ON THE DIMENSION OF SOLUTIONS OF NONLINEAR EQUATIONS

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ABSTRACT. We study the covering dimension of (positive) solutions to varoius classes of nonlinear equations based on the nontriviality of the fixed point index of a certain condensing map. Applications to semilinear equations and to nonlinear perturbations of the Wiener-Hopf integral equations are given.

1. Introduction

Let K be a retract of a Banach space X (e.g., K is a closed and convex subset of X, say a cone). Then K is closed. Let $D \subset R^m \times K$ be an open bounded subset and $F : \overline{D} \subset R^m \times K \to X$ be continuous and ϕ-condensing. Our objective is to prove that the set of positive solutions of the equation

$$x - F(\lambda, x) = f \tag{1.1}$$

has the covering dimension at least m under suitable conditions on F. Equation (1.1) is undetermined and under suitable assumptions we shall prove that its set of positive solutions has a covering dimension at least m. Begining with the work of Fitzpatrick-Massabo-Pejsachowicz [2], when $K = X$, equation (1.1) has been studied by many authors using algebraic topology arguments (see [2], [5], [9,11] for an extensive literature on the subject). In this work, we shall prove our results using the fixed point index method for multivalued ϕ-condensing maps instead in conjunction with the selection results of Michael [7] and Saint-Raymond [15]. In this approach, we introduce a notion of a complementing map by a continuous multivalued compact map. It differs from the notion of complementing maps by a finite dimensional single valued map introduced in [2]. We prove that if a certain restriction of F has a nonzero fixed point index, then $I - F$ is a complementing map. When $K = X$, a degree argument for condensing maps has been used by Gelman [3] to study the dimension of the fixed point set of such maps. Still another approach based only on the above mentioned selection theorems was used by Ricceri [14] to study the dimension of the solution set of certain compact perturbations of surjective linear maps. In the second part of Section 2, we prove the dimension of the solution set of a semilinear equation of the form

$$Lx - Nx = f \tag{1.2}$$

is bigger then or equal then the dimension of the null space of the linear surjection $L : X \to Y$, where N is a k-set contraction, or is such that $L + N : X \to Y$ is A-proper. When N is a k-contraction, we have that the solution set is also arcwise connected. An application of our results is given to a nonlinear perturbation of the Wiener-Hopf equation in Section 3.

2. Dimension results.

Recall that if D is a topological space, and k is a positive integer, then D has the covering dimension equal to k provided that k is the smallest integer with the property that whenever U is a family of open subsets of D whose union covers D, there exists a refinement, U', of U whose union also covers D and no subfamily of U' consisting of more than k+1 members has nonempty intersection. If D fails to have this refinment property for each positive integer, then D is said to have infinite dimension. In the absence of a manifold structure on D, the concept of dimension is a natural way in which to describe its size.

Let X be a Banach space X and K(X) be closed convex subsets of X. We need the following continuous selection results of Michael [7] and Saint-Raymond [15]

Theorem 2.1 *a) ([7]) Let T be a paracompact topological space, X be a Banach space and $G : T \to K(X)$ be a lower semicontinuous multivalued map. Then, for each closed subset A of T and each continuous selection ψ of $G|_A$, there is a continuous selection ϕ of G such that $\phi|_A = \psi$.*

b) ([15]) Let Y be a compact metrisable subspace of dimension at most $n-1$ of a Banach space X, $F : Y \to K(X)$ be a multivalued lower semicontinuous map such that $0 \in F(x)$ and dim $F(x) \geq n$ for each $x \in Y$. Then there is a continuous selection f of F such that $f(x) \neq 0$ for all $x \in Y$.

Let K be a retract of a Banach space X (e.g., K is a closed and convex subset of X, say a cone). Let $CV(K)$ be compact and convex subsets of K, $D \subset K$ be an open subset of K (in the relative topology on K).

Recall that the *set measure of noncompactness* of a bounded set $D \subset X$ is defined as $\gamma(D) = \inf\{d > 0 \mid D$has a finite covering by sets of diameter less than d$\}$. The *ball-measure of noncompactness* of D is defined as $\chi(D) = \inf\{r > 0 \mid D \subset U_{i=1}^{n} B(x_i, r),\ x \in X, n \in N\}$. Let ϕ denote either the set or the ball measure of noncompactnes. Then a mapping $T : D \subset X \to X$ is said to be k-ϕ-*contractive (ϕ-condensing)* if $\phi(T(Q)) \leq k\phi(Q)$ (respectively, $\phi(T(Q)) < \phi(Q)$) whenever $Q \subset D$ (with $\phi(Q) \neq 0$). Recall also that a map $T : D \subset X \to Y$ is A-proper w.r.t. a projectional scheme $\Gamma = \{X_n, P_n, Y_n, Q_n\}$ for (X,Y) if $Q_n T : D \cap X_n \to Y_n$ is continuous for each n and whenever $\{x_{n_k} \in D \cap X_{n_k}\}$ is bounded and $\|Q_{n_k} T x_{n_k} - Q_{n_k}\| \to 0$, a subsequence $x_{n_k(i)} \to x$ with Tx=f.

Theorem 2.2 *Let $F : \overline{D} \subset K \to CV(K)$ be an upper semicontinuous condensing map, $x \notin F(x)$ for each $x \in \partial D$ and the fixed point index $i(F, D, K) \neq 0$. Suppose that there is an open neighborhood U in K with Fix $(F, D) \subset U \subset D$ and a lower semicontinuous map $G : U \to CV(X)$ such that $G(x) \subset F(x)$, dim $G(x) \geq n$ for each $x \in U$ and $x \in G(x)$ for each $x \in$Fix (F, D). Then dim Fix (F,D)$\geq n$.*

Proof. Suppose that the claim is false, i.e., dim Fix $(F, D) \leq n-1$. Since F is u.s.c. and condensing, it is easy to show that Fix (F, D) is a compact metric subspace of X. Let $H : U \to CK(X)$ be given by $H = I - G$ and

$H_1 = I - G|_{\text{Fix }(F,D)}$. Since G is lower semicontinuous from K to CK(X), then $H_1 : \text{Fix}(F, D) \to CV(X)$ is lower semicontinuous from K to CV(X), $0 \in H_1(x)$ and dim $H_1(x) \geq n$ for each $x \in$ Fix (F, D). Then, by Saint Raymond's Theorem 2.1-b) there is a continuous selection $h_1 : \text{Fix }(F, D) \to X$ of H_1, with $h_1 = I - f_1 : \text{Fix }(F, D) \to X$, $0 \neq h_1(x) \in H_1(x)$ for each $x \in$ Fix (F, D). Since U is paracompact and $H : U \to CV(X)$ is lower semicontinuous and U is paracompact, by Michael's Theorem 2.1-a) there is a continuous selection $h : U \to X$, $h(x) \in H(x)$ for each $x \in U$, such that $h|_{\text{Fix }(F,\overline{D})} = h_1$ and $h(x) \neq 0$ for each $x \in U$ since $0 \notin H(x)$ if $x \in U \setminus$ Fix (F, D). Moreover, $h(x) = x - f(x) \in H(x) \subset x - F(x)$ with $f(x) \in G(x) \subset K$ for each $x \in U$.

Define a new multivalued map $F_1 : \overline{D} \subset K \to CV(K)$ by $F_1(x) = f(x)$ for $x \in U$ and $F_1(x) = F(x)$ for $x \notin U$. It is easy to see that F_1 is an u.s.c. condensing multivalued map with $x \notin F_1(x) \subset F(x)$ for all $x \in \overline{D}$. Then $i(F_1, D, K) = i(F, D, K) \neq 0$. Hence, $0 \in F_1(x)$ for some $x \in D$, in contradiction to the definition of F_1. Hence, dim $\text{Fix}(F, D) \geq n$.

Note that if $G : K \to CV(K)$ is lower semicontinuous, then $G : K \to CV(X)$ is also lower semicontinuous. Indeed, let U_1 be open in X with $G_1(x_0) \cap U_1 \neq \emptyset$, then $U = U_1 \cap K$ is open in K and $G_1(x_0) \cap (U_1 \cap K) = (G_1(x_0) \cap K) \cap U_1 = G_1(x_0) \cap U_1 \neq \emptyset$ since $G_1(x_0) \subset K$. Hence, the conclusion. If F in Theorem 2.2 is also lower semicontinuous and therefore continuous, then we can take G=F in Theorem 2.2. When K=X, Theorem 2.2 was proved by Gelman [3] using the degree theory for the multivalued map I-F.

Next, we will apply this result to n-parameter equation (1.1) and to semilinear equations (1.2) with a nontrivial null space of the linear part. To that end we need to introduce a notion of a complementing map. Let $D \subset R^m \times K$ be an open subset (in the relative topology) and $F : \overline{D} \to K$ be a continouous condensing map. We say that F is complemented by a continuous compact multivalued map $G : \overline{D} \to CV(R^m)$ if the fixed point index for the multivalued condensing map $H : \overline{D} \cap (R^m \times K) \to CV(R^m \times K)$ given by $H(\lambda, x) = (\lambda - (G(\lambda, x), F(\lambda, x))$, $i(H, D, R^m \times K) \neq 0$. Note that $(G(\lambda, x), F(\lambda, x)) = (\lambda, x) - (\lambda - G(\lambda, x), F(\lambda, x)) = (I - H)(\lambda, x)$ is a condensing perturbation of the identity. Note that in this case $S(H, D) = \{(\lambda, x) \mid (\lambda, x) \in H(\lambda, x)\} \subset S(F, D) = \{(\lambda, x) \mid F(\lambda, x) = x\}$.

Theorem2.3 *Let X be a Banach space and $F : \overline{D} \subset R^m \times K \to K$ be a continuous condensing map complemented by a continuous multivalued compact map $G : \overline{D} \to CK(R^m)$ with dim $G(x) = m$ for each $x \in \overline{D}$. Then dim $S(F, D) \geq n$.*

Proof. Since F is complemented by G, the map $H : \overline{D} \to G(\lambda, x) \times F(\lambda, x) \in CK(R^m \times K)$ is a multivalued continuous condensing map with compact convex values, dim$H(\lambda, x) \geq n$ for each $(\lambda, x) \in \overline{D}$ and has a nonzero fixed point index, $i(H, D, R^m \times K)$. Since $S(H, D) \subset S(F, D)$, dim $S(H, \overline{D}) \geq m$ by Theorem 2.2 and therefore dim $S(F, \overline{D}) \geq n$ by the monotonicity property of dimension.

Proposition 2.1 *Let $F : \overline{D} \subset R^m \times K \to K$ be continuous and condensing, $D_0 = \overline{D} \cap (0 \times K)$ and $F_0(x) = F(0,x) : D_0 \subset K \to K$ be such that $i(F_0, D_0, K) \neq 0$. Then F is complemented by the continuous compact multivalued map $G(x) = \overline{B}(0,r) \subset R^m$ for all $x \in \overline{D}$ and some fixed $r > 0$.*

Proof. Define $H_r : \overline{D} \to CK(R^m \times K)$ by $H_r(\lambda, x) = \overline{B}(0,r) \times F(\lambda, x)$. We claim that H_r has no fixed points in ∂D for some $r > 0$. If not, then there would exist $(\lambda_k, x_k) \in \partial D$ such that $(\lambda_k, x_k) \in H_k(\lambda_k, x_k) = \overline{B}(0, 1/k) \times F(\lambda_k, x_k)$ for each positive integer k. Since F is condensing, it is easy to see that $I - F$ is proper on bounded closed subsets of $R^m \times K$. Hence, we may assume that $(\lambda_k, x_k) \to (0, x_0)$ with $x_0 \in \partial D_0$ and $x_0 = F(0, x_0)$ in contradiction to our assumption on F. Thus, for some $r > 0$, $(\lambda, x) \notin H_r(\lambda, x)$ for all $(\lambda, x) \in \partial D$. Since $h(\lambda, x) = (0, F(\lambda, x))$ is a continuous selection of $H_r(\lambda, x)$, we have that the fixed point index

$$i(H_r, D, B(0,r) \times K) = i(h, D, B(0,r) \times K).$$

Since $h : D \subset R^m \times K \to K$ and K is a retract of $R^m \times K$, the permanence property of the index implies that $i(h, D, B(0,r) \times K) = i(h, D_0, 0 \times K)$. Hence, $i(H_r, D, B(0,r) \times K) = i(F_0, D_0, K) \neq 0$, proving that F is complemented by the constanr multivalued map $G(\lambda, x) = \overline{B}_m(0,r) \subset R^m$ for some $r > 0$.

Theorem 2.4 *Let X be a Banach space, $F : \overline{D} \subset R^n \times K \to K$ be continuous and condensing, $\overline{D}_0 = \overline{D} \cap (0 \times K)$ and $F_0(x) = F(0,x) : \overline{D}_0 \subset K \to K$ be such that $i(F_0, D_0, K) \neq 0$. Then dim $S(F, D) \geq m$.*

Proof. By Proposition 2.1, F is complemented by a continuous compact multivalued map $G(x) = \overline{B}_m(0,r) \subset R^m$ for all $x \in \overline{D}$ and some $r > 0$. Hence, $H : \overline{D} \to \overline{B}_m(0,r) \times F(\lambda, x)$ is a continuous multivalued condensing map with compact convex values, dim$H(x) \geq n$ for each $x \in \overline{D}$ and has a nonzero index. By Theorem 2.3 dim $S(F, D) \geq m$.

We need the following result to study the unboundedness of the solution set. If X is a Banach space, define a norm of the Banach space $X_1 = R \times X$ by $\|(t,x)\| = (|t|^2 + \|x\|^2)^{1/2}$. Let $B(0,r)$ and $S_r = \partial B(0,r)$ be a ball and its boundary in X_1.

Lemma 2.1 *Let K be a closed convex subset of a Banach space X containing zero, $F : K_1 = R^1 \times K \to K$, and $\|F(t,x)\| \leq r$ for all $(t,x) \in S_r \cap K_1$ and satisfy either one of the following conditions*

(a) F is continuous and condensing on $\overline{B}(0,r) \cap K_1$, i.e., $\phi(F([0,r] \times Q)) < \phi(Q)$ for all $Q \subset \overline{B}(0,r) \cap K$ with $\phi(Q) > 0$.

(b) The map $H : K_1 \to K$ given by $H(t,x) = x - F(t,x)$ is A-proper on $\overline{B}(0,r) \cap K$ w.r.t. $\Gamma = \{R \times X_n, X_n, P_n\}$ with $P_n(K) \subset K$.

Then the equation $F(t,x) = x$ has a solution in $S_r \cap K_1$, and in case a) dim $S(F, \overline{B}_r) =$ dim $\{(t,x) \in \overline{B}(0,r) \mid F(t,x) = x\} \geq 1$ provided that $\|F(0,x)\| < r$ for $\|x\| = r$.

Proof. a) Define the map $G : \overline{B}_K = \overline{B}(0,r) \cap K \subset X \to K$ by $G(x) = F((r^2 - \|x\|^2)^{1/2}, x)$. Then G maps $\overline{B}_K$ into itself and is ϕ-condensing. Indeed, let $Q \subset \overline{B}_K$ with $\phi(Q) > 0$. Then $\phi(G(Q)) \leq \phi(F([0,r] \times Q)) < \phi(Q)$. Hence, $G(x) = F((r^2 - \|x\|^2)^{1/2}, x) = x$ for some x and therefore $F(t,x) = x$ with $t^2 = r^2 - \|x\|^2$ and $\|(t,x)\| = r$. Note also that if $\|F(0,x)\| < r$ for $\|x\| = r$, then the index of $F_0 = F_{|\overline{B}_K} : \overline{B}_K \to K$, $i(F_0, B_K, K) = 1$ since the homotopy $H(t,x) = x - tF(0,x) \neq 0$ for $\|x\| = r$ and $t \in [0,1]$. Hence, dim $S(F, \overline{B}(0,r)) \geq 1$ by Theorem 2.4.

b) Define G as in a). Then $I - G$ is A-proper with respect to $\Gamma = \{X_n, P_n\}$ for X. Indeed, let $x_{n_k} \in \overline{B}(0,r) \cap X_n \cap K$ and $x_{n_k} - P_n G x_{n_k} \to f$, i.e., $x_{n_k} - P_n F((r^2 - \|x_{n_k}\|)^{1/2}, x_{n_k}) \to f$. Then a subsequence of $\{(r^2 - \|x_{n_k}\|)^{1/2}, x_{n_k}\}$ converges to $(r^2 - \|x\|)^{1/2}, x)$ with $G(x) = F((r^2 - \|x\|)^{1/2}, x) = f$ by the A-properness of $H(t,x)$. Since $P_n G(\overline{B}(0,r) \cap X_n \cap K) \subset \overline{B}(0,r) \cap X_n \cap K$, by part a) $P_n G(x_n) = P_n F((r^2 - \|x_n\|^2)^{1/2}, x_n) = x_n$ for some $x_n \in \partial B(0,r) \cap X_n \cap K$ and all large n. Hence, $P_n F(t_n, x_n) = x_n$ with $t_n^2 = r^2 - \|x_n\|^2$ and $\|(t_n, x_n)\| = r$. By the A-properness of H, a subsequence of $\{\|(t_n, x_n)\| = r\}$ converges to $(t,x) \in K_1$ with $F(t,x) = x$ and $\|(t,x)\| = r$.

For a mapping $T : X \to Y$, let Σ be the set of all points $x \in X$ where T is not locally invertible, and let card$T^{-1}(\{f\})$ be the cardinal number of the set $T^{-1}(\{f\})$. Thoughout the paper we assume that a nonlinear mapping N is quasibounded with the quasinorm

$$|N| = \limsup_{||x|| \to \infty} ||Nx||/||x|| < \infty$$

Theorem 2.5 *Let X be a Banach space, K be a closed subset of X containing zero and $F : R^m \times K \to K$ be continuous and condensing, such that for some $a \in [0,1)$ and $b > 0$*

$$\|F(\lambda, x)\| \leq a\|x\| + b \text{ for all } (\lambda, x) \in R^m \times K.$$

Then $S(F, R^m \times K) = \{(\lambda, x) \mid F(\lambda, x) = x\}$ is unbounded and dim $S(F, R^m \times K) \geq m$. If $K = X$, the same conclusions hold for $S(F - f, R^m \times X)$ for each $f \in X$. If $m = 0$, then $S(F, K) \neq \emptyset$ and bounded. If $n = 0$ and $K = X$, then the cardinality of $S(F - f, X)$ is positive, finite and constant for each f in connected components of $X \setminus (I - F)(\Sigma)$.

Proof. For all $r > b/(1-a)$, $\|F(\lambda, x)\| \leq r$ for each $(\lambda, x) \in \overline{B}(0,r) \subset R^n \times K$. Indeed, for such (λ, x)

$$\|F(\lambda, x)\| \leq a\|x\| + b \leq ar + b$$

and, if $r > b/(1-a)$, then $\|F(\lambda, x)\| \leq r$. Moreover, there is an $r_0 > 0$ such that for each $r > r_0$, $H(t,x) = x - tF(0,x) \neq 0$ for all $t \in [0,1]$ and $\|x\| = r$ in K. If not, then there would exist $t_n \to t$ and x_n with $\|x_n\| \to \infty$ such that $H(t_n, x_n) = 0$ for all n. Hence, $\|x_n\| \leq \|F(0, x_n)\| \leq a\|x_n\| + b$. Dividing by

$\|x\|$ and taking the limit we get a contardiction $a \geq 1$. Thus such an r_0 exists and by the homotopy theorem for condensing maps, $i(F(0,.), B(0,r), K) = 1$ for each $r > r_0$. We have shown that i$(F_0, B_0, K) \neq 0$ and by Theorem 2.4, dim$S(F, \overline{B}_r) \geq m$.

To prove that $S(F, R^m \times K)$ is unbounded, it is enough to look at $F : R \times K \to K$. Let $r_1 >$max$\{r_0, b/(1-a)\}$ and $\overline{B}_0 = \overline{B}(0,r) \cap (0 \times K)$ for $r > r_1$. Then, by Lemma 1, for each large $r > r_1$, $F(t,x) = x$ for some $(t,x) \in \partial B(0,r) \subset R \times K$ and therefore $S(F, R^m \times K)$ is unbounded. If $m = 0$, then the above proof shows that $S(F,K) \neq \emptyset$. Moreover, for $x \in S(F,K)$,

$$\|x\| \leq \|F(x)\| \leq a\|x\| + b$$

which implies that $S(F,K)$ is bounded. If $K = X$, then $F_f = F - f$ satisfies all conditions of F for each $f \in X$ and the conclusions of the theorem hold for F_f. If $m = 0$ and $K = X$, then $I - F$ is locally proper and satisfies condition (+), i.e., $\{x_n\}$ is bounded whenever $x_n - Fx_n \to y$ in X. Hence, the cardinality of $(I-F)^{-1}(f)$ positive, finite and constant for each $f \in X \setminus (I-F)(\Sigma)$ by Theorem 3.5 in [12].

Next, we shall apply Theorem 2.4 to semilinear equations (1.2)

Theorem 2.6 *Let X and Y be Banach spaces, $L : X \to Y$ be a continuous linear surjection, $X_0 = ker\,(L)$ be closed and have a closed complement X_1 in X, and $N : X \to Y$ be a k-set contraction with $ck < 1$ such that $I - NL^+ : Y \to Y$ satisfies condition (+). Then $L + N : X \to Y$ is surjective and, for each $f \in Y$, $S(f) = \{x \mid Lx + Nx = f\}$ is unbounded and*

$$dim\ S(f) \geq dim\ ker\,(L).$$

If L is a homeomorphism, then $S(f) \neq \emptyset$ compact set for each $f \in Y$ and the cardinal number of $S(f)$ is constant, finite on each connected component of $Y \setminus (L+N)(\Sigma)$.

Proof. Let $n = \dim(X_0)$ if X_0 is finite dimensional and $n < \dim(X_0)$ be any nonnegative integer otherwise. Let X_n be an n-dimensional subspace of X_0. Since $N_f x = Nx - f$ has the same properties as N, we may assume $f = 0$ and study the equation $Lx + Nx = 0$. Define a map $F : X_n \times Y \to Y$ by $F(x,y) = N(x + L^+ y)$. We claim that F is ck-set contractive. Let $Q \subset X_n \times Y$ be bounded. Then, without loss of generality, we can assume that $Q = Q_1 \times Q_2$ with both $Q_1 \subset X_n$ and $Q_2 \subset Y$ bounded. Moreover, $Q_3 = \{x + L^+(y) \mid (x,y) \in Q\}$ is also bounded and $Q_3 \subset L^{-1}(Q_2)$. Hence

$$\gamma(F(Q)) = \gamma(N(Q_3)) \leq \gamma(NL^+(Q_2)) \leq kc\gamma(Q_2) = kc\gamma(Q)$$

since Q_1 is compact. Then (x,y) is a solution of $N(x + L^+ y) = y$ if and only if $u = x + L^+ y$ is a solution of $Lu + Nu = 0$. The map $A : X_n \times Y \to X_n \times X_1$ defined by $A(x,y) = x + L^+ y$ is a bijection. Its surjectivity is clear. It is injective since $(x_1, y_1) \neq (x_2, y_2)$ implies that $A(x_1, y_1) \neq A(x_2, y_2)$. In

particular, A is a homeomorphism between the solution sets $S(0)$ and $S(F, X_n \times Y) = \{(x, y) \mid F(x, y) = y\}$. We claim that there is an $r > 0$ such that $H(t, (0, y)) = (0, y) - tF(0, y) \neq 0$ for all $(0, y) \in \partial B_n(0, 1) \times \partial B_Y(0, r)$. If not, then there would exist $t_k \in [0, 1]$, $y_k \in Y$ such that $\|y_k\| \to \infty$ and $H(t_k, 0, y_k) = 0$ for each k. This contradicts condition (+) for $I - F(0, .) = I - NL^+$. Hence, the homotopy $H_1 : [0, 1] \times (0 \times Y) \to Y$ given by $H_1(t, (0, y)) = y - tF(0, y)$ is not zero for $t \in [0, 1]$ and $y \in \partial B_Y(0, r)$ for some $r > 0$. Thus, the degree $\deg(I - F(0, .), 0 \times B_Y(0, r), 0) = 1$. By Proposition 2.1, F is complemented by the costant compact map $(x, y) \to \overline{B}(0, r)$ on $\overline{B}_n(0, 1) \times \overline{B}_Y(0, r)$, where $\overline{B}_n(0, 1)$ is the closed unit ball in X_n. The map $H : \overline{B}_n(0, 1) \times \overline{B}_Y(0, r) \to F(x, y) \times \overline{B}_n$ is ck-set contractive with nonzero index, compact convex values and $\dim H(x) \geq n$ for each $x \in \overline{B}_n(0, 1) \times \overline{B}_Y(0, r)$. By Theorem 2.2, dim $S(H, \overline{B}_n(0, 1) \times Y\} \geq n$ and therefore dim $(F, X_n \times Y) \geq \dim S(F, \overline{B}_n(0, 1) \times Y) \geq n$. Since $S(F, X_n \times Y)$ is homeomorphic to $S(0)$, by the monotonicity of the dimension

$$\dim S(0) \geq \dim \{(x, y) \mid N(x + L^+ y) = y\} \geq n.$$

Since n was arbitrary, we get that

$$\dim S(0) \geq \dim \ker (L).$$

To see that $S(0)$ is unbounded observe that $x \in S(f)$ if and only if $x = u + L^+ y$ is a solution of $y - N(u + L^+ y) = 0$, where $F(x, y) = N(u + L^+ y)$ is ck-set contractive as shown above. Hence, we can use Lemma 2.1. If L is a homeomorphism, then $S(f) \neq \emptyset$ by the above proof. The finite solvability on connected components follows from Theorem 3.5 in [12].

We need the following nonlinear Fredholm alternative.

Theorem 2.7 ([10])*Let X and Y be separable Banach spaces with a projectionally complete scheme $\Gamma = \{X_n, Y_n, Q_n\}$, $L : X \to Y$ be a continuous linear Fredholm map of index zero with dim ker $(L) = m > 0$ and $N : X \to Y$ be a continuous map such that $|N| < c/\delta$, the range $R(N) \subset R(L)$, and $L + N : X \to Y$ be A-proper w.r.t. $\Gamma = \{X_n, Y_n, Q_n\}$ with X_0 =ker (L)$\subset X_n$ and $Y_0 \subset Y_n$. Then, for each $f \in R(A)(= N(A*)^\perp)$, and only such ones, there is a connected closed subset C of $S(f) = \{x \mid Lx + Nx = f\}$ whose dimension at each point is at least m and the projection P maps C onto ker (L).*

An immediate consequence of this result is

Theorem 2.8 *Let X and Y be separable Banach spaces, $L : X \to Y$ be continuous, linear, and surjective with dim X_0 = ker $(L) = m > 0$ and $N : X \to Y$ be a continuous map such that $|N| < c/\delta$, $L + N$ be A-proper w.r.t. $\Gamma_m = \{X_0 \times X_n, Y_n, Q_n\}$. Then, for each $f \in Y$ there is a connected closed subset C of $S(f) = \{x \mid Lx + Nx = f\}$ whose dimension at each point is at least m and the projection P maps C onto ker (L).*

Proof. The maps $L_e : X \to Y \times R^m$, $L_e x = (Lx, 0)$ and $N : X \to Y \subset Y \times R^m$ satisfy all conditions of Theorem 2.7 with $\Gamma_e = \{X_0 \times X_n, Y_n \times R^m, (Q_n, I)\}$ hence the conclusion.

Theorem 2.8 applies to many different nonlinearities N. Soome of them are given in the following example.

Example 2.1 ([8]) Let $L : X \to Y$ be continuous, linear, and surjective with dim $X_0 = \ker\,(L) = m > 0$ and $N : X \to Y$ be a continuous map. Then $L + N$ is A-proper w.r.t. $\Gamma_m = \{X_0 \times X_n, Y_n, Q_n\}$ if N is k-ball contractive with k sufficiently small, and in particular if N is compact or k-contractive.

In case of k-contractive map N, we can say more. Recall that a topological space V is an *absolute extensor* for the paracompact (respectively, normal) spaces if for each paracompact (respectively, normal) topological space U, each closed subset A of U and each continuous function $\psi : A \to V$, there exists a continuous function $\phi : U \to V$ such that $\phi_{|A} = \psi$.

Theorem 2.9 *Let X and Y be separable Banach spaces with a projectionally complete scheme $\Gamma = \{X_n, Y_n, Q_n\}$, $L : X \to Y$ be a continuous linear surjection and $N : X \to Y$ be a k-contraction with $ck < 1$, where $c = \sup\{\inf\{\|x\| \mid x \in L^{-1}(y)\} : y \in Y, \|y\| \leq 1\}$. Let $S(f) = \{x \in X \mid Lx + Nx = f\}$. Then for each $f \in Y$ there is a connected closed subset C of $S(f) = \{x \mid Lx + Nx = f\}$ whose dimension at each point is at least m and the projection P maps C onto ker (L). Moreover, the set $S(f)$ is a nonempty absolute extensor for paracompact spaces, and, in particular, it is arcwise connected.*

Proof. The dimension assertion follows from Theorem 2.8 and Example 2.1. The absolute extensor property of S(f) was proved in Ricceri [13].

When the null space of L has a complement, then we have the following extension of Theorem 2.8.

Theorem 2.10 *Let X and Y be separable Banach spaces with a projectionally complete scheme $\Gamma = \{X_n, Y_n, Q_n\}$, $L : X \to Y$ be a continuous linear surjection such that its null space X_0 has a complement X_1, and $N : X \to Y$ be a continuous quasibounded nonlinear map with sufficiently small quasinorm $|N|$ such that for each finite dimensional subspace U of X_0, $L + N : U \times X_1 \to Y$ is A-proper w.r.t. $\Gamma = \{U \times X_n, Y_n = L(X_n), Q_n\}$. Then $S(f) = \{x \in X \mid Lx + Nx = f\}$ is unbounded and dim $S(f) \geq$ dim ker(L). Moreover, if dim ker L is finite, then there is a connected closed subset C of $S(f)$ whose dimension at each point is at least dim ker L and the projection P onto X_0 maps C onto X_0. If L is a homeomorphism, then $S(f) \neq \emptyset$ compact set for each $f \in Y$ and the cardinal number of $S(f)$ is constant, finite on each connected component of $Y \setminus (L + N)(\Sigma)$.*

Proof. Since $N_f x = Nx - f$ has the same properties as N, we may assume that $f = 0$. Let $\epsilon > 0$ be such that $\|Nx\| \leq (|N| + \epsilon)\|x\|$ for all $\|x\| \geq r$. Let $U \subset H_0$ be a finite dimensional subspace, $L^{-1} : Y \to X_1$ be a partial inverse of L, and define a map $G : U \oplus Y \to Y$ by $G(u, y) = y + N(u + L^{-1}y)$. Then the map G is A-proper w.r.t. $\Gamma = \{U \times Y_n, Q_n\}$ with $Y_n = L(X_n)$. Indeed, let $\{(u_n, y_n) \in U \times Y_n\}$ be bounded and $Q_n G(u_n, y_n) = y_n + Q_n N(u_n + L^{-1}y_n) \to f$ in Y. Then $y_n = Lx_n$ for $x_n \in X_n$ and $Lx_n + Q_n N(u_n + x_n) \to f$ and, by the A-properness of $L+N$ a subsequence $u_{n_k} + x_{n_k} \to u + x$ with $Lx + N(u + x) = f$.

Then $y_{n_k} = Lx_{n_k} \to Lx = y$ and $u_{n_k} \to u$ and therefore $y + N(u + L^{-1}y) = f$. By Lemma 2.1, the equation $N(te+L^{-1}(y)) = y$ has a solution $(t, y) \in \partial B(0, r)$ for any unit vector $e \in U$. Then $x = te + L^{-1}(y)$ is a solution of $Lx + Nx = 0$. Since $t^2 + \|y\|^2 = r^2$, then either $|t| > r/\sqrt{2}$ or $\|y\| > r/\sqrt{2}$. If $\|y\| > r/\sqrt{2}$, then $\|y\| = \|L(x)\| \leq \|L\|\|x\|$, or $\|x\| \geq r/(\sqrt{2}\|L\|)$. If $|t| > \sqrt{r}/(\sqrt{2})$, then

$$\|x\| \geq \|t\| - \|q(y)\| \geq r/\sqrt{2} - m\|y\| \geq r/\sqrt{2} - m\|L\|\|x\|$$

and so

$$\|x\| \geq r/(\sqrt{2}(1 + m\|L\|)).$$

Hence, in ether case $\|x\| \to \infty$ as $r \to \infty$. The existence of a closed connected subset C of $S(f)$ follows from Theorem 2.8. If L is a homeomorphism, then $S(f) \neq \emptyset$ by the above proof. Moreover, $\|L^{-1}\| \ |N| < 1$ implies that $\|x\| \leq \|L^{-1}f\| + \|L^{-1} \ |N|\|x\|$ for each $x \in S(f)$ and therefore $S(f)$ is bounded. The finite solvability on connected components follows from Theorem 3.5 in [12].

3. Nonlinear perturbations of the Wiener-Hopf equations

Consider a nonlinear perturbation of the Wiener-Hopf equation

$$\lambda x(s) - \int_0^\infty k(s-t)x(t)dt + (Nx)(s) = y(s) \tag{3.1}$$

where $k : R \to C$ is in $L_1(R, C)$ and $y(s) \in L_1(R^+, C)$ and N is a suitable nonlinear mapping. The corresponding linear Wiener-Hopf equations is

$$\lambda x(s) - \int_0^\infty k(s-t)x(t)dt = y(s). \tag{3.2}$$

Let $i(\lambda)$ be the index of the homogeneous equation corresponding to (3.2) with y(s)=0. Then, if $i(\lambda) > 0$, the homogeneous equation has an $i(\lambda)$-dimensional space of solutions, and the nonhomogeneous linear equation (3.2) has infintely many solutions for each y in a suitable space. If $i(\lambda) = 0$, then (3.2) has a unique solution for each y. These results have been proven in the seminal paper by Krein [K]. Detailed study of Eq. (3.2) can be found in Corduneanu [1]. Many problems in mathematical physics lead to Eq. (3.2). In particular, it appears in studying questions of transfer of radiant energy. We also note that the study of Eq. (3.2) with the integral over R is much simpler and is based on using integral Fourier transform.

In this section, we shall extend these results to the nonlinear perturbed Wiener-Hopf equation (3.1). As mentioned above, if $x(s)$ is a solution of (3.2), then one also has the $i(\lambda)$-dimensional plane of solutions of (3.2) consisting of $\{x(s) + x_0(s) \mid x_0(s) \text{ in the null space of } A\}$. The results for Eq.(3.1) we will prove consist in describing the manner in which this $i(\lambda)$-dimensional aspect of the solutions of Eq. (3.2) persists in the global desription of the set of solutions of the nonlinear Eq. (3.1).

Since N is nonlinear, one cannot expect the solutions of (3.1) to have any linear structure. We prove various dimension results for the solution set of (3.1) with $i(\lambda) > 0$, where by dimension we will mean the natural extension of the linear concept of dimension, namely, the Lebesgue covering dimension. Moreover, if $i(\lambda) = 0$, we prove that Eq. (3.1) has a constant number of solutions on certain connected components for almost all f.

Let us first recall some facts about the Wiener-Hopf equation (3.2) where $k : R \to C$ is in $L_1(R, C)$ and $y(s) \in L_1(R^+, C)$. Let $Y = L_1(R, C)$ and $x(s) = y(s) = 0$ for $s < 0$. Then Eq. (1.2) becomes

$$\lambda x(s) - \int_{-\infty}^{\infty} k(s-t)x(t)dt = z(s)$$

where $z(s) = y(s)$ for $s \geq 0$ and $z(s) = -\int_0^{\infty} k(s-t)x(t)dt$ for $s < 0$. Let $\hat{k}(\xi)$ be the Fourier transform of k, i.e.,

$$\hat{k}(\xi) = \int_{-\infty}^{+\infty} k(t)e^{i\xi t}dt.$$

Applying the Fourier transform to Eq. (3.1), we get $\lambda\hat{x}(\xi) - \hat{k}(\xi)\hat{x}(\xi) = \hat{z}(\xi)$ on R. Let $X = L_p(R^+, C)$ and $K : X \to X$ be given by

$$Kx(s) = \int_0^{\infty} k(s-t)x(t)dt, \quad s \in R^+.$$

It turns out that $\lambda I - K : X \to X$ is a Fredholm mapping if and only if $\lambda - \hat{k}(\xi) \neq 0$ in $R \cup \{-\infty, +\infty\}$ and the index $i(\lambda)$ =index$(\lambda I - K) = -w(\Gamma_\lambda, 0)$ for $\Gamma_\lambda = \{\lambda - \hat{k}(\xi) : -\infty \leq \xi \leq +\infty\}$. Γ_λ is a closed curve and $w(\Gamma_\lambda, 0)$ is the winding number. If $i(\lambda) \geq 0$, then dim$N(\lambda I - K) = i(\lambda)$ and the range $R(\lambda I - K) = X$. If $i(\lambda) < 0$, then dim$N(\lambda I - K) = \{\emptyset\}$ and codim$R(\lambda I - K) = -i(\lambda)$. Suppose that $\lambda - \hat{k}(\xi) \neq 0$. Then $\lambda I - K$ is of index zero if, for example, $k(t-s)$ $(0 \leq t, s < \infty)$ is a symmetric kernel, that is, if $k(t)$ (-$\infty < t < \infty$) is an even function. In this case $\hat{k}(\xi)$ is also an even function. Another interesting case is when $k(t-s)$ is a hermitian kernel, i.e., $k(-t) = \overline{k(t)}$. Then $\hat{k}(\xi)$ is real, and $\lambda - \hat{k}(\xi) \neq 0$ if and only if it is positive. Hence $\lambda I - K$ is of index zero in this case if $\lambda - \hat{k}(\xi) > 0$. Let L stand for the operator $\lambda I - K$ for some $\lambda \in C$ and $P : X \to N(L)$ is the projection onto $N(L)$.

The next result follows from Theorem 2.10.

Theorem 3.1 (Nonlinear Fredholm Alternative) *Let $L = \lambda I - K : X \to X$ be a Fredholm mapping of index $i(L) \geq 0$ induced by Eq. (3.2) and $N : X \to X$ be a k-ball contractive mapping with k and $|N|$ sufficiently small.*

Then, either

(i) the equation $Lx = 0$ has a unique zero solution, i.e, $i(L) = 0$, in which case Eq. (3.1) is approximation solvable for each $y \in X$ and $(L + N)^{-1}(\{y\})$

is compact for each $y \in X$ and the cardinal number card$(L+N)^{-1}(\{y\})$ is constant, finite and positive on each connected component of $X \setminus (L+N)(\Sigma)$,

or

(ii) $N(L) \neq \{0\}$, i.e., $i(L) = dimN(L) > 0$, in which case, for each $y \in X$, there is a connected closed subset C of $(L+N)^{-1}(y)$ whose dimension at each point is at least $m = i(L)$ and the projection P maps C onto $N(L)$.

If $i(A) = 0$ and N a k-contaction then Eq. (3.1) is uniquely solvable by the contraction principle as shown by Corduneanu [1].

For odd nonlinearities, the following result follows from [8].

Theorem 3.2 *Let L be a Fredholm mapping of index $i(L) > 0$ induced by Eq. (3.2) and $N : X \to X$ be a continuous odd k-ball contactive mapping with k sufficiently small. Let S_0 be the solution set of Eq. (3.1) with $y = 0$. Then, for any positive real number r and $B(0,r) = \{x \in X \mid ||x|| < r\}$, the dimension of $S_0 \cap \partial B(0,r)$ is at least $i(L) - 1$, when $i(L) > 1$, and $S_0 \cap \partial B(0,r)\}$ contains at least two points when $i(L) = 1$.*

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