tag{7}$$

and

$$\begin{aligned} \lim_{\tau\to t-} G_{ij}(x,t;y,\tau) &= -\frac{1}{2\pi\rho}(\delta_{ij} - \frac{2(x_i-y_i)(x_j-y_j)}{|x-y|^2})\frac{1}{|x-y|^2} \\ &= \frac{1}{\rho}\frac{\partial^2 G_l(x,y)}{\partial x_i \partial x_j} = \frac{1}{\rho}\frac{\partial^2 G_l(x,y)}{\partial y_i \partial y_j}. \end{aligned} \tag{8}$$

For the fundamental solutions of pressure, we have

$$p_i(x,t;y,\tau) = -\frac{\partial G_l(x,y)}{\partial x_i}\delta(t-\tau), \quad i = 1,2; \qquad \frac{\partial p_j}{\partial y_i} = \frac{\partial p_i}{\partial y_j}. \tag{9}$$

For $x \neq y$ and $\tau < t$, it is easy to see that $G_{ij} = G_{ij}(x,t;y,\tau)$ satisfies the forward heat equation in x and t and backward heat equation in y and τ:

$$-\rho\frac{\partial G_{ij}}{\partial \tau} - \mu\nabla_y^2 G_{ij} = 0, \quad (y,\tau) \in (D - B_\epsilon(x)) \times [0,t). \tag{10}$$

From (4), $\frac{\partial G_{ij}}{\partial x_i} = -\frac{\partial G_{ij}}{\partial y_i}$, and $G_{ij} = G_{ji}$, we have

$$\frac{\partial G_{ij}}{\partial y_i} = \frac{\partial G_{ji}}{\partial y_i} = 0. \tag{11}$$

We now provide some estimates of $G_{ij} = G_{ij}(x,t;y,\tau)$, whose proof is similar to that of the heat kernel in [11] (see also [8]).

Lemma 2.1. (Estimates of unsteady Stokeslet)

1. *For any fixed $t > \tau$ and $x \in \mathbb{R}^2$, G_{ij} is bounded and thus absolutely integrable as a function of y in any bounded domain $D \in \mathbb{R}^2$.*

2. *For any $0 < \alpha < 1$,*

$$|G_{ij}(x,t;y,\tau)| \leq \frac{const.}{|t-\tau|^{\alpha}|x-y|^{2-2\alpha}}. \tag{12}$$

   Thus, for a fixed $x \in D$, G_{ij} is absolutely integrable on $D \times [0,t]$ as a function of y and τ; and for a fixed $x \in S$ (S is a curve in $\mathbb{R}^2$), G_{ij} is absolutely integrable on $S \times [0,t]$ as a function of y and τ.

3. *For any $1/2 < \alpha < 1$,*

$$|\frac{\partial}{\partial y_k} G_{ij}(x,t;y,\tau)| \leq \frac{const.}{|t-\tau|^{\alpha}|x-y|^{3-2\alpha}}. \tag{13}$$

   Thus, for a fixed $x \in D$, $\frac{\partial}{\partial y_k} G_{ij}$ is absolutely integrable on $D \times [0,t]$ as function of y and τ.

The following integrals will be used subsequently, all of which can be calculated using polar coordinates.

Lemma 2.2. *Let S_a be a circle of radius a centered at the origin and B_a be a ball of radius a centered at the origin. Then*

$$\int_{S_a} \mu \frac{\partial G_{ij}(x,t)}{\partial x_k} n_k ds_x = -\delta_{ij} \frac{a^2 e^{-a^2/4\nu t}}{8\nu t^2}, \tag{14}$$

$$\int_{B_a} \rho G_{ij}(x,t) dx = \frac{1}{2}\delta_{ij}(1 - e^{-a^2/4\nu t}), \tag{15}$$

and

$$\int_{S_a} \frac{\partial G_l(x,0)}{\partial x_i} n_j ds_x = -\frac{1}{2}\delta_{ij}. \tag{16}$$

3 Green's formula for the unsteady Stokes flow

Consider the unsteady Stokes flow with a nonhomogeneous term in a bounded domain D with boundary S:
(1) the governing equation:

$$\rho\frac{\partial u_i}{\partial t} = -\frac{\partial p}{\partial x_i} + \mu\Delta u_i + f_i(\mathbf{x}, t), \quad (x,t) \in D \times [0,T], \quad i = 1,2; \quad (17)$$

(2) the incompressibility condition:

$$\frac{\partial u_i}{\partial x_i} = 0, \quad (x,t) \in D \times [0,T]; \quad (18)$$

(3) initial condition:

$$u_i(x,0) = u_{0i}(x), \quad x \in D, \quad t = 0. \quad (19)$$

Multiplying G_{ij} with the governing equation with the variables changed to y and τ in (17), we obtain

$$G_{ij}(x,t;y,\tau)\left\{\rho\frac{\partial u_j}{\partial \tau} + \frac{\partial p}{\partial y_j} - \mu\Delta_y u_j - f_j(y,\tau)\right\} = 0. \quad (20)$$

Consider first the case that $x \in D$. Integrating the above equation over the domain $D - B_\epsilon(x)$ with $B_\epsilon(x)$ a small ball of radius ϵ centered at x, we obtain

$$\int_0^t \int_{D-B_\epsilon(x)} G_{ij}(x,t;y,\tau)\left\{\rho\frac{\partial u_j}{\partial \tau} + \frac{\partial p}{\partial y_j} - \mu\Delta_y u_j - f_j(y,\tau)\right\} dyd\tau = 0. \quad (21)$$

Applying integration by parts on the first term, Green's theorem on the second term, and Green's second identity on the third term, we obtain

$$\begin{aligned}
&\int_0^t \int_{D-B_\epsilon(x)} u_j\left(-\rho\frac{\partial G_{ij}(x,t;y,\tau)}{\partial \tau} - \mu\Delta_y G_{ij}\right) dyd\tau - \int_0^t \int_{D-B_\epsilon(x)} p\frac{\partial G_{ij}}{\partial y_j} dyd\tau \\
&= \mu\int_0^t \int_{S-S_\epsilon} (G_{ij}\frac{\partial u_j}{\partial y_k} - u_j\frac{\partial G_{ij}}{\partial y_k}) n_k ds_y d\tau - \int_0^t \int_{S-S_\epsilon} pG_{ij}n_j ds_y d\tau \\
&- \int_{D-B_\epsilon(x)} \rho G_{ij}(x,t;y,\tau)u_j(y,\tau)|_0^t dy + \int_0^t \int_{D-B_\epsilon(x)} G_{ij}(x,t;y,\tau)f_j(y,\tau)dyd\tau.
\end{aligned} \quad (22)$$

Since the Stokeslet satisfies the backward heat equation in y and τ and divergence free condition (see (10), (11)), the left hand side of (22) is zero. Thus

$$\begin{aligned} 0 = &\int_0^t \int_{S-S_\epsilon} G_{ij} \left(\mu \frac{\partial u_j}{\partial y_k} n_k - p n_j \right) ds_y d\tau - \int_0^t \int_{S-S_\epsilon} \mu \frac{\partial G_{ij}}{\partial y_k} n_k u_j ds_y d\tau \\ &- \int_{D-B_\epsilon} \rho G_{ij}(x,t;y,\tau) u_j(y,\tau)|_0^t dy + \int_0^t \int_{D-B_\epsilon} G_{ij}(x,t;y,\tau) f_j(y,\tau) dy d\tau. \end{aligned} \tag{23}$$

Furthermore,

$$\begin{aligned} &- \int_{D-B_\epsilon} \rho G_{ij}(x,t;y,\tau) u_j(y,\tau)|_0^t dy \\ &= \int_{D-B_\epsilon} \rho G_{ij}(x,t;y,0) u_{0j}(y) dy - \int_{D-B_\epsilon} \frac{\partial^2 G_l(x,y)}{\partial y_i \partial y_j} u_j(y,t) dy \\ &= \int_{D-B_\epsilon} \rho G_{ij}(x,t;y,0) u_{0j}(y) dy - \int_{D-B_\epsilon} \frac{\partial}{\partial y_j} \left[\frac{\partial G_l(x,y)}{\partial y_i} u_j(y,t) \right] dy \\ &= \int_{D-B_\epsilon} \rho G_{ij}(x,t;y,0) u_{0j}(y) dy - \int_{S-S_\epsilon} \frac{\partial G_l(x,y)}{\partial y_i} u_j(y,t) n_j ds_y. \end{aligned} \tag{24}$$

where (8) is used to obtain the first equality, the incompressibility of u_j is used to derive the second equality, and in the third equality we have applied Green's theorem.
Combining (23), (24), we obtain

$$\begin{aligned} 0 = &\int_0^t \int_{S-S_\epsilon} G_{ij} \left(\mu \frac{\partial u_j}{\partial y_k} n_k - p n_j \right) ds_y d\tau \\ &- \int_0^t \int_{S-S_\epsilon} \mu \frac{\partial G_{ij}}{\partial y_k} n_k u_j ds_y d\tau - \int_{S-S_\epsilon} \frac{\partial G_l(x,y)}{\partial y_i} u_j(y,t) n_j ds_y \\ &+ \int_{D-B_\epsilon} \rho G_{ij}(x,t;y,0) u_{0j}(y) dy \\ &+ \int_0^t \int_{D-B_\epsilon} G_{ij}(x,t;y,\tau) f_j(y,\tau) dy d\tau. \end{aligned} \tag{25}$$

(1) When $|x-y|=\epsilon$ is very small, since the velocity, its first derivative, and the pressure are all continuous, we have

$$\begin{aligned}
&\left|\int_0^t\int_{S_\epsilon} G_{ij}\left(\mu\frac{\partial u_j}{\partial y_k}n_k - pn_j\right)ds_y d\tau\right| \\
&\le M\int_0^t\int_0^{2\pi}\frac{1}{(t-\tau)^\alpha\epsilon^{2-2\alpha}}\epsilon d\theta d\tau \\
&\le M\epsilon^{2\alpha-1}\to 0, \quad \text{as} \quad \epsilon\to 0 \quad \text{for} \quad 1/2<\alpha<1,
\end{aligned} \tag{26}$$

where (12) is used in the first inequality.
(2) Using (14), we have

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_0^t\int_{S_\epsilon}\mu\frac{\partial G_{ij}}{\partial y_k}n_k u_j(y,\tau)ds_y d\tau \\
&=\lim_{\epsilon\to 0}\int_0^t u_j(x,\tau)\int_{S_\epsilon}\mu\frac{\partial G_{ij}}{\partial y_k}n_k ds_y d\tau \\
&=\lim_{\epsilon\to 0}\int_0^t u_j(x,\tau)\left(-\delta_{ij}\frac{\epsilon^2 e^{-\epsilon^2/4\nu(t-\tau)}}{8\nu(t-\tau)^2}\right)d\tau \\
&=-\frac{1}{2}\lim_{\epsilon\to 0}\int_{\epsilon^2/4\nu t}^\infty u_i(x,t-\epsilon^2/4\nu\lambda)e^{-\lambda}d\lambda \\
&=-\frac{1}{2}u_i(x,t).
\end{aligned} \tag{27}$$

(3) Using (16), we have

$$\begin{aligned}
&\int_{S_\epsilon}\frac{\partial G_l(x,y)}{\partial y_i}u_j(y,t)n_j ds_y \to u_j(x,t)\int_{S_\epsilon}\frac{\partial G_l(x,y)}{\partial y_i}n_j ds_y \\
&=u_j(x,t)(-\frac{1}{2}\delta_{ij}) = -\frac{1}{2}u_i(x,t), \quad \epsilon\to 0.
\end{aligned} \tag{28}$$

Taking the limit of $\epsilon\to 0$ in (25) and then using (26), (27), (28), we obtain

$$\begin{aligned}
u_i(x,t) = &\int_0^t\int_S G_{ij}\left(\mu\frac{\partial u_j}{\partial y_k}n_k - pn_j\right)ds_y d\tau \\
&-\int_0^t\int_S\mu\frac{\partial G_{ij}}{\partial y_k}n_k u_j ds_y d\tau - \int_S\frac{\partial G_l(x,y)}{\partial y_i}u_j(y,t)n_j ds_y \\
&+\int_D \rho G_{ij}(x,t;y,0)u_{0j}(y)dy \\
&+\int_0^t\int_D G_{ij}(x,t;y,\tau)f_j(y,\tau)dyd\tau.
\end{aligned} \tag{29}$$

To symmetrize the first two terms on the right hand side, we use the incompressibility conditions for both G_{ij} and u_i. We have

$$\begin{aligned}
&\int_0^t \int_S G_{ij} \frac{\partial u_k}{\partial y_j} n_k ds_y d\tau = \int_0^t \int_D \frac{\partial}{\partial y_k} \left(G_{ij} \frac{\partial u_k}{\partial y_j} \right) dy d\tau \\
&= \int_0^t \int_D \frac{\partial G_{ij}}{\partial y_k} \frac{\partial u_k}{\partial y_j} dy d\tau = \int_0^t \int_D \frac{\partial G_{ik}}{\partial y_j} \frac{\partial u_j}{\partial y_k} dy d\tau \\
&= \int_0^t \int_D \frac{\partial}{\partial y_k} \left(u_j \frac{\partial G_{ik}}{\partial y_j} \right) dy d\tau = \int_0^t \int_S u_j \frac{\partial G_{ik}}{\partial y_j} n_k ds_y d\tau,
\end{aligned} \tag{30}$$

where the first and the last equalities follow from Green's theorem, the second and the fourth equalities follow from the incompressibility of u_k and G_{ik}, respectively, and the third equality follows from the interchange of the dummy summation variables j and k.
Furthermore, since $p_i(x,t;y,\tau) = \frac{\partial G_l(x,y)}{\partial y_i} \delta(t-\tau)$, we may write

$$\int_S \frac{\partial G_l(x,y)}{\partial y_i} u_j(y,t) n_j ds_y = \int_0^t \int_S p_i(x,t;y,\tau) u_j(y,\tau) n_j ds_y d\tau, \tag{31}$$

with the understanding of $\int_0^t u_j(y,\tau)\delta(t-\tau)d\tau = u_j(y,t)$.
Combining (29), (30), and (31), we obtain

$$\begin{aligned}
u_i(x,t) = &\int_0^t \int_S G_{ij}(x,t;y,\tau)\sigma_j(y,\tau) ds_y d\tau \\
&- \int_0^t \int_S T_{ij}(x,t;y,\tau) u_j(y,\tau) ds_y d\tau \\
&+ \int_D \rho G_{ij}(x,t;y,0) u_{0j}(y) dy \\
&+ \int_0^t \int_D G_{ij}(x,t;y,\tau) f_j(y,\tau) dy d\tau,
\end{aligned} \tag{32}$$

where

$$\begin{aligned}
\sigma_j &= \mu \left(\frac{\partial u_j}{\partial y_k} + \frac{\partial u_k}{\partial y_j} \right) n_k - p n_j \\
&= \left[\mu \left(\frac{\partial u_j}{\partial y_k} + \frac{\partial u_k}{\partial y_j} \right) - p\delta_{jk} \right] n_k \\
&= \sigma_{jk} n_k(y)
\end{aligned} \tag{33}$$

is the jth component of the surface force and

$$\begin{aligned} T_{ij}(x,t;y,\tau) &= \mu\left(\frac{\partial G_{ij}}{\partial y_k}+\frac{\partial G_{ik}}{\partial y_j}\right)n_k + p_i n_j \\ &= -\left[\mu\left(\frac{\partial G_{ij}}{\partial x_k}+\frac{\partial G_{ik}}{\partial x_j}\right) - p_i(x,t;y,\tau)\delta_{jk}\right]n_k(y) \\ &= T_{ijk}n_k \end{aligned} \tag{34}$$

is the stress tensor associated with the Stokeslet.

When $x \notin \bar{D}$, it is easy to see that the left side of (32) should be zero. Moreover, the jump relation of the double layer potential (69) shows that the left side of (32) should be $\frac{1}{2}u_i(x,t)$ for $x \in S$. Combining all these facts, we obtain

$$\begin{aligned} &\int_0^t \int_S G_{ij}(x,t;y,\tau)\sigma_j(y,\tau)ds_y d\tau - \int_0^t \int_S T_{ij}(x,t;y,\tau)u_j(y,\tau)ds_y d\tau \\ &+ \int_D \rho G_{ij}(x,t;y,0)u_{0j}(y)dy + \int_0^t \int_D G_{ij}(x,t;y,\tau)f_j(y,\tau)dyd\tau \\ &= \begin{cases} u_i(x,t), & x \in D, \\ \frac{1}{2}u_i(x,t), & x \in S, \\ 0, & x \notin \bar{D}. \end{cases} \end{aligned} \tag{35}$$

We call (35) the Green's formula for the unsteady Stokes flow (there is a short, nonrigorous and incorrect derivation for this formula in [22]). It actually provides the so called "direct" boundary integral equation formulation for the unsteady Stokes flow. The first term of the left side of (35) is called the single layer potential of the unsteady Stokes flow, the second term is called the double layer potential, the third term is called the initial potential, the last term is called the volume potential which deals with the nonhomogeneous term of the governing equation.

We now analyze these four terms and study various problems associated with the unsteady Stokes flow.

4 Pure initial-value problem

Theorem 4.1. (Solution of initial-value problem). *Suppose that* $\mathbf{f} \in C^2(\mathbb{R}^2)$ *has compact support in* $\mathbb{R}^2$ *and satisfies the condition*

$$\nabla \cdot \mathbf{f} = \frac{\partial f_i}{\partial x_i} = 0. \tag{36}$$

Define

$$u_i(x,t) = \int_{\mathbb{R}^2} \rho G_{ij}(x,t;y,0) f_j(y) dy. \tag{37}$$

Then

1. $\mathbf{u} \in C^2_\infty(\mathbb{R}^2 \times (0,\infty))$,
2. $\rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \mu \Delta \mathbf{u}$, *where* p *is an arbitrary function of* t *only,*
3. $\nabla \cdot \mathbf{u} = 0$, $t > 0$,
4. $$\lim_{t\to 0+} u_i(x,t) = \lim_{t\to 0+} \int_{\mathbb{R}^2} \rho G_{ij}(x,t;y,0) f_j(y) dy = f_i(x). \tag{38}$$

Proof. 1. For $t > 0$, G_{ij} is infinitely differentiable with respect to t, so is $u_i(x,t)$. To show that u_i is twice differentiable with respect to x, we simply make change of variable and write

$$u_i(x,t) = \int_{\mathbb{R}^2} \rho G_{ij}(y,t) f_j(x-y) dy. \tag{39}$$

2. To show that $\mathbf{u}$ and $p = g(t)$ satisfy the governing equation, we merely note that (a). G_{ij} satisfies the heat equation and $\nabla p = 0$; (b). the second derivatives of G_{ij} are actually bounded at $y = x$ for $t > 0$.

3. The fact that $\mathbf{u}$ is divergence free can also be easily proved by noting that G_{ij} is divergence free and its first derivatives are bounded at $y = x$ for $t > 0$.

4. Fix $x \in \mathbb{R}^2$, $\epsilon > 0$. Choose $\delta > 0$ such that

$$|f_j(y) - f_j(x)| < \epsilon \qquad \text{if} \quad |y - x| < \delta. \tag{40}$$

Let $B_\delta(x)$ be the ball of radius δ centered at x. We split the integral into two parts

$$u_i(x,t) = \int_{\mathbb{R}^2 - B_\delta(x)} + \int_{B_\delta(x)} := I_1 + I_2, \tag{41}$$

and split I_1 further into two parts with one part being time-independent

$$I_1 = J_1 + J_2, \tag{42}$$

where

$$J_1 = \int_{\mathbb{R}^2 - B_\delta(x)} \frac{\partial^2 G_l(x,y)}{\partial y_i \partial y_j} f_j(y) dy \tag{43}$$

and

$$\begin{aligned} J_2 = \int_{\mathbb{R}^2 - B_\delta(x)} \Bigg(& \frac{1}{2\pi} \left[\delta_{ij} - \frac{2(x_i - y_i)(x_j - y_j)}{|x-y|^2} \right] \frac{e^{-|x-y|^2/4\nu t}}{|x-y|^2} \\ & + \frac{1}{4\pi} \left[\delta_{ij} - \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^2} \right] \frac{e^{-|x-y|^2/4\nu t}}{\nu t} \Bigg) f_j(y) dy. \end{aligned} \tag{44}$$

(a) For J_1, we have

$$\begin{aligned} J_1 &= \int_{\mathbb{R}^2 - B_\delta(x)} \frac{\partial}{\partial y_j} \left(\frac{\partial G_l(x,y)}{\partial y_i} f_j(y) \right) dy \\ &= - \int_{S_\delta(x)} \frac{\partial G_l(x,y)}{\partial y_i} f_j(y) n_j ds_y \\ &\to -f_j(x) \int_{S_\delta(x)} \frac{\partial G_l(x,y)}{\partial y_i} n_j ds_y \\ &= \frac{1}{2} f_i(x), \end{aligned} \tag{45}$$

where we used the divergence free condition of f_j in the first equality, Green's theorem in the second equality, and (16) to obtain the final result.

(b) Since δ is fixed, the integrand of J_2 tends to 0 as $t \to 0$. Hence $J_2 \to 0$ as $t \to 0$.

(c) For I_2, we have

$$I_2 \to f_j(x) \int_{B_\delta(x)} \rho G_{ij}(x,t;y,0) dy = f_j(x) \frac{1}{2} \delta_{ij} (1 - e^{-\delta^2/4\nu t}), \tag{46}$$

where the second equality follows from (15).
Again since δ is fixed, we have

$$I_2 \to \frac{1}{2} f_i(x) \quad \text{as} \quad t \to 0. \tag{47}$$

And (38) follows by combining (41)–(47).

□

5 Nonhomogeneous problem and the volume potential

Theorem 5.1. (Solution of nonhomogeneous problem). *Suppose that* $\mathbf{f} \in C_1^2(\mathbb{R}^2 \times [0,\infty))$ *and* $\mathbf{f}$ *has compact support. Define* $\mathbf{u}$ *by the formula*

$$u_i(x,t) = \int_0^t \int_{\mathbb{R}^2} G_{ij}(x,t;y,\tau) f_j(y,\tau) dy d\tau \tag{48}$$

and p *by the formula*

$$\begin{aligned} p(x,t) &= \int_0^t \int_{\mathbb{R}^2} p_j(x,t;y,\tau) f_j(y,\tau) dy d\tau \\ &= \int_{\mathbb{R}^2} \frac{\partial G_l(x,y)}{\partial y_j} f_j(y,t) dy \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x_j - y_j}{|x-y|^2} f_j(y,t) dy = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{y_j}{|y|^2} f_j(x-y,t) dy. \end{aligned} \tag{49}$$

Then

1. $\mathbf{u} \in C_1^2(\mathbb{R}^2 \times (0,\infty))$, $p \in C_1^2(\mathbb{R}^2 \times (0,\infty))$,
2. $\rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \mu \Delta \mathbf{u} + \mathbf{f}(x,t)$,
3. $\nabla \cdot \mathbf{u} = 0$,
4. $\lim_{t\to 0} u_i(x,t) = 0$ *for each point* $x \in \mathbb{R}^2$.

Proof. 1. Since G_{ij} has a singularity at $(0,0)$, we cannot directly justify differentiating under the integral sign.

First we change variables, to write

$$u_i(x,t) = \int_0^t \int_{\mathbb{R}^2} G_{ij}(y,\tau) f_j(x-y,t-\tau) dy d\tau. \tag{50}$$

As $\mathbf{f} \in C_1^2(\mathbb{R}^2 \times [0,\infty))$ has compact support and $G_{ij} = G_{ij}(y,\tau)$ is bounded near $\tau = t > 0$, we compute

$$\begin{aligned} \frac{\partial u_i(x,t)}{\partial t} = &\int_0^t \int_{\mathbb{R}^2} G_{ij}(y,\tau) \frac{\partial f_j(x-y,t-\tau)}{\partial t} dy d\tau \\ &+ \int_{\mathbb{R}^2} G_{ij}(y,t) f_j(x-y,0) dy d\tau, \end{aligned} \tag{51}$$

$$\frac{\partial u_i(x,t)}{\partial x_k} = \int_0^t \int_{\mathbb{R}^2} G_{ij}(y,\tau) \frac{\partial f_j(x-y,t-\tau)}{\partial x_k} dy d\tau, \tag{52}$$

and

$$\frac{\partial^2 u_i(x,t)}{\partial x_k \partial x_l} = \int_0^t \int_{\mathbb{R}^2} G_{ij}(y,\tau) \frac{\partial^2 f_j(x-y,t-\tau)}{\partial x_k \partial x_l} dy d\tau. \tag{53}$$

Since $\mathbf{f} \in C_1^2(\mathbb{R}^2 \times [0,\infty))$ has compact support D, $G_{ij}(y,\tau)$ is absolutely integrable on $D \times (0,t)$, and $G_{ij}(y,t)$ is absolutely integrable on D for $t > 0$, we see that u_i, $\frac{\partial u_i(x,t)}{\partial t}$, $\frac{\partial u_i(x,t)}{\partial x_k}$, $\frac{\partial^2 u_i(x,t)}{\partial x_k \partial x_l}$ are all continuous on $\mathbb{R}^2 \times (0,\infty)$, i.e., $\mathbf{u} \in C_1^2(\mathbb{R}^2 \times (0,\infty))$.
The proof for p is similar as we note that (a) time derivative can be directly passed into integration; (b) $\frac{\partial G_l}{\partial y_j} = \frac{x_j - y_j}{2\pi|x-y|^2}$ is absolutely integrable on a bounded domain and thus after change of variable spatial derivatives can also be directly passed into the integral sign to f_j.

2. We then calculate

$$\begin{aligned}
&\rho \frac{\partial u_i(x,t)}{\partial t} + \frac{\partial}{\partial x_i} p(x,t) - \mu \Delta_x u_i(x,t) \\
&= \int_0^t \int_{\mathbb{R}^2} G_{ij}(y,\tau)[(\rho \frac{\partial}{\partial t} - \mu \Delta_x) f_j(x-y,t-\tau)] dy d\tau \\
&+ \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{y_j}{|y|^2} \frac{\partial f_j(x-y,t)}{\partial x_i} dy \\
&+ \rho \int_{\mathbb{R}^2} G_{ij}(y,t) f_j(x-y,0) dy d\tau \\
&:= T_1 + T_2 + T_3.
\end{aligned} \tag{54}$$

We now show that $T_1 = -T_2 - T_3 + f_i(x,t)$ and thus the governing equation is satisfied.

$$\begin{aligned}
T_1 &= \int_0^t \int_{B_R(0)} G_{ij}(y,\tau)[(-\rho \frac{\partial}{\partial \tau} - \mu \Delta_y) f_j(x-y,t-\tau)] dy d\tau \\
&= \int_0^\delta \int_{B_R(0)} + \int_\delta^t \int_{B_R(0) - B_\epsilon(0)} + \int_\delta^t \int_{B_\epsilon(0)} \\
&:= J_1 + J_2 + J_3,
\end{aligned} \tag{55}$$

where $B_R(0)$ is a ball of fixed radius R centered at the origin which contains the compact support of $\mathbf{f}$, $B_\epsilon(0)$ is a ball of radius ϵ.

Using the estimate (12), we have

$$\begin{aligned} |J_1| &\leq (\|\mathbf{f}_t\|_{L^\infty} + \|D^2\mathbf{f}\|_{L^\infty}) \int_0^\delta \int_{B_R} \frac{\text{const.}}{|\tau|^\alpha |y|^{2-2\alpha}} dy d\tau \\ &\leq C\delta^{1-\alpha}, \quad 0 < \alpha < 1. \end{aligned} \tag{56}$$

Similarly,

$$\begin{aligned} |J_3| &\leq (\|\mathbf{f}_t\|_{L^\infty} + \|D^2\mathbf{f}\|_{L^\infty}) \int_\delta^t \int_{B_\epsilon} \frac{\text{const.}}{|\tau|^\alpha |y|^{2-2\alpha}} dy d\tau \\ &\leq C\epsilon^{2\alpha}, \quad 0 < \alpha < 1. \end{aligned} \tag{57}$$

Integrating by parts, we have

$$\begin{aligned} J_2 = &\int_\delta^t \int_{B_R - B_\epsilon} [(\rho \frac{\partial}{\partial \tau} - \mu \Delta_y) G_{ij}(y,\tau)] f_j(x-y, t-\tau) dy d\tau \\ &- \rho \int_{B_R - B_\epsilon} G_{ij}(y,t) f_j(x-y,0) dy \\ &+ \rho \int_{B_R - B_\epsilon} G_{ij}(y,\delta) f_j(x-y,t-\delta) dy \\ &- \mu \int_\delta^t \int_{S_\epsilon} G_{ij}(y,\tau) \frac{\partial f_j(x-y,t-\tau)}{\partial y_k} n_k ds_y d\tau \\ &+ \mu \int_\delta^t \int_{S_\epsilon} \frac{\partial G_{ij}(y,\tau)}{\partial y_k} f_j(x-y,t-\tau) n_k ds_y d\tau \\ &:= 0 - T_3 + K_1 + K_2 + K_3, \end{aligned} \tag{58}$$

where the first term on the right side vanishes since G_{ij} satisfies the heat equation.

we have

$$
\begin{aligned}
\lim_{\delta\to 0} K_1 &= \rho \int_{B_R - B_\epsilon} G_{ij}(y,0) f_j(x-y,t) dy \\
&= \int_{B_R - B_\epsilon} \frac{\partial^2 G_l(y)}{\partial y_i \partial y_j} f_j(x-y,t) dy \\
&= \int_{B_R - B_\epsilon} \left[\frac{\partial}{\partial y_i} \left(\frac{\partial G_l(y)}{\partial y_j} f_j(x-y,t) \right) - \frac{\partial G_l(y)}{\partial y_j} \frac{\partial f_j(x-y,t)}{\partial y_i} \right] dy \\
&= -\int_{S_\epsilon} \frac{\partial G_l(y)}{\partial y_j} f_j(x-y,t) n_i ds_y + \int_{B_R - B_\epsilon} \frac{\partial G_l(y)}{\partial y_j} \frac{\partial f_j(x-y,t)}{\partial x_i} dy \\
&= \frac{1}{2\pi} \int_{S_\epsilon} \frac{y_j}{|y|^2} f_j(x-y,t) n_i ds_y - \frac{1}{2\pi} \int_{B_R - B_\epsilon} \frac{y_j}{|y|^2} \frac{\partial f_j(x-y,t)}{\partial x_i} dy \\
&\to \frac{1}{2} f_i(x,t) - T_2, \quad \text{as} \quad \epsilon \to 0.
\end{aligned}
\tag{59}
$$

Using the estimate (12), we have

$$
\begin{aligned}
|K_2| &\le C \|D\mathbf{f}_t\|_{L^\infty} \int_\delta^t \int_{S_\epsilon} \frac{\text{const.}}{|\tau|^\alpha |\epsilon|^{2-2\alpha}} |ds_y| d\tau \\
&\le C\epsilon^{2\alpha - 1}, \quad 1/2 < \alpha < 1.
\end{aligned}
\tag{60}
$$

Using (14), we have

$$
\begin{aligned}
K_3 &\to \delta_{ij} \int_\delta^t f_j(x, t-\tau) \frac{\epsilon^2 e^{-\epsilon^2/4\nu\tau}}{8\nu\tau^2} d\tau \\
&\to \int_0^t f_i(x, t-\tau) \frac{\epsilon^2 e^{-\epsilon^2/4\nu\tau}}{8\nu\tau^2} d\tau \quad \text{as} \quad \delta \to 0 \\
&= \frac{1}{2} \int_{\epsilon^2/4\nu t}^\infty f_i(x, t - \epsilon^2/4\nu\lambda) e^{-\lambda} d\lambda \\
&\to \frac{1}{2} f_i(x,t), \quad \text{as} \quad \epsilon \to 0.
\end{aligned}
\tag{61}
$$

Combining (54)–(61), we obtain

$$
\rho \frac{\partial u_i(x,t)}{\partial t} + \frac{\partial}{\partial x_i} p(x,t) - \mu \Delta_x u_i(x,t) = f_i(x,t). \tag{62}
$$

3. The proof of divergence free condition is entirely similar.

4. Finally, we note

$$\begin{aligned} \|\mathbf{u}(\cdot,t)\|_{L^\infty} &\le \|\mathbf{f}\|_{L^\infty} \int_0^t \int_{B_R} \frac{\text{const.}}{|\tau|^\alpha |y|^{2-2\alpha}} dy d\tau, \quad 0<\alpha<1 \\ &\le C\|\mathbf{f}\|_{L^\infty} t^{1-\alpha} \to 0 \quad \text{as} \quad t \to 0. \end{aligned} \tag{63}$$

□

Remark 5.2. *We can certainly combine Theorems 4.1 and 5.1 to solve an initial value problem with nonhomogeneous term.*

6 Dirichlet problem and properties of the double layer potential

Theorem 6.1. (Solution of the Dirichlet problem). *Suppose that* $\mathbf{f}$ *is a continuous function on* $S \times [0,T]$ *satisfying the following compatibility conditions*

$$\begin{cases} \mathbf{f}(x,0) = \mathbf{0}, & x \in S, \\ \displaystyle\int_S \mathbf{f}(y,t)\cdot n(y) ds(y) = 0, & t \in [0,T]. \end{cases} \tag{64}$$

Suppose further that $\boldsymbol{\phi}$ *is a continuously differentiable function on* $S \times [0,T]$*. Define the double layer potential for the velocity by the formula:*

$$u_i(x,t) = \int_0^t \int_S T_{ij}(x,t;y,\tau)\phi_j(y,\tau) ds_y d\tau, \tag{65}$$

where the kernel T_{ij} *is defined in* (34)*.*
Define the associated pressure field by the formula:

$$\begin{aligned} p(x,t) = \rho \int_S G_l(x,y) n_j(y) \frac{\partial \phi_j(y,t)}{\partial t} dy \\ + 2\mu \int_S \frac{\partial^2 G_l(x,y)}{\partial y_j \partial y_k} n_k(y)\phi_j(y,t) dy. \end{aligned} \tag{66}$$

Then

1.

$$\rho \frac{\partial \mathbf{u}(x,t)}{\partial t} = -\nabla p(x,t) + \mu \Delta \mathbf{u}(x,t), \quad (x,t) \in D \times (0,T]; \tag{67}$$

2.

$$\nabla \cdot \mathbf{u}(x,t) = 0, \quad (x,t) \in D \times (0,T]; \tag{68}$$

3. for $x \in S$, the integrals in the definition of $\mathbf{u}$ exist in the usual sense;

4. $\mathbf{u}$ satisfies the following jump relation:

$$\lim_{\epsilon \to 0+} u_i(x \pm \epsilon n(x), t) = u_i(x,t) \pm \frac{1}{2}\phi_i(x,t), \quad x \in S; \tag{69}$$

5. $\mathbf{u}$ satisfies the following boundary condition

$$\mathbf{u}(x,t) = \mathbf{f}(x,t), \quad (x,t) \in S \times (0,T] \tag{70}$$

if the density $\boldsymbol{\phi}$ satisfies the equations:

$$-\frac{1}{2}\phi_i(x,t) + \int_0^t \int_S T_{ij}(x,t;y,\tau)\phi_j(y,\tau)ds_y d\tau = f_i(x,t), \quad i = 1,2. \tag{71}$$

Proof. 1. We first show the velocity defined by (72) and the pressure defined by (66) satisfy the governing unsteady Stokes equation.

Substituting (34) into (65), we obtain

$$\begin{aligned} u_i(x,t) = &\int_S \frac{x_i - y_i}{2\pi|x-y|^2} n_j(y)\phi_j(y,t)ds_y \\ &- \mu \int_S \int_0^t \left(\frac{\partial G_{ij}}{\partial x_k} + \frac{\partial G_{ik}}{\partial x_j} \right) n_k(y)\phi_j(y,\tau)d\tau ds_y \end{aligned} \tag{72}$$

For $x \in D$, differentiation can be passed into integration and we calculate

$$\begin{aligned} &\rho \frac{\partial u_i(x,t)}{\partial t} = \rho \int_S \frac{\partial G_l(x,y)}{\partial y_i} n_j(y) \frac{\partial \phi_j(y,t)}{\partial t} ds_y \\ &- \rho\mu \int_S \left(\frac{\partial G_{ij}(x,t;y,t)}{\partial x_k} + \frac{\partial G_{ik}(x,t;y,t)}{\partial x_j} \right) n_k(y)\phi_j(y,t)ds_y \\ &- \rho\mu \int_S \int_0^t \left(\frac{\partial^2 G_{ij}(x,t;y,\tau)}{\partial x_k \partial t} + \frac{\partial^2 G_{ik}(x,t;y,\tau)}{\partial x_j \partial t} \right) n_k(y)\phi_j(y,\tau)d\tau ds_y \\ &= \rho \int_S \frac{\partial G_l(x,y)}{\partial y_i} n_j(y) \frac{\partial \phi_j(y,t)}{\partial t} ds_y \\ &- 2\mu \int_S \frac{\partial^3 G_l(x,y)}{\partial x_i \partial x_j \partial x_k} n_k(y)\phi_j(y,t)ds_y \\ &- \rho\mu \int_S \int_0^t \left(\frac{\partial^2 G_{ij}(x,t;y,\tau)}{\partial x_k \partial t} + \frac{\partial^2 G_{ik}(x,t;y,\tau)}{\partial x_j \partial t} \right) n_k(y)\phi_j(y,\tau)d\tau ds_y, \end{aligned}$$

$$(73)$$

where the second equality follows (8). Similarly,

$$\begin{aligned}
-\mu\Delta u_i(x,t) &= -\mu\int_S \frac{\partial}{\partial x_i}\Delta_x G_l(x,y)n_j(y)\phi_j(y,t)ds_y \\
&\quad + \mu^2\int_S\int_0^t\left(\frac{\partial\Delta_x G_{ij}}{\partial x_k}+\frac{\partial\Delta_x G_{ik}}{\partial x_j}\right)n_k(y)\phi_j(y,\tau)d\tau ds_y \\
&= \rho\mu\int_S\int_0^t\left(\frac{\partial^2 G_{ij}}{\partial x_k\partial t}+\frac{\partial^2 G_{ik}}{\partial x_j\partial t}\right)n_k(y)\phi_j(y,\tau)d\tau ds_y,
\end{aligned} \tag{74}$$

where the second equality follows from the fact that G_l is harmonic for $x \neq y$ and G_{ij} satisfy the heat equation for $x \neq y$ and $t \neq \tau$. Also,

$$\begin{aligned}
\frac{\partial p(x,t)}{\partial x_i} &= \rho\int_S \frac{\partial G_l(x,y)}{\partial x_i}n_j(y)\frac{\partial\phi_j(y,t)}{\partial t}ds_y \\
&\quad + 2\mu\int_S \frac{\partial^3 G_l(x,y)}{\partial x_i\partial y_j\partial y_k}n_k(y)\phi_j(y,t)ds_y \\
&= -\rho\int_S \frac{\partial G_l(x,y)}{\partial y_i}n_j(y)\frac{\partial\phi_j(y,t)}{\partial t}ds_y \\
&\quad + 2\mu\int_S \frac{\partial^3 G_l(x,y)}{\partial x_i\partial x_j\partial x_k}n_k(y)\phi_j(y,t)ds_y.
\end{aligned} \tag{75}$$

Adding the above three equations (73)-(75), we obtain

$$\rho\frac{\partial u_i(x,t)}{\partial t}+\frac{\partial p(x,t)}{\partial x_i}-\mu\Delta u_i(x,t)=0. \tag{76}$$

2. The fact that the velocity is divergence free can be easily proved by a similar computation.

3. We now show that even for $x \in S$, the integrals in (72) exist in the usual sense.
   We divide the kernel in the second integral in (72) into two parts:

$$-\mu\left(\frac{\partial G_{ij}}{\partial x_k}+\frac{\partial G_{ik}}{\partial x_j}\right)n_k(y) = K_{ij}(x,t;y,\tau)+\tilde{K}_{ij}(x,t;y,\tau), \tag{77}$$

where K_{ij} and $\tilde{K}_{ij}$ are defined by the formulae

$$K_{ij}(x,t;y,\tau) = \frac{n_i(y)(x_j-y_j)}{2\pi|x-y|^2}\frac{|x-y|^2 e^{-\frac{|x-y|^2}{4\nu(t-\tau)}}}{4\nu(t-\tau)^2}$$
$$-\frac{n_i(y)(x_j-y_j)+n_j(y)(x_i-y_i)}{2\pi|x-y|^2}\frac{4\nu}{|x-y|^2}\left(1-e^{-\frac{|x-y|^2}{4\nu(t-\tau)}}-\frac{|x-y|^2}{4\nu(t-\tau)}e^{-\frac{|x-y|^2}{4\nu(t-\tau)}}\right), \tag{78}$$

$$\tilde{K}_{ij}(x,t;y,\tau) = \frac{(x_k-y_k)n_k(y)}{2\pi|x-y|^2}\left[\left(\delta_{ij}-\frac{2(x_i-y_i)(x_j-y_j)}{|x-y|^2}\right)\frac{|x-y|^2 e^{-\frac{|x-y|^2}{4\nu(t-\tau)}}}{4\nu(t-\tau)^2}\right.$$
$$\left.-\left(\delta_{ij}-\frac{4(x_i-y_i)(x_j-y_j)}{|x-y|^2}\right)\frac{4\nu}{|x-y|^2}\left(1-e^{-\frac{|x-y|^2}{4\nu(t-\tau)}}-\frac{|x-y|^2}{4\nu(t-\tau)}e^{-\frac{|x-y|^2}{4\nu(t-\tau)}}\right)\right]. \tag{79}$$

Combining (72), (77), (79) with a further splitting, we obtain

$$\begin{aligned} u_i(x,t) &= \int_S B_{ij}(x,y)\phi(y,t)ds_y \\ &\quad + \int_S\int_0^t K_{ij}(x,t;y,\tau)[\phi_j(y,\tau)-\phi_j(y,t)]d\tau ds_y \\ &\quad + \int_S\int_0^t \tilde{K}_{ij}(x,t;y,\tau)\phi_j(y,\tau)d\tau ds_y, \end{aligned} \tag{80}$$

where B_{ij} is defined by the formula

$$B_{ij}(x,y) = \frac{(x_i-y_i)n_j(y)}{2\pi|x-y|^2} + \int_0^t K_{ij}(x,t;y,\tau)d\tau. \tag{81}$$

(a) For fixed $x \in S$, we claim that B_{ij} is absolutely integrable on S as a function of y.
First, applying change of variable $\lambda = \frac{|x-y|^2}{4\nu(t-\tau)}$, we have $d\lambda = \frac{|x-y|^2}{4\nu(t-\tau)^2}d\tau$,

$$\begin{aligned} \int_0^t \frac{|x-y|^2 e^{-\frac{|x-y|^2}{4\nu(t-\tau)}}}{4\nu(t-\tau)^2}d\tau &= \int_{|x-y|^2/4\nu t}^{\infty} e^{-\lambda}d\lambda \\ &= e^{-\frac{|x-y|^2}{4\nu t}}, \end{aligned} \tag{82}$$

and

$$\int_0^t \frac{4\nu}{|x-y|^2}\left(1-e^{-\frac{|x-y|^2}{4\nu(t-\tau)}}-\frac{|x-y|^2}{4\nu(t-\tau)}e^{-\frac{|x-y|^2}{4\nu(t-\tau)}}\right)d\tau$$
$$=\int_{|x-y|^2/4\nu t}^{\infty}\frac{1-e^{-\lambda}-\lambda e^{-\lambda}}{\lambda^2}d\lambda \tag{83}$$
$$=\frac{1-e^{-\frac{|x-y|^2}{4\nu t}}}{|x-y|^2/4\nu t}.$$

Combining (79), (81), (82), and (83), we obtain

$$B_{ij}(x,y)=\frac{(x_i-y_i)n_j(y)}{2\pi|x-y|^2}+\int_0^t K_{ij}(x,t;y,\tau)d\tau$$
$$=\frac{1}{2\pi|x-y|^2}\Bigg((x_i-y_i)n_j(y)+(x_j-y_j)n_i(y)e^{-\frac{|x-y|^2}{4\nu t}}$$
$$-[(x_j-y_j)n_i(y)+(x_i-y_i)n_j(y)]\frac{1-e^{-\frac{|x-y|^2}{4\nu t}}}{|x-y|^2/4\nu t}\Bigg). \tag{84}$$

Since $e^{-\lambda}\to 1$ and $(1-e^{-\lambda})/\lambda\to 1$ as $\lambda\to 0$, we see the $B_{ij}(x,y)\le M|x-y|$ and thus is absolutely integrable on S.

(b) Next, for fixed $x\in S$ and $t>0$, we show that $\tilde{K}_{ij}$ is absolutely integrable on $S\times(0,t)$ and thus the associated integral operator is compact. First, a well known result in classical potential theory (see, for example, page 79 of [25]) states that

$$|(x_k-y_k)n_k(y)|\le M|x-y|^2. \tag{85}$$

Second, it is obvious that

$$|(x_i-y_i)(x_j-y_j)|\le M|x-y|^2. \tag{86}$$

Third, note that for any $\lambda>0$, $\lambda^\phi e^{-\lambda}\le M$ for any $\phi\ge 0$ and $|(1-e^{-\lambda}-\lambda e^{-\lambda})/\lambda^\phi|\le M$ for any $0<\phi\le 2$. With $\lambda=|x-y|^2/4\nu(t-\tau)$, we easily see that

$$\left|\frac{|x-y|^2e^{-\frac{|x-y|^2}{4\nu(t-\tau)}}}{4\nu(t-\tau)^2}\right|\le\frac{M}{|t-\tau|^\alpha|x-y|^{2-2\alpha}},\quad 1/2<\alpha<1 \tag{87}$$

and

$$\begin{aligned}&\left|\frac{4\nu}{|x-y|^2}\left(1-e^{-\frac{|x-y|^2}{4\nu(t-\tau)}}-\frac{|x-y|^2}{4\nu(t-\tau)}e^{-\frac{|x-y|^2}{4\nu(t-\tau)}}\right)\right|\\&\leq\frac{M}{|t-\tau|^\alpha|x-y|^{2-2\alpha}},\quad 1/2<\alpha<1.\end{aligned}\tag{88}$$

Combining (79), (85)–(88), we obtain

$$|\tilde{K}_{ij}(x,t;y,\tau)|\leq\frac{M}{|t-\tau|^\alpha|x-y|^{2-2\alpha}},\quad 1/2<\alpha<1,\tag{89}$$

and thus $\tilde{K}_{ij}$ is absolutely integrable on $S\times(0,t)$.

(c) Since $\boldsymbol{\phi}$ is continuously differentiable, it is Holder continuous with exponent β $(0<\beta<1)$. Then a similar derivation shows that

$$|K_{ij}(x,t;y,\tau)[\phi_j(y,\tau)-\phi_j(y,t)]|\leq\frac{M}{|t-\tau|^\alpha|x-y|^{3-2\alpha-\beta}},\quad 1-\beta/2<\alpha<1,\tag{90}$$

and we see that $K_{ij}(x,t;y,\tau)[\phi_j(y,\tau)-\phi_j(y,t)]$ is absolutely integrable on $S\times(0,t)$.

4. We now derive the jump relation of the double layer potential. We write

$$\begin{aligned}u_i(x,t)=&\int_S B_{ij}(x,y)\phi_j(y,t)ds_y\\&+\int_S\int_0^t K_{ij}(x,t;y,\tau)[\phi_j(y,\tau)-\phi_j(y,t)]d\tau ds_y\\&+\int_S\int_0^t \tilde{K}_{ij}(x,t;y,\tau)[\phi_j(y,\tau)-\phi_j(y,t)]d\tau ds_y\\&+\int_S\int_0^t \tilde{K}_{ij}(x,t;y,\tau)\phi_j(y,t)d\tau ds_y.\end{aligned}\tag{91}$$

From the above estimates about B_{ij}, K_{ij}, and $\tilde{K}_{ij}$, it is easy to see that the first three integrals in the above equation are continuous across the boundary S. We will denote the last integral by $J_i(x,t)$, i.e.,

$$J_i(x,t)=\int_S\int_0^t \tilde{K}_{ij}(x,t;y,\tau)\phi_j(y,t)d\tau ds_y.\tag{92}$$

Integrating over τ first, we find

$$
\begin{aligned}
J_i(x,t) &= \int_S \frac{(x_k-y_k)n_k(y)}{2\pi|x-y|^2}\left[\left(\delta_{ij}-\frac{2(x_i-y_i)(x_j-y_j)}{|x-y|^2}\right)e^{-\frac{|x-y|^2}{4\nu t}}\right.\\
&\left.-\left(\delta_{ij}-\frac{4(x_i-y_i)(x_j-y_j)}{|x-y|^2}\right)\frac{1-e^{-\frac{|x-y|^2}{4\nu t}}}{|x-y|^2/4\nu t}\right]\phi_j(y,t)ds_y\\
&= \int_S \frac{(x_k-y_k)n_k(y)}{2\pi|x-y|^2}\phi_i(y,t)\left[e^{-\frac{|x-y|^2}{4\nu t}}-\frac{1-e^{-\frac{|x-y|^2}{4\nu t}}}{|x-y|^2/4\nu t}\right]ds_y\\
&+\int_S \frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)n_k(y)}{\pi|x-y|^4}\left[2\frac{1-e^{-\frac{|x-y|^2}{4\nu t}}}{|x-y|^2/4\nu t}-e^{-\frac{|x-y|^2}{4\nu t}}\right]\phi_j(y,t)ds_y\\
&:= J_i^1(x,t)+J_i^2(x,t).
\end{aligned}
\tag{93}
$$

Since $\frac{(x_k-y_k)n_k(y)}{2\pi|x-y|^2}=\frac{\partial G_l(x,y)}{\partial n(y)}$ and the jump relation of the double layer potential of the Laplace equation (see, for example, page 80 of [25]) states that

$$
\lim_{\epsilon\to 0+}\int_S \frac{\partial G_l(x\pm\epsilon n(x),y)}{\partial n(y)}\phi(y)ds_y=\int_S \frac{\partial G_l(x,y)}{\partial n(y)}\phi(y)ds_y\pm\frac{1}{2}\phi(x),\quad x\in S,
\tag{94}
$$

we find that

$$
\lim_{\epsilon\to 0+} J_i^1(x\pm\epsilon n(x),t)=J_i^1(x,t),\quad x\in S,
\tag{95}
$$

as

$$
\lim_{y\to x}\left[e^{-\frac{|x-y|^2}{4\nu t}}-\frac{1-e^{-\frac{|x-y|^2}{4\nu t}}}{|x-y|^2/4\nu t}\right]=0.
\tag{96}
$$

Similarly, we find that (see, for example, page 61 of [37])

$$
T_{ijk}^S=4\frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)n_k(y)}{|x-y|^4},
\tag{97}
$$

where T_{ijk}^S is the stress tensor for 2D steady Stokes flow. We also find (see, for example, page 61 of [37]) that jump relation for the stress

tensor T^S_{ijk} is as follows:

$$\lim_{\epsilon\to 0+}\int_S T^S_{ijk}(x\pm\epsilon n(x),y)n_k(y)\phi_j(y)ds_y = \pm 2\pi\phi_i(x)+\int_S T^S_{ijk}(x,y)n_k(y)\phi_j(y)ds_y. \tag{98}$$

Note that we have changed notations in [37] so that (97) and (98) match our notation. Note further that

$$\lim_{y\to x}\left[2\frac{1-e^{-\frac{|x-y|^2}{4\nu t}}}{|x-y|^2/4\nu t}-e^{-\frac{|x-y|^2}{4\nu t}}\right]=1. \tag{99}$$

Combining (88), (92)-(94), we obtain

$$\lim_{\epsilon\to 0+} J_i^2(x\pm\epsilon n(x),t) = J_i^2(x,t)\pm\frac{1}{2}\phi_i(x,t), \quad x\in S. \tag{100}$$

Combining (81)-(83), (95), (100), we obtain the jump relation of the double layer potential for the unsteady Stokes flow as follows:

$$\lim_{\epsilon\to 0+} u_i(x\pm\epsilon n(x),t) = u_i(x,t)\pm\frac{1}{2}\phi_i(x,t), \quad x\in S. \tag{101}$$

5. This simply follows from the jump relation of the double layer potential.

□

Remark 6.2. *For exterior problems, the boundary integral equations have the form (as compared with* (71) *for interior problem)*

$$\frac{1}{2}\phi_i(x,t)+\int_0^t\int_S T_{ij}(x,t;y,\tau)\phi_j(y,\tau)ds_y d\tau = f_i(x,t), \quad i=1,2. \tag{102}$$

7 Neumann problem and properties of the single layer potential

Define the single layer potential for the velocity by the formula

$$u_i(x,t)=\int_0^t\int_S G_{ij}(x,t;y,\tau)\phi_j(y,\tau)ds_y d\tau, \tag{103}$$

and the associated pressure by the formula

$$\begin{aligned} p(x,t) &= \int_0^t \int_S p_j(x,t;y,\tau)\phi_j(y,\tau)ds_y d\tau \\ &= \int_S \frac{\partial G_l(x,y)}{\partial y_j}\phi_j(y,t)ds_y. \end{aligned} \tag{104}$$

We first have the following theorem regarding the single layer potential itself.

Theorem 7.1. *The kernel G_{ij} is absolutely integrable as a function of y and τ for $x \in \mathbb{R}^2$. Thus the single layer potential operator is compact from $C(S \times [0,T])$ to $C(S \times [0,T])$ and the single layer potential is continuous across the boundary.*

Proof. The absolute integrability simply follows from the estimate (12) and the subsequent statement in Lemma 2.1. Since the kernel G_{ij} is absolute integrable, the single layer potential operator is compact. The fact that the potential is continuous across the boundary S can also easily be shown using the fact that the estimate (12) is true for both $x \in D$ and $x \in S$. □

It is also easy to see that the following theorem holds by a similar calculation as in Section 6.

Theorem 7.2. *Suppose that $\boldsymbol{\phi}$ is a continuous function on $S \times [0,T]$ and that $\mathbf{u}$ and p are defined by* (103) *and* (104)*, respectively. Then*

1.
$$\rho\frac{\partial \mathbf{u}(x,t)}{\partial t} = -\nabla p(x,t) + \mu\Delta\mathbf{u}(x,t), \quad x \notin S, \quad t \in (0,T], \tag{105}$$

2.
$$\nabla \cdot \mathbf{u}(x,t) = 0, \quad x \notin S, \quad t \in (0,T], \tag{106}$$

3.
$$\lim_{t\to 0} u_i(x,t) = 0, \quad x \in \mathbb{R}^2, \quad i = 1,2. \tag{107}$$

In the Neumann problem of fluid dynamics, the surface force is specified on the boundary, that is,

$$\left[\mu\left(\frac{\partial u_i(x,t)}{\partial x_k} + \frac{\partial u_k(x,t)}{\partial x_i}\right) - p(x,t)\delta_{ik}\right] n_k(x) = f_i(x,t), \quad (x,t) \in S \times (0,T].$$

(108)

Thus we will not consider the jump relation of the normal derivative of the single layer potential. Instead, we consider the stress tensor

$$\begin{aligned}\sigma_{ik}(x,t) &= \mu\left(\frac{\partial u_i(x,t)}{\partial x_k} + \frac{\partial u_k(x,t)}{\partial x_i}\right) - p(x,t)\delta_{ik} \\ &= \int_0^t \int_S T^*_{ijk}(x,t;y,\tau)\phi_j(y,\tau)ds_y d\tau,\end{aligned} \tag{109}$$

where T^*_{ijk} is defined by the formula

$$T^*_{ijk}(x,t;y,\tau) = \mu\left(\frac{\partial G_{ij}(x,t;y,\tau)}{\partial x_k} + \frac{\partial G_{kj}(x,t;y,\tau)}{\partial x_i}\right) - p_j(x,t;y,\tau)\delta_{ik}. \tag{110}$$

We also introduce T^*_{ij} given by the formula

$$\begin{aligned}T^*_{ij}(x,t;y,\tau) &= T^*_{ijk}(x,t;y,\tau)n_k(x) \\ &= \left[\mu\left(\frac{\partial G_{ij}(x,t;y,\tau)}{\partial x_k} + \frac{\partial G_{kj}(x,t;y,\tau)}{\partial x_i}\right) - p_j(x,t;y,\tau)\delta_{ik}\right] n_k(x) \\ &= T_{ji}(y,t;x,\tau),\end{aligned} \tag{111}$$

where T_{ij} is defined by (34).
We have the following theorem regarding the jump relation of the single layer potential in a form that is suitable for handling the boundary condition (108). Its proof is entirely similar to that of Theorem 6.1.

Theorem 7.3. (Jump relation of the single layer potential operator) *Suppose that $\boldsymbol{\phi}$ is a continuously differentiable function on $S \times [0,T]$. Then*

1. *the integrals $\int_S \int_0^t T^*_{ij}(x,t;y,\tau)\phi_j(y,\tau)ds_y d\tau$ $(i = 1,2)$ exist in the usual sense for $x \in S$,*

2. *for $x \in S$, the following jump relation holds:*

$$\lim_{\epsilon \to 0+} \sigma_{ik}(x \pm \epsilon n_k(x),t)n_k(x) = \sigma_{ik}(x,t)n_k(x) \mp \frac{1}{2}\phi_i(x,t), \tag{112}$$

3. $\mathbf{u}$ *and* p *are solutions of initial-boundary value problem specified by equations* (105)–(108) *if the density* $\boldsymbol{\phi}$ *satifies the following equations:*

$$\frac{1}{2}\phi_i(x,t) + \int_0^t \int_S T^*_{ij}(x,t;y,\tau)\phi_j(y,\tau)ds_y d\tau = f_i(x,t), \quad i = 1,2. \tag{113}$$

Remark 7.4. *For exterior problems, the boundary integral equations have the form (as compared with* (113) *for interior problems)*

$$-\frac{1}{2}\phi_i(x,t) + \int_0^t \int_S T^*_{ij}(x,t;y,\tau)\phi_j(y,\tau)ds_y d\tau = f_i(x,t), \quad i = 1,2. \tag{114}$$

Remark 7.5. *The existence and uniqueness of the solution to the boundary integral equation* (113) *associated with interior Neumann problem was proved in [38] for* $f_i \in L^2(S)$ *(*$i = 1,2$*) when* S *is a Lipschitz curve. For exterior Neumann problems, simple integration by parts shows that the nullspace of* (114) *contains all functions of the form* $g(t)n(x)$ *with* $g(t)$ *an arbitrary function in* $L^2([0,T])$ *and* $n(x)$ *the unit outward normal function on* S*. Since the kernel* T^*_{ij} *is adjoint to* T_{ij} *in spatial variable, it is easy to see that the boundary integral equation* (102) *associated with exterior Dirichlet problem has a unique solution; and that the boundary integral equation* (71) *associated with interior Dirichlet problem has a nullspace containing functions of the form* $g(t)\eta(x)$ *with* $g(t)$ *an arbitrary function in* $L^2([0,T])$ *and* η *a particular function in* $L^2(S)$*. Finally, the existence of the solution to* (71) *can be proved in a similar fasion as in [38].*

8 Conclusions and discussions

We have analyzed the properties of the unsteady Stokeslet and developed an analytical machinery for unsteady Stokes flows in two dimensions. It is shown that the unsteady Stokeslet is a linear combination of the Laplace kernel and the heat kernel (and their derivatives and integrals). The initial value problem, the nonhomogeneous problem, the Dirichlet problem, and the Neumann problem have been analyzed and each of them has a simple integral equation formulation via the initial potential, the volume potential, double layer potential, and single layer potential, respectively. Since the problem is linear, the more general initial-boundary value problem with nonhomogeneous problem can be solved via integral equations by simple superpositions.

The analysis here can be easily carried out for three dimensional unsteady Stokes flows and will be discussed in a subsequent paper. The numerical applications of these integral equation formulations are also currently under investigation and will be reported on a later date. We would like to point out that a full numerical machinery for solving the heat equation using integral equations has been developed (see, for example, [19, 28, 15, 43, 44]). The resulting algorithm is accurate, robust, and nearly optimal in terms of computational speed and storage requirement. Though details will be different, we expect that the numerical algorithm for the unsteady Stokes flow based on the integral equation formulations developed in this paper will also be accurate, robust, and nearly optimal in computational complexity. Finally, we expect that the integral equation formulations developed in this paper can be used to solve the fully nonlinear Navier-Stokes equation in a simlilar fasion as in [12].

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