Managing Warranty Costs with variable Usage rates

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Abstract: We investigate a Copula based approach to model expected warranty costs and their corresponding minimization under a two-dimensional warranty regime defined by a base warranty period and a maximum allowable use limit. Contrary to the standard paradigm adopted by most researchers on the subject, we assume that customers' individual usage rates - although assumed constant, is unknown to the seller except through a distributional specification of usage rates among all customers. A family of warranty strategies that partition the warranty period into three intervals, which have been considered by several researchers (Jack et al.(2009), Yun et al.(2008), Banerjee and Bhattacharejee ((2012-a) & (2012-b)) is explored with minimal repairs and a replacement option, via a Copula approach and illustrated using an accelerated Weibull lifetime conditional on usage rate, with the latter being uniformly distributed. We provide a numerical example, which is identical to the example used by others - to contrast our results with theirs.

1 Introduction and Summary

Most of the recent research on modeling and optimization of servicing costs for non-renewing free replacement warranties (NR-FRW) from a manufacturer's viewpoint assumes that a consumer's rate of use of the product is constant and known. Such an assumption is unrealistic for many moderately high value consumer durables (e.g., although a buyer of a new automobile may have a constant usage rate determined by his/her driving needs; it would be unreasonable to expect the buyer to disclose such information to the dealer, even if the buyer can accurately estimate it). In such cases, it would be pragmatic to assume that the manufacturer/seller is uncertain about customer's usage rate of the product, and that his (i.e., the seller's) corresponding beliefs are summarized by a probability distribution of the usage rate of his target customers. In particular, we consider the family of pragmatic warranty servicing strategies that partition the effective warranty period into three intervals - that rectify the first failure, if any, in the middle interval by a replacement; while all other failures undergo minimal repairs.

Research reported here is a part of ongoing doctoral research of the first author. It seeks to model and minimize the expected costs of pragmatic servicing strategies for 'NR-FRW' warranties, using a "*Copula*" based approach to capture the adverse impact of increasing product usage rate on its time-to-failure, which

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is believed to be new. A copula based approach for a joint distribution is well known to be able to allow for a flexible nonparametric dependency structure, and requires only a knowledge of the underlying marginal distributions. We develop the Copula corresponding to the conditional AFT (accelerated failure time) model and an arbitrary absolutely continuous distribution of usage rates; establish its equivalence to a joint distribution formulation for computational purposes.

Since exact analytical solutions to our models are typically not obtainable; numerical methods using MATLAB and 'Simulated Annealing' algorithm for globally optimal cost minimization are used for computational solution. Our methods and results are illustrated numerically using an increasing failure rate (IFR) Weibull AFT model and a uniform distribution of usage rate which show a substantial cost reduction in optimal costs relative to the situation when the usage rate is known and not sufficiently low.

2 The Background of Warranty Servicing Models & Copulas

2.1 The Warranty Strategy

The two-dimensional (2-D) warranty offered at the time of sale of the product is defined by the rectangular warranty region

$$[0, W) \times [0, U)$$

in the (age,accumulated use) plane. The parameter W is referred as the *base warranty* that comes with a new product and is provided by the manufacturer. The warranty expires when either the product reaches age W or, total accumulated use U whichever occurs first. The effective warranty period is thus

$$W_y = min(W, U/y) \tag{2.1}$$

for a customer whose usage rate is y. For us, y is unknown, except for its distribution profile.

For a one-dimensional warranty defined by $[0, W)$ together with minimal repair and replacement options; Jack, N. and Van der Duyn Schouten(2000) conjectured and Jiang et al.(2006) subsequently proved the minimal cost optimality of the strategy that divides the time axis into a three interval partition

$$[0, K), [K, L), [L, W). \tag{2.2}$$

All repairs in the first and third intervals are minimal, while failures in the middle interval $[K, L)$ are rectified by a choice between minimal repair or, replacement via a criterion depending on the age at failure. In the sequel, such strategies will be referred to as the *family of (K,L) - strategies.* As the optimal strategy requires continuous monitoring and is not easy to practically implement; Jack and Murthy (2001) proposed a nearly optimal strategy using the same three-interval partition (2.2) such that all repairs are minimal repairs, except that the first failure (if any) in the middle interval $[K, L)$ is rectified by a replacement. In subsequent research reported in the literature over the last several years (Jack et al.(2009), Yun et al.(2008), Iskander et al.(2005), Banerjee & Bhattacharjee ((2012-a) & (2012-b))), attention has been focused on this three-interval pragmatic strategy, referred to henceforth as the (K, L)-policy. For 2-D warranties, W in (2.2) is replaced by effective warranty period W_y for a customer with usage rate y. It may be noted that the Chukova with two other coauthors have considered an alternative approach to modeling of warranty costs with a probabilistic specified usage rate (Chukova & Johnston (2006) and Chukova & Varnosafaderani (2010)). As their formulation, which is "splits the standard 2-D rectangular warranty region into several nested rectangular subregions with equal or possibly different ratios of the sides of the nested rectangles" , is thus notably different from ours; we will not consider it further.

Notations

X	: product's lifelength (time-to-failure)
Y	: usage rate (*varies between customers, but fixed for each customer*)
G(y)(resp. g(y))	: cdf and pdf of usage rate y as a random variable
c_m	: cost of minimal repair
c_r	: cost of replacement
$F(x\|y)$	: conditional cdf of time to failure,given usage rate y
$f(x\|y)$	: conditional pdf of time to failure,given usage rate y
$h(x\|y)$	: failure(hazard) rate of $F(x\|y)$
$H(x\|y) = \int_0^x h(t\|y)\, dt$	: cumulative hazard function of $F(x\|y)$
γ	: accelerated failure time (AFT) parameter
$J(K, L; y)$	: Expected cost of warranty servicing policy with known usage rate y
$I(K, L)$	: Expected cost of warranty servicing policy when usage rate is uncertain
$S(x, y)[s(x, y) resp.]$	: Joint cdf (pdf,respectively) of X(time to failure) and Y(usage rate)

2.2 Copulas

When a customer's usage rate is unknown to the manufacturer/seller; the cost of warranty servicing, from their perspective, must be averaged over the usage rate profile of the target group of customers. It is here that "Copulas" can play a significant role in capturing the impact of usage rate on the product lifelength for modeling warranty costs. Below, we briefly summarize these salient facts and basic results for bivariate Copulas that will be needed.

Definition 2.1 *A Copula C is a map $C : I^2 \to I \equiv [0, 1]$ such that:*

- *For every u,v ϵ I,*
  $C(u,0) = C(0,v) = 0$*;* $C(u,1) = u$ *and* $C(1,v) = v$

- C *is "2-increasing":*
  $C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \equiv \Delta_{v_1}^{v_2} \Delta_{u_1}^{u_2} C(u,v) \geq 0$*, for all pairs* (u_i, v_i) *satisfying* $(0,0) \leq (u_i, v_i) \leq (1,1);\ i = 1,2.$

Thus, a Copula is a joint distribution of a random vector (X,Y) on the unit square, with uniformly distributed marginals. The "*2 -increasing*" condition is equivalent to requiring $P(u_1 < X \leq u_2\,, v_1 < Y \leq v_2) \geq 0$, for all points $(u_1, u_2)\ \varepsilon\ I^2$, $(v_1, v_2)\ \varepsilon\ I^2$. The connection between arbitrary joint distributions and Copulas is provided by the following result [see Nelson(2006)]

Theorem 2.1 (Sklar's Theorem) *Let* S *be a joint distribution function with margins* F *and* G*. Then there exists a Copula* C *such that for all* x, y*,*

$$S(x,y) = C(F(x), G(y)) \tag{2.3}$$

If F *and* G *are continuous, then* C *is unique. Otherwise, the Copula* C *is uniquely determined on* $Range(F) \times Range(G)$*.*
Conversely, if C *is a Copula and* F *and* G *are distribution functions, then the function* S *defined above, is a joint distribution function with margins* F *and* G*.*

In virtue of Sklar's theorem, the Copula C corresponding to a joint distribution $S(x,y)$ can be computed as:

$$C(u,v) = S(F^{-1}(u), G^{-1}(v)), \quad 0 \leq u, v \leq 1 \tag{2.4}$$

where F^{-1}, G^{-1} are uniquely defined for $0 < u,\ v < 1$ as the inverse functions of the continuous cdfs F, G, and $F^{-1}(0)$, $F^{-1}(1)$ are respectively the leftmost and rightmost endpoints of the *support* of F (formally $F^{-1}(0) :=\ \inf\{x : F(x) > 0\}$ and $F^{-1}(1) :=\ \sup\{x : F(x) < 1\}$, with $G^{-1}(0)$ and $G^{-1}(1)$ defined similarly). Thus (2.4) provides a method of constructing Copulas from joint distributions.

If a bivariate Copula C is absolutely continuous with respect to Lebesgue measure on the unit square, then its density function (denoted by c) exists, and is given by :

$$c(u,v) = \frac{\partial^2 C(u,v)}{\partial u \partial v}. \tag{2.5}$$

If X_1 and X_2 are random variables with marginal densities f_1 , f_2 and with distribution functions F_1 , F_2 respectively, then the joint density function of the pair (X_1, X_2) associated with a Copula C, is :

$$f_{12}(x_1, x_2) = c(F_1(x_1), F_2(x_2))f_1(x_1)f_2(x_2). \tag{2.6}$$

Copulas and association

Dependence properties and measures of association are interrelated. The latter numerically quantify the extent of dependence between the components of a random vector. Among the most widely known scale-invariant measures of association for Copulas, are the population versions of Kendall's tau (τ) and Spearman's rho (ϱ) that measure a form of dependence known as concordance:

Definition 2.2 *Let (x_i, y_i) and (x_j, y_j) denote two observations from a vector (X, Y) of continuous random variables. We say that (x_i, y_i) and (x_j, y_j) are concordant if $x_i < x_j$ and $y_i < y_j$, or if $x_i > x_j$ and $y_i > y_j$. Similarly, (x_i, y_i) and (x_j, y_j) are said to be discordant if they are not concordant; i.e., if $x_i < x_j$ and $y_i > y_j$, or if $x_i > x_j$ and $y_i < y_j$.*

Definition 2.3 *"Kendall's tau" measure τ_{XY} of a pair (X, Y) , is the difference between the probabilities of concordance and discordance for two iid (independent and identically distributed) pairs (X_1, Y_1) and (X_2, Y_2) each with a common joint distribution S, i.e.,*

$$\tau_{XY} = P\{(X_1 - X_2)(Y_1 - Y_2) > 0\} - P\{(X_1 - X_2)(Y_1 - Y_2) < 0\} \tag{2.7}$$

These probabilities can be evaluated by integrating over the distribution of (X_2, Y_2).

If C is the Copula uniquely determined by (X, Y), then one can show (see Nelson(2006)), the Kendall's tau for (X, Y) is given as:

$$\tau_C = 4\ [\int_0^1 \int_0^1 C(u, v)\, dC(u, v)\ ] - 1 \tag{2.8}$$

where C is the Copula associated with (X, Y).

Definition 2.4 *Let (X_1, Y_1), (X_2, Y_2) and (X_3, Y_3) be three independent random vectors with a common joint distribution function S. The "$Spearman's\ rho$" coefficient associated with a pair (X, Y), distributed as S, is defined as*

$$\varrho_{XY} = P\{(X_1 - X_2)(Y_1 - Y_3) > 0\} - P\{(X_1 - X_2)(Y_1 - Y_3) < 0\}, \tag{2.9}$$

the difference between the probabilities of concordance and discordance of the random vectors (X_1, Y_1), (X_2, Y_3).

Note that in (2.9), X_1 and Y_1 have a joint distribution S with an associated Copula C where as X_2 and Y_3 are iid uniform $(0,1)$. In terms of a Copula C, Spearman's rho(ϱ_C) is :

$$\varrho_C = 12\ [\int_0^1 \int_0^1 (C(u,v) - uv)\, du\, dv\ ] \tag{2.10}$$

Sklar's result proving a 1:1 correspondence between joint distributions with absolutely continuous marginals and Copulas (distributions on the unit square with continuous uniform marginals) dates back to the late fifties (Sklar (1959)) but received little attention until the 1990s that witnessed an explosive growth of interest in Copulas for exploring their use in finance, in attempts to understand and fairly price complex financial instruments based on a basket of positively dependent assets and associated risks; efforts which continue and are too numerous to reference. See Cherubini et al. ((2004) & (2012)) as examples of book length references on the subject. Notwithstanding such substantial efforts to exploit Copulas in finance that have seen both successes as well as major failures in terms of real life market behavior; one can find relatively few instances of the use of Copulas in other fields. An internet search reveals that such efforts have been sporadic and have been attempted only in a few areas ; e.g., see Yi et al.(1998), Favre et al.(2004), Salvadori et al.(2007), Dupis(2007), Quinn(2007), Challotte(2008), Vedenov(2008), Muhaisen et al.(2009) for some applications to management science, civil engineering/hydrology, agricultural and health economics. Many of these works have still not been published and appear as working papers or technical reports. It would thus appear that Copulas as a modeling tool still has an unexpected potential in many areas. Our attempts to use Copulas to model and explain warranty costs fits in this broad context.

Although our computational solutions for the numerical illustration (sec.3.5) were implemented using MATLAB; alternative softwares can also be used for such purposes. In particular, see Yan et al. (2007) on open source implementation of Copula computations using a R-package.

3 The AFT model with uncertain usage rate

We consider the standard AFT(Accelerated failure time) model with uncertain usage rate to illustrate (i) our methodology and (ii) the associated Copula. Modeling warranty costs starting with a given dependency structure between usage and productive lifetime via parametric Copula families with negative association as measured by Kendall's tau (τ) and the corresponding cost optimization is the subject of ongoing research to be reported in the future.

3.1 The AFT model

We essentially follow Jack et al.'s(2009) notations with some obvious simplifications and change of terminology. If F_o denotes a nominal baseline usage rate (assumed, without loss of generality, to be 1), then the conditional distribution of the time(T) to failure given the usage rate y, differs from F_o by the scale parameter

$$\alpha(y) := y^{-\gamma}, \qquad \gamma \geq 1$$

such that

$$F(x|y) = P(T \leq x|y) = F_o(y^\gamma x), \qquad x \geq 0 \tag{3.1}$$

with conditional hazard rate $h(\cdot|y)$ and cumulative *hazard function*

$$H(x|y) = \int_0^x h(t|y)\, dt.$$

Note that for usage rates heavier than the baseline usage rate (i.e. for $y \geq 1$), the product failure time distribution $F(\cdot|y)$ is stochastically smaller than the baseline distribution F_o and thus reflects greater "wear and tear".

3.2 Expected Warranty Cost

The total expected cost under a (K, L)-strategy, as defined in section 3.2, with uncertain usage rate Y is:

$$I(K,L) = \int_0^\infty J(K,L,y)\, dG(y), \tag{3.2}$$
$$where,\ J(K,L,y) = E\{I(K,L)|y\},$$

is the cost of a (K, L)-strategy conditional on usage rate y. If T_1 denotes the time of first failure, if any, in the time interval (K, L); then proceeding as in Jack et al.(2009), the conditional expected cost is

$$\begin{aligned} J(K,L,y) &= c_m H(K|y) + \int_K^L [c_r + c_m \int_x^{W_y} h((W_y - t)|y)]\, dP(T_1 \leq t|y, K < T_1 \leq L) \\ &\quad + c_m \int_L^{W_y} h(x|y)\, dx\, P(T_1 > L|y) \\ &= c_m \left\{ H(K|y) + \int_K^L [\rho + H((W_y - x)|y)] \frac{f(x|y)}{\overline{F}(K|y)} t dx + [H(W_y|y) - H(L|y)] \frac{\overline{F}(L|y)}{\overline{F}(K|y)} \right\} \end{aligned} \tag{3.3}$$

where $\rho = \frac{c_r}{c_m}$ is the relative cost of replacement vs. minimal repair, and $\overline{F} := (1 - F)$. (3.2) and (3.3) together yield our total expected cost as the objective function to be minimized with respect to the arguments K, L subject to $0 \leq K \leq L \leq$ ess sup W_Y, where 'ess sup' denotes the smallest right endpoint (including $+\infty$) of the support of a random variable.

The existence of an optimal pair (K, L) follows by arguments similar to those of Jack et al.(2009) and are not detailed here. From (3.2) and (3.3), it is clear in any case that in general the optimal choice of K and L are not amenable to an analytic solution, but must be obtained numerically. For illustrative purposes, we have used a Weibull family with an acceleration parameter as the distribution of failure time $F(\cdot|y)$ conditional on the usage rate via the AFT model, and a uniform distribution of the usage rate Y, which basically reflects absence of any information available to the sellers about customer's usage rates except its range. Numerical parameter values of the Weibull family chosen for our illustration are the same as in Jack et al.(2009) for normative comparison purposes. For computation of the expected cost function $I(K, L)$, averaged over the usage rate distribution, '*Romberg Integration*' (Mysovskikh(2002)) was used, while for optimization of $I(K, L)$ to find $\min_{K,L} I(K, L)$ and the corresponding optimal arguments $(K^*,\ L^*)$; computations were carried out in MATLAB with a global optimization toolbox, using probabilistic search *via* 'simulated annealing'.

3.3 The computational format

When the usage rate distribution G is continuous with a density g; in virtue of (3.2)-(3.3), the objective cost function $I(K, L)$ to be minimized, can be expressed as:

$$\begin{aligned} c_m^{-1} I(K,L) &= \int_0^\infty \left\{ H(K|y) + [H(W_y|y) - H(L|y)] \frac{\overline{F}(L|y)}{\overline{F}(K|y)} \right\} g(y)\, dy \\ &+ \int_0^\infty \frac{1}{\overline{F}(K|y)} \left(\int_K^L [\rho + H((W_y - x)|y)] f(x|y)\, dx \right) g(y) dy \\ &= J_1 + J_2. \end{aligned} \tag{3.4}$$

Using the fact, $f(x|y) = s(x, y)/g(y)$; rewrite the second term as:

$$J_2 = \int_0^\infty \int_K^L [\rho + H((W_y - x)|y)] \frac{s(x,y)}{\overline{F}(K|y)}\, dx\, dy. \tag{3.5}$$

The simplest and the most efficient way to compute the warranty cost $I(K, L)$ is to use (3.4) - (3.5). In our numerical illustration below, the cost function $I(K, L)$ was evaluated using the above representation *via* Romberg Integration for each admissible candidate (K, L) ranging over a relatively fine grid on the plane. Cost optimization with respect to the decision parameters K,L were then implemented using Simulated Annealing. We can express the term $H(\cdot|y)$, $\overline{F}(\cdot|y)$, and the joint density $s(x, y)$ in terms of the underlying Copula induced by the AFT-model and usage distribution, using Sklar's theorem and equation (2.3). Such an exercise however will not facilitate the numerical evaluation of the cost function, with which we can proceed directly using (3.4)-(3.5). The underlying Copula for a specific usage distribution and the Weibull AFT-model is of interest however, to illustrate the former's structure of dependency.

3.4 AFT Copula family

Any continuous candidate distribution of the usage rate Y,together with the AFT-model (section 3.1) of the continuously distributed time to failure conditional on Y, identifies a unique Copula, which we will refer to as the *family of AFT - Copulas*. Such Copulas are of interest since they allow flexibility of the underlying dependency structure, which are in particular summarized by *Kendall's* τ and *Spearman's* ϱ. As an illustration, we derive below the AFT Copula corresponding to a uniform distribution of the usage rate and Weibull AFT lifetimes conditional on rate of use y. The corresponding Weibull CDF and PDF conditional on y are:

$$\overline{F}(x|y) = \exp(-(x(y^{\gamma}))^{\beta}),$$
$$f(x|y) = -\frac{d\overline{F}}{dx}(x|y) = \beta y^{\gamma\beta} x^{\beta-1} \exp(-(x(y^{\gamma}))^{\beta}).$$

Hence, the joint probability density (of time to failure X, and usage rate Y) is:

$$\begin{aligned} s(x,y) &= g(y)f(x|y) \\ &= \beta y^{\gamma\beta} x^{\beta-1} \exp(-(x(y^{\gamma}))^{\beta})\, g(y) \qquad x > 0,\, y > 0 \end{aligned} \tag{3.6}$$

The joint cdf of (X,Y) is then:

$$\begin{aligned} S(x,y) &= \int_0^x \int_0^y s(t,r)\, dr\, dt \\ &= \beta \int_0^x t^{\beta-1} \left\{ \int_0^y \exp(-(t(r^{\gamma}))^{\beta}) r^{\gamma\beta} g(r)\, dr \right\} dt, \end{aligned} \tag{3.7}$$

with marginal cdf of time to failure X, given y is,

$$F(x) = P(X \leq x, Y < \infty) = S(x, \infty).$$

The Copula that uniquely corresponds to $S(x,y)$ is ,

$$C(u,v) = S(F^{-1}(u), G^{-1}(v)).$$

In particular, for uniform distribution on (a,b), for the usage rate, we get

$$C(u,v) = S(F^{-1}(u), a + (b-a)v),$$

where $x \equiv F^{-1}(u)$,$0 < u < 1$,is the unique solution of $u \equiv S(x,b)$, such that

$$
\begin{aligned}
u &= \frac{\beta}{b-a}\int_0^x t^{\beta-1}\int_a^b \exp(-(t(r^\gamma))^\beta) r^{\gamma\beta}\,dr\,dt \\
&= \frac{1}{b-a}\int_a^b \int_0^x t^{\beta-1}\exp(-(t(r^\gamma))^\beta)\,dt r^{\gamma\beta}\,dr \\
&= \frac{1}{b-a}\int_a^b \int_0^{(x(r^\gamma))^\beta} \exp(-z)\,dz\,dr \qquad (setting (x(r^\gamma))^\beta = z) \\
&= \frac{1}{b-a}\int_a^b (1-\exp(-(x(r^\gamma))^\beta))\,dr
\end{aligned} \tag{3.8}
$$

The final form of our AFT-Copula is ,

$$
C(u,v) = \frac{\beta}{b-a}\int_0^x \int_a^{a+(b-a)v} \exp(-(t(r^\gamma))^\beta) t^{\beta-1} r^{\gamma\beta}\,dr\,dt \tag{3.9}
$$

where $x \equiv F^{-1}(u)$ (solves (3.8)). Clearly, both $F^{-1}(u)$ and $C(u,v)$ in (3.9) can in general be numerically evaluated.

3.5 Numerical illustration and results

For illustration and comparison purposes, we use the same example as in Jack et al.(2009), in which

- $W = U = 2$, $c_m = 1$, $c_r = 2$;
- Weibull AFT model with $\beta = \gamma = 2$; and
- $Y \sim Uniform(0.1, 5)$, if Y is unknown.

For a minimal repairs only strategy, the expected warranty cost is

$$
J_y^m = \begin{cases} 4y^4, & \text{if } y \le 1, \\ 2y^4, & \text{if } y > 1. \end{cases}
$$

The averaged warranty cost under minimal repairs is then computed as $\int_0^\infty J_y^m\, dG(y) = 33.2932$, when G is *Uniform* $(0.1, 5)$.Table 1 also shows that Jack et al.'s results of parameters and minimal costs for various values of y. The optimal value for our expected cost with unknown Y uniformly distributed on (0.1,5), obtained by simulated annealing is 7.7923.

Simulated annealing, a probabilistic search algorithm based on MCMC (Markov Chain Monte Carlo) methods to locate the global minimum of a function, is based on an analogy between minimization and the process of annealing. The latter is a "physical process of heating a solid metal above the melting point; then cooling it down so slowly that highly excited atoms can settle down into a global minimum energy state, yielding a single crystal with a regular structure ”, since “fast cooling by rapid quenching may result in widespread irregularities and defects in the crystal structure analogous to being too hasty to find global minimum” (Won Young Yang et al.(2005)).

In our example, Figure 1 shows convergence to the optimal cost value which was basically achieved around 150 iterations (MATLAB output shows convergence up to three decimal accuracy was achieved at 146-th iteration) with optimal cost function value 7.792. Figure 2 further shows the stability of the convergence for a total of 1223 iterations; the final optimal cost function value of 7.7923 up to four decimal accuracy being achieved after 230-th iteration.

The inputs to the simulated annealing process are the expected values of cost function at combinations of (K, L), which are numerically obtained from equation (3.3) using *Romberg numerical integration* Mysovskikh(2002), see http://en.wikipedia.org/wiki/Romberg’s_method.

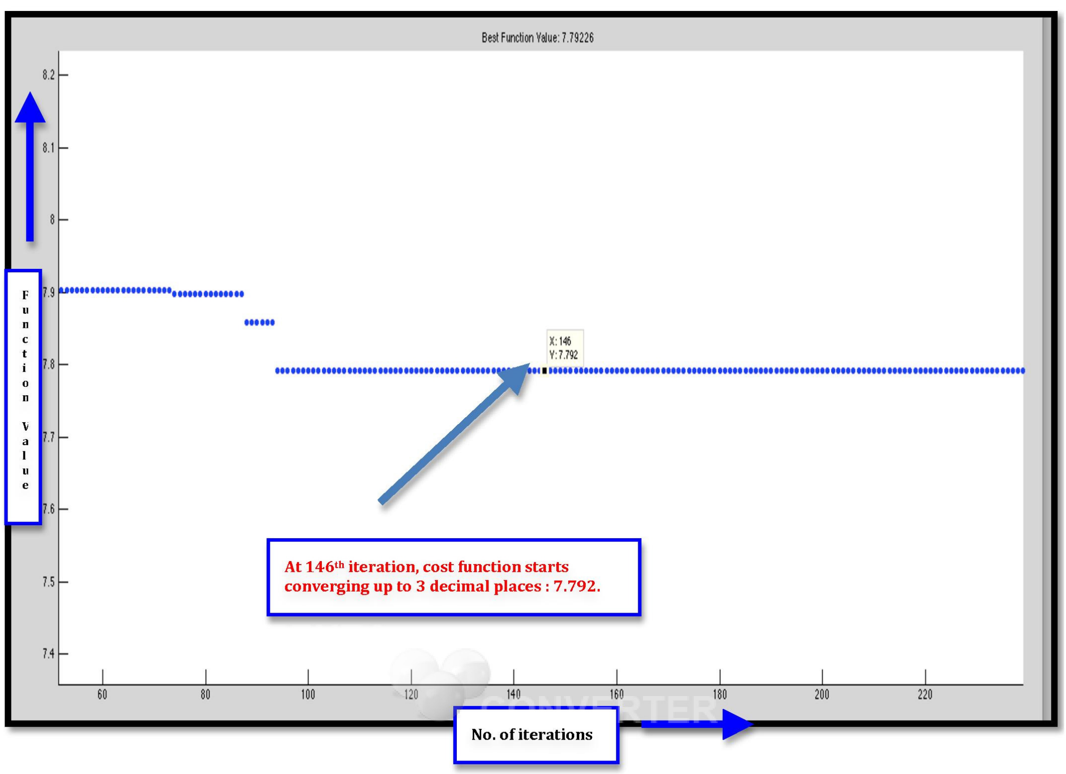

Figure 1: Simulated Annealing convergence(initial iterations)

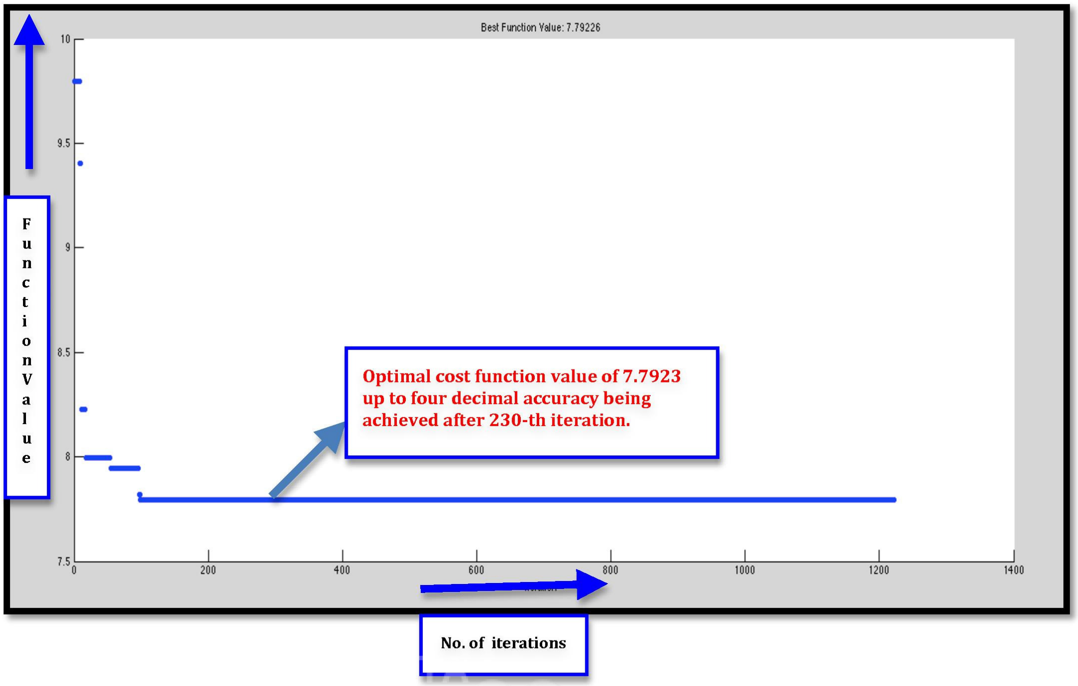

Figure 2: Simulated Annealing convergence(subsequent iterations)

(Y **known**) *Source*: Jack et al.(2009)					
y	K_y^*	L_y^*	$J(K_y^*, L_y^*)$	J_y^m	$\%reduction$
0.845	0.87	1.14	2.0382	2.0393	0.05
0.85	0.82	1.21	2.0785	2.0880	0.45
0.9	0.66	1.49	2.4614	2.6244	6.21
1.0	0.66	1.71	3.2312	4	19.22
1.2	0.60	1.51	4.0980	5.76	28.85
1.4	0.56	1.33	5.1032	7.84	34.91
1.6	0.51	1.19	6.2790	10.24	38.68
1.8	0.48	1.06	7.6157	12.96	41.24
2.0	0.44	0.97	9.1197	16	43.00
2.5	0.34	0.78	13.7987	25	44.81
3.0	0.26	0.66	19.7906	36	45.03
3.5	0.20	0.57	27.6721	49	43.53
4.0	0.16	0.50	37.4444	64	41.49
4.5	0.13	0.44	48.2577	81	40.42
5.0	0.11	0.40	61.6739	100	38.33
(Y **unknown**, uniform $(0.1, 5)$)					
y	K^*	L^*	$I(K^*, L^*)$	EJ_y^m	$\%reduction$
uniform(0.1,5)	0.0000	0.6417	7.7923	33.2932	76.60

Table 1: Cost Comparison: usage rate known vs. unknown

Comparing our cost value 7.7923 with uncertain usage rate ($Uniform$ in (0.1,5)) against Jack et al.'s (2009) optimal costs averaged over y is clearly of interest. Although it is tempting to average Jack et al.'s (*ibid.*) optimal costs for various y values shown in Table 1, to compare it against our optimal cost; such a comparison is misleading- since the average value (computed as 16.7108) of these costs in Table 1 is based on only fifteen y values, to be considered as a good estimate of $\int_0^\infty J(K_y^*, L_y^*, y)g(y)\,dy$ with $G(\cdot)$ $Uniform$ (0.1,5), even if one assumes these y values were randomly chosen, for a valid Monte Carlo estimate of the average expected costs represented by the integral. A more precise idea of the *opportunity cost* (of the optimal strategy with uncertain usage rate vs. the expected value of conditional optimal costs) can be computed as the difference

$$\Delta = I(K^*, L^*) - \int_0^\infty J(K_y^*, L_y^*, y)g(y)\,dy, \tag{3.10}$$

where the second term is easily shown to be a lower bound on our optimal cost $I(K^*, L^*)$. The optimal cost $I(K^*, L^*)$ is obviously bounded above by the corresponding expected value of minimal repair costs J_y^m averaged over y. Thus

$$\int_0^\infty J(K_y^*, L_y^*, y)g(y)\,dy \le I(K^*, L^*) \le \int_0^\infty J_y^m\,dG(y). \tag{3.11}$$

For the numerical example with uniformly distributed usage rate over (0.1, 5), the estimated numerical values corresponding to the inequality (3.11) were obtained as

$$2.2063 \le I(K^*, L^*) \equiv 7.7923 \le 33.2932.$$

The evaluation of the upper bound in (3.11) is straightforward. The lower bound in (3.11) was evaluated numerically by first choosing 1000 equally spaced y -values in the support (0.1,5) of the uniform distribution of Y and Romberg Integration to compute the conditional optimal costs $J(K_y^*, L_y^*, y)$ for each selected y; and then estimating the integral representing the lower bound by Simulated Annealing.

The plots of the Copula $C(u, v)$ in (3.9) corresponding for our numerical example (see p.10) are shown for 10 and 50 values for each of the arguments in Figure 3 and Figure 4 respectively.

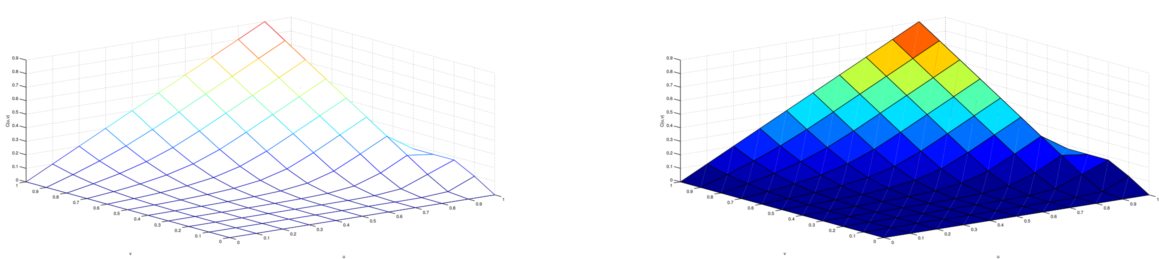

Figure 3: Mesh and Surf plots of Copula CDF for 10 values of (u, v) (where x-axis represents 'u', y-axis represents 'v' and z-axis represents 'C(u,v)')

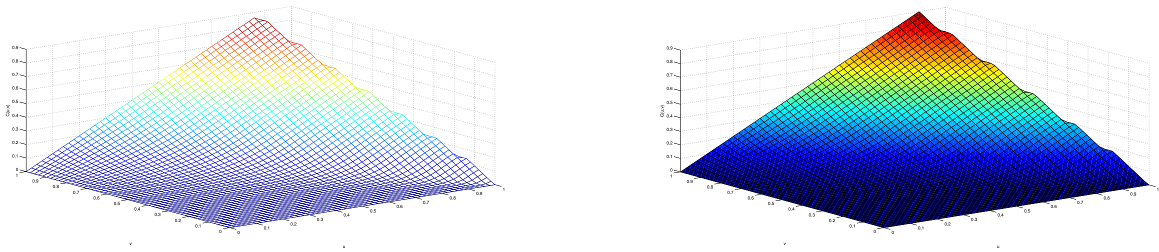

Figure 4: Mesh and Surf plots of Copula CDF for 50 values of (u, v) (where x-axis represents 'u', y-axis represents 'v' and z-axis represents 'C(u,v)')

For the Copula above, the association between usage rate and time-to-failure were numerically computed as:

$$Kendall's\ tau\ :\ \tau\ =\ -0.8862,$$
$$Spearman's\ rho\ :\ \varrho\ =\ -0.9938.$$

These values confirm our intuitive belief that increasing usage rate has an increasingly negative impact on the time to failure.

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