

# EXCITABLE NODES ON RANDOM GRAPHS: RELATING DYNAMICS TO NETWORK STRUCTURE \*

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**Abstract.** Rhythmic activity in complex systems is generated and sustained through interactions among the constituent units. In this paper we study the interplay between topology and dynamics of excitable nodes on random networks. The nodal dynamics are discrete, each node being in three possible states, active, refractory or silent. Loading rules are defined whereby a subset of active nodes may be able to convert a silent node into active one at the next time step. In the case of simple loading (*SL*) a silent node becomes active if it receives input from any neighbor. In the majority rules (*MR*) loading, a silent node fires when the majority of its neighbors are active. We address the question of whether a particular network design pattern confers dynamical advantage for the generation and sustainment of rhythmic activity. We find that the intrinsic properties of a node and the rules for interaction between them determine which structural features of the graph permit sustained activity. With *SL* the level of activity in the graph increases monotonically with the probability of connections between nodes, while for *MR*, the level of activity may be either monotonic or non-monotonic depending on parameters.

**Key words.** discrete dynamics, periodic orbit, phase transition

**AMS subject classifications.** 05C80, 37B15

**1. Introduction.** Networks of interconnected and interacting nodes (elements) are found throughout the physical, technological, and industrial world [15, 41, 35]. Social [34] and economic [12] systems also provide instances of interacting networks. Biological systems offer many examples that can be framed in terms of network. Gene networks [43], metabolic networks [24], evolutionary networks [22] and neuronal networks [40] all provide situations where nodes (genes, molecular species or neurons) interact with one another through some coupling rule.

For many decades, graph theory has provided the mathematical foundation to study topological features of complex network. Various types of networks have been defined and studied. In many networks, the state of individual network nodes does not change over time. However the connectivity between nodes may change leading to questions of how the graph evolves. The structure of these networks is dictated by certain rules on how nodes and connections between nodes are added to the network. In this regard, Erdős and Rényi [17] revolutionized the study of random graphs which are networks of elements in which some random subset makes random connections with another, not necessarily disjoint, subset of elements. These graphs, defined below, will be the focus of our study. Other types of networks have been defined by the degree distribution (number of connections) of nodes such as scale-free networks [4] or using a combination of path-length (related to the average distance between nodes) and cluster coefficients (related to the average connectivity of each node) such

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as in small world networks [42]. Certain topological features have been shown to have an impact on contact process such epidemic spreading [33, 38], communication and information processing [3].

Other studies have been conducted on how dynamics evolve on nodal networks. In these networks, the state of individual nodes changes with time while the connectivity between nodes typically does not. For example, the state of a neuron can be defined based on the voltage difference across the cell membrane which changes as a function of ion flow across the membrane over time. Important processes that are studied within this framework include the phenomenon of synchronization and the generation of sustained complex physiological rhythms [8, 18, 20].

A typical question that is asked in the context of networks is whether the network can support sustained activity, by which we mean the existence of a periodic solution. Periodic network activity plays an important role in sustainment of rhythmic processes and is often considered to be a putative pacemaker of the network. Heart beat rhythms, for instance, are generated by the activity of excitable heart cells [20]. In some studies, the underlying degree distribution has been probed to see if certain types of networks confer an advantage over others in sustaining activity in the network [11]. Hütt and Lense [21] studied the spread of excitation pattern on hierarchical graphs to understand the global interplay and network effect. Lewis and Rinzel utilized ring architectures to address questions of sustained activity in neuronal networks [29]. In [19], Gansert *et al.* consider a set of 80 neurons that are randomly connected through gap junctions. Gap junctions are physical connections between neurons that allow current to flow directly between them. Through simulations they found several examples of what they called the “kernel” of sustained activity. Namely, for each of the random networks that could support sustained activity, they identified the minimal subset of neurons that are necessary for the existence of a periodic solution on the network. Their study suggests several interesting questions of a general nature.

In a random network consisting of nodes that can change their state as a function of time, are there specific architectural features that promote sustained activity? How important are the intrinsic dynamics of individual nodes in the sustainment? Do the rules for nodal interactions relate network architecture to the existence of sustained activity? A primary goal of this paper is to begin to address these questions in a systematic manner in the context of random graphs.

A random graph consists of nodes and randomly placed edges between a subset of these nodes. Any two nodes that share an edge are said to be neighbors. The degree of a node equals the number of neighbors (or alternatively, the number of edges) it has. There are two standard ways to define a family of random graphs. One way is to let  $\mathcal{G}(n, p)$  denote the family of all random graphs that have  $n$  nodes and a probability  $p$  of an edge existing between any two nodes. The other is to let  $\mathcal{G}(n, M)$  denote the family of all random graphs that have  $n$  nodes and exactly  $M \leq n(n-1)/2$  edges randomly distributed between these nodes. In the limit of  $n$  large enough, these two sets of graphs share roughly the same properties [23], so it is common to use the set that is easier to work with for the particular application being considered. In the following, we work with  $\mathcal{G}(n, p)$ .

A property of the family of graphs  $\mathcal{G}(n, p)$  is called monotone if given that the property holds on a graph  $G_1$ , then it also holds on any graph  $G_2$  that contains  $G_1$ . For example, the property that a random graph has a  $k$ -cycle,  $k$  nodes being connected by  $k$  edges into a closed path, is monotone. A remarkable and fundamental result established by Erdős and Rényi [17] and further refined by Bollobás [7] is that

graphs with a monotone property, call it  $\mathcal{P}$ , have, in most cases, a threshold (or phase transition) for this property. Namely, there exists a probability of connection  $p_c$  such that if  $p \gg p_c$  then almost surely any graph contained in  $\mathcal{G}(n, p)$  enjoys property  $\mathcal{P}$ . Alternatively if  $p \ll p_c$  then almost surely any graph in  $\mathcal{G}(n, p)$  does not have property  $\mathcal{P}$ . We will make this precise in the methods section.

We consider a family of random graphs and ask how architecture and dynamics are related. We define a discrete dynamical system in which nodes can take on three different types of states. One type is called the refractory state, defined by the node taking on a negative integer value, where input from other nodes has no effect on the value of the node at the next time step. A second type is the silent state consisting of a single value 0. The third is the active state, defined by the node taking on a positive integer value. We refer to these as  $r : b$  nodes where  $r$  denotes the total number of possible states (positive-, negative- and zero-valued) and  $b$  denotes the number of active states (positive-valued). Two different loading rules are defined whereby some subset of active nodes may be able to convert a silent node into an active one at the next time step. In simple loading (*SL*), a silent node becomes active when it receives an input from at least one active neighbor whereas in majority rules (*MR*), a majority of neighbors are required to be in the active state for a silent node to become active. We compute the average activity of the network which is a measure of the fraction of initial conditions that lie on or converge to a periodic solution of the network averaged over many realizations of the random graph.

When the length of the active state of a node is short relative to the refractory state,  $r > 2b$ , we find that closed pathways are necessary for propagation of activity with both *SL* and *MR*. The latter case requires, however, that these closed cycles be minimal in a sense that each node on the cycle has only two neighbors. As a consequence in these situations, *SL* on average supports more dynamical activity than *MR*. The *SL* activity curves are monotone increasing as a function of edge probability, while those for *MR* are not. We show the existence of thresholds in the edge probability  $p$  that correspond to the onset of activity. In situations where the nodes have longer active duration relative to their refractory state,  $r \leq 2b$ , the presence of cycles is not required; instead a link between pairs of nodes can support activity in the network. The *SL* activity curves are still monotone, while the *MR* curves can be non-monotonic or monotonic as a function of network size depending on the fraction  $f = b/r$ . We provide numerical evidence that suggests that as the number of nodes  $n$  tends to infinity there exists a critical ratio  $f^*$  such that if  $f > f^*$ , the *MR* curves are monotonic, while if  $1/2 \leq f < f^*$ , the curves are non-monotonic. We show how the existence of various structural thresholds signal a switch to different types of activity within the network.

The inclusion of dynamics on the graph makes searching for the existence of phase transitions somewhat more complicated, the primary reason being that certain dynamic properties are not monotone. However, as we will show, it is still possible to use threshold information from the structural portion of the graphs to infer the existence of thresholds for some of the dynamic properties of the graphs. For example, the thresholds for the emergence of single edges and  $k$ -cycles play an important role in determining the type of dynamics that exist. Moreover, these and other structural features have greater or lesser importance depending on the loading rule, the ratio  $f$  and the size  $n$  of the network.

This paper is organized as follows. In Section 2, we formally define what is meant by a family of random graphs, nodal dynamics, and describe dynamical evolution

on these graphs which proceeds via a set of loading rules. In Section 3, we describe several important structural features of the graphs. These features, previously shown to exist [17], help to explain the link to dynamics. We then follow this with a numerical study utilizing different kinds of nodal dynamics and loading rules to determine which structures of the graph are likely to contribute to sustained periodic activity. Analytic arguments are presented to understand many of the numerical results. The paper concludes with a summary and discussion in Section 4.

**2. Model and Methods.** Define  $\mathcal{G}(n, p)$  as the set of all random graphs that have  $n$  nodes and a probability  $p$  that any two nodes share an edge. To define a threshold for a property  $\mathcal{P}$  of  $\mathcal{G}(n, p)$  we follow the notation and definitions given by Diestel [14]. To do so, we consider the edge probability  $p$  as a function of the number of nodes  $n$ . We shall say that a graph  $G \in \mathcal{P}$  if the graph has property  $\mathcal{P}$ . The real function  $t = t(n)$ ,  $t(n) \neq 0$  for all  $n$  is called a threshold function for  $\mathcal{P}$  if for all  $p = p(n)$  and  $G \in \mathcal{G}(n, p)$

$$(2.1) \quad \lim_{n \rightarrow \infty} P(G \in \mathcal{P}) = \begin{cases} 0 & \text{if } p/t \rightarrow 0 \text{ as } n \rightarrow \infty \\ 1 & \text{if } p/t \rightarrow \infty \text{ as } n \rightarrow \infty. \end{cases}$$

Note that  $t(n)$  is only unique up to a constant positive multiple as  $c(t(n))$  will also be a threshold for any  $c > 0$ . The value of  $c$  can be estimated in certain contexts, but will not be so relevant for our purposes. We shall often say that a property  $\mathcal{P}$  has threshold  $O(p(n))$  meaning that the phase transition for the value of  $P(G \in \mathcal{P})$  given in (2.1) occurs at  $O(p(n))$ . In this context, we shall sometimes use the notation  $p_c$  to denote the threshold.

To create an element of  $\mathcal{G}(n, p)$ , we choose two nodes  $i$  and  $j$ . We then generate a uniform random number  $x \in [0, 1]$ . If  $x < p$ , an edge is introduced between this pair of nodes  $i$  and  $j$ . This process is repeated for all distinct pairs of  $i$  and  $j$ . The information of network connectivity is represented by the elements  $A_{ij}$  of the adjacency matrix. If there is a connection between nodes  $i$  and  $j$ ,  $A_{ij} = 1$  otherwise  $A_{ij} = 0$ . As information between any two neighboring nodes can be exchanged bidirectionally, we are actually generating an undirected graph, and thus the adjacency matrix is symmetric with  $A_{ij} = A_{ji}$ . All topological information of the network architecture can be obtained from this adjacency matrix. Erdős and Rényi computed the thresholds for various monotone properties of  $G(n, p)$ . Three properties that are of particular interest to us are for the emergence of the first edge at  $O(1/n^2)$ , emergence of cycles at  $O(1/n)$  and the disappearance of the last isolated node at  $O((\log n)/n)$  [17].

Dynamics are placed on  $\mathcal{G}(n, p)$  by specifying the state or value of each node  $\sigma_i(t)$  at each discrete time  $t$ . We consider  $r : b$  nodes where  $r$  is the total number of states of the node and  $b$  denotes the allowed number of active states. Thus for each  $t$ ,  $\sigma_i(t) \in \{-(r-b-1), -(r-b-2), \dots, -1, 0, 1, 2, \dots, b\}$ . A node is said to be refractory if its state has negative value, silent if in state 0 and active if it has any positive value. We will also call a node inactive if it is either in the refractory or silent state. We define  $f = b/r$  to denote the fraction of states for which a neuron can be active. To provide insight onto why such dynamics are chosen, we note that  $r : 1$  nodes mimic the behavior of spiking neurons. Here the nodes have a single active state that corresponds to the spike or action potential of a neuron. The  $r - 2$  refractory states correspond to the time during which the neuron cannot fire. A  $r : b$  node mimics the behavior of a bursting neuron that exhibits  $b$  consecutive spikes before entering a refractory period of length  $r - b - 1$ . It is well known that real biological systems show both spiking and bursting behaviors [16, 37] and thus it is of interest

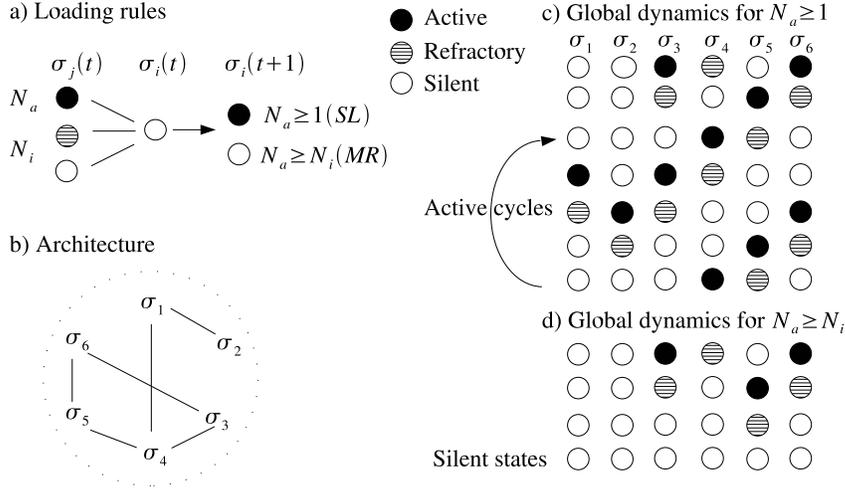


FIG. 2.1. *Network Dynamics:* (a) Each node can be in the active (solid ball), refractory (hashed ball) or silent (empty ball) state. A silent node can become active depending on the states of its neighbors and the loading rule *SL* or *MR*. In the former case, since  $N_a(t) \geq 1$ , the node becomes active. In the latter since,  $N_a(t) < N_i(t)$ , the node remains silent. (b) Example of a random graph with six nodes and 6 edges. (c) Evolution of a particular initial condition along the graph shown in (b) under the *SL* rule showing convergence to a period 4 solution. (d) Evolution of the same initial condition as in (c) along the graph shown in (b) under the *MR* rule showing convergence to the 0 fixed point.

to see how these differences may affect the global collective dynamics in networks of such systems.

The dynamical evolution of the nodal state  $\sigma_i(t) \neq 0$  takes place as follows

$$(2.2) \quad \sigma_i(t+1) = \begin{cases} \sigma_i(t) + 1 & \text{if } \begin{cases} 1 \leq \sigma_i(t) \leq b-1 \text{ or} \\ -(r-b-1) \leq \sigma_i(t) < 0 \end{cases} \\ -(r-b-1) & \text{if } \sigma_i(t) = b. \end{cases}$$

These transitions happen independent of the number of neighbors a node may have and can be thought of as the intrinsic dynamics of each node.

When  $\sigma_i(t) = 0$ , the determination of when the node switches from 0 to 1 is a function of the number of neighbors it has together with what we will call loading rules of the node. At any time  $t$ , denote the number of active neighbors that a node has by  $N_a(t)$ . Similarly denote by  $N_i(t)$  the number of inactive neighbors the node has. Clearly the degree  $d = N_a(t) + N_i(t)$  and while the degree is constant the individual terms  $N_a$  and  $N_i$  need not be. We consider two different loading rules; see Fig. 2.1. The first rule is called simple loading, *SL*, in which a node in state 0 will switch to state 1 if it has at least one active neighbor

$$(2.3) \quad \sigma_i(t) = 0 \rightarrow \sigma_i(t+1) = 1 \text{ if } N_a(t) \geq 1.$$

The second rule is called majority rules, *MR*, in which a node in state 0 will switch to state 1 if a majority of its neighbors are active

$$(2.4) \quad \sigma_i(t) = 0 \rightarrow \sigma_i(t+1) = 1 \text{ if } N_a(t) \geq N_i(t).$$

These rules are obviously special cases of all possible loading rules. Nonetheless, as will be seen, they allow us to gain significant insight into how architecture and dynamics can be related.

Note that  $MR$  is defined to consider only active neighbors and ignores silent and refractory nodes. The reason for this choice is that in networks of gap-junctionally coupled neurons, any time current is injected into a particular neuron (node), some amount of current will leak through the gap-junctions to neighboring cells that are either silent or refractory. As a result, less current will be available to fire the neuron into which current was injected. Analogously, we take any neighboring node in the silent or refractory state to create a load that inhibits the ability of the chosen node to become active.

We define  $A(t)$ ,  $R(t)$ , and  $S(t)$  as the sets of nodes that are in active, refractory, and silent states at each time  $t$ . That is for each  $t$ , let

$$(2.5) \quad A(t) = \{i : \sigma_i(t) > 0\}, \quad S(t) = \{i : \sigma_i(t) = 0\}, \quad R(t) = \{i : \sigma_i(t) < 0\}.$$

Note that at any time  $t$ ,  $|A(t)| + |S(t)| + |R(t)| = n$ . The states of all nodes are collected into the  $n$ -vector

$$(2.6) \quad \vec{\sigma}(t) = \{\sigma_1(t), \sigma_2(t), \dots, \sigma_n(t)\}.$$

As the state of each node  $i$  is updated synchronously, the different values of  $\vec{\sigma}(t)$  over time  $t$  constitute a dynamical trajectory in the phase space. Since the dynamical evolution takes place in a finite phase space consisting of  $r^n$  states, all trajectories eventually tend to an attractor, either a periodic orbit or a stable fixed point. For any initial condition, the trajectory after passing through transient states may return to previously visited points, implying the existence of a periodic solution

$$(2.7) \quad \vec{\sigma}(t) = \vec{\sigma}(t + q)$$

where  $q > 1$  is the smallest integer value and corresponds to the period of the orbit. For a given initial condition, if there is no value of  $q$  for which (2.7) is satisfied, then the trajectory tends to the unique fixed point  $\vec{\sigma} \equiv 0$ .

At a given value of  $p$  for fixed  $n$ , we randomly choose 10,000 initial conditions  $\vec{\sigma}(0)$  and compute the steady state response of the network for each  $\vec{\sigma}(0)$ . We obtain the dynamical activity in the network  $A_f$  which is defined as the fraction of initial conditions that tend to a periodic solution. Note that  $A_f$  can also be interpreted as the probability that a randomly chosen initial condition lies in the basin of attraction of a periodic solution. We partitioned the set  $[0, 1]$  of probability values and unless otherwise stated, we generated 2000 different random graphs for each value of  $p$  and obtain an expected value of dynamical activity  $A_f$  of  $\mathcal{G}(n, p)$ . Many threshold phenomena arise at lower values of  $p$ , so the partition was chosen to be much finer there to resolve the numerics. In some cases, we were interested in properties of  $\mathcal{G}(n, p)$  independent of the dynamics on the network. Here we simply took the ensemble average for the quantity in which we were interested over the total number of graphs generated. Our typical simulations involving the structure of  $\mathcal{G}(n, p)$  utilize  $n = 30$ . The theoretical results for random graphs hold when  $n$  is asymptotically large. Despite our relatively small number of nodes, there is reasonable agreement between our numerical results involving graph structure and the analytic counterparts. However, once dynamics are placed on the graph, the size of the network  $n$  can play a role in determining the properties of  $A_f$  as illustrated in section 3.3, where we have used  $n = 30, 50$  and  $100$ .

### 3. Results.

**3.1. Structural properties of random graphs.** We first review some of the seminal results of Erdős and Rényi [17] concerning the existence of certain thresholds in  $\mathcal{G}(n, p)$ . The property that a graph  $G \in \mathcal{G}(n, p)$  contains an edge is a monotone property. Indeed if a subgraph contains an edge, then any larger graph, containing this subgraph, will also contain the edge. The threshold for the emergence of the first edge in  $\mathcal{G}(n, p)$  is  $O(1/n^2)$  [17]. Similarly, the property that  $G$  contains a  $k$ -cycle, (a cyclic sequence of  $k$  nodes formed by connecting a sequence of  $k$  edges) is also a monotone property of  $\mathcal{G}(n, p)$ . The threshold at which  $k$ -cycles emerge is  $O(1/n)$ , independent of  $k$  [17]. We shall call a  $k$ -cycle minimal if it is a  $k$ -cycle that consists of  $k$  nodes each of which has degree exactly equal to two. That is, the nodes in a minimal  $k$ -cycle have edges with only two others in the cycle and no other nodes outside of the cycle. Thus by definition a minimal  $k$ -cycle is isolated from other nodes in the network. The giant component of  $\mathcal{G}(n, p)$  refers to the largest subset of nodes within the graph that share an edge with at least one other node in the subset. Erdős and Rényi showed that the size of the giant component also undergoes a phase transition above  $O(1/n)$  [17]. Namely for  $p < O(1/n)$ , the graph is very sparsely connected and the largest component is small. Whereas if  $p > O(1/n)$ , a large cluster forms that contains a large fraction of nodes. This result implies that with very small changes in  $p$ , the graph suddenly switches from one in which there exist many small islands of sparsely connected clusters of nodes to one in which a dominant and large cluster emerges. The giant component grows in size until another threshold occurs for the disappearance of the last isolated node, or for when the graph becomes connected. This occurs at  $O((\log n)/n)$ . The graph becomes fully connected, or complete, when  $p = 1$  and all edges between nodes exist.

We have numerically computed various structural aspects of  $\mathcal{G}(n, p)$  for  $n = 30$  nodes and have recovered the results mentioned above, thereby providing validation for our method. In later sections, we use larger values for  $n$  when considering dynamics on these graphs. We have also computed a number of other statistics that will help us determine the role of architecture in generating sustained activity.

**Evolution of edges:** Determining the number of edges for the cases  $p = 0$  and  $p = 1$  is trivial. For  $p = 0$ , all nodes are isolated whereas if  $p = 1$ , the system is fully connected. Fig. 3.1(a) shows the proportion of nodes that have an edge as the probability of connections  $p$  is varied from  $p = 0$  to  $p = 1$ . The black (red) curve shows the average fraction of nodes which (do not) have an edge. As  $p$  increases, the number of nodes which have edges gradually increases while the number of isolated nodes decreases. At  $p = p_1$  these two curves intersect and, on average, half of the nodes have edges while the other half are isolated. As  $p$  increases above  $p_1$ , more and more isolated nodes gradually obtain edges until the graph reaches a stage  $p = p_2 \sim (O((\log n)/n))$ , from where the evolution saturates. After this value  $p = p_2$ , in every realization of the graph, all nodes are connected to a cluster and hence the largest cluster is formed. We also plot the average degree of nodes in the random network as a function of  $p$ ; see blue line in Fig. 3.1(a). As can be expected, this is a monotone increasing function of  $p$ .

**Evolution of clusters:** At lower values of  $p$ , the cluster  $C_2$  (size 2) appears first. It is then followed by the appearance of clusters of size  $C_3, \dots, C_n$  respectively giving rise to the formation of different size clusters at different values of  $p$ . Fig. 3.1(b) shows the distribution of the number of isolated clusters in the system as a function of  $p$ . The curve  $C$  showing the number of isolated clusters increases with  $p$  (largest contribution

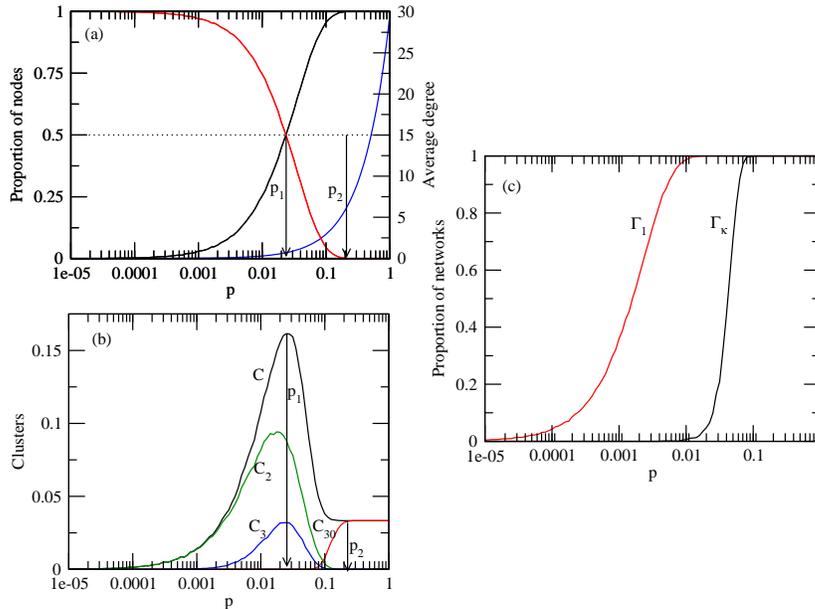


FIG. 3.1. Structural features of  $\mathcal{G}(n, p)$  illustrated with  $n = 30$  nodes.: (a) The black curve plots the fraction of nodes that have at least one edge and red curve plots the fraction of isolated nodes as a function of the probability of edge existence. The values  $p_1$  and  $p_2$  are defined in the text. The blue curve shows the variation of average degree of nodes in the random network vs.  $p$ . (b) The number of clusters of size 2, 3 and 30 ( $C_2$  green,  $C_3$  blue and  $C_{30}$  red) normalized by the number of nodes  $n$  (behavior of other clusters computed but not shown). The black curve  $C$  shows the total number of clusters normalized by  $n$ . (c) The red curve  $\Gamma_1$  shows the fraction of graphs that contains at least one edge while the black curve  $\Gamma_k$  shows the fraction that contains a  $k$ -cycle.

arises from  $C_2$ ) until it reaches a maximum value and then decreases to a value from where it starts to saturate. The peak of the distribution of isolated clusters occurs around the point  $p = p_1$  where almost half of the nodes are connected. Above this value, the addition of new edges leads to the formation of larger clusters by connecting small isolated clusters and isolated nodes simultaneously. In this transition, there is a dramatic change of connectivity leading to the emergence of the giant component. This then leads to a decrease in the number of isolated clusters. The behavior of  $C_2$  and  $C_3$  is also shown in Fig. 3.1(b). Both these curves increase, reach a maxima and then decrease to zero. The distribution of all  $C_i$ s, with  $2 \leq i \leq n-1$ , shows a similar behavior. Only the largest size cluster  $C_n$  shows a sigmoidal behavior. The largest cluster  $C_{30}$  (red curve in Fig. 3.1(b)) begins to emerge after  $p_1$  and increases until it reaches the plateau of the  $C$  curve. In the plateau region, every realization of the network leads to the formation of the largest size cluster.

**Emergence of single edges and  $k$ -cycles:** In Fig. 3.1(c), we have plotted the threshold for the emergence of the first edge  $\Gamma_1$  (red curve) and for the emergence of  $k$ -cycles  $\Gamma_k$  (black curve). The analytical value for these two thresholds are  $O(1/n^2)$  and  $O(1/n)$  respectively. Our numerical study consisting of  $n = 30$  nodes yields thresholds of the same respective orders of magnitude despite  $n$  being small. These two curves will serve as left (or upper) bounds of certain activity curves that we will define below in the dynamics section.

**3.2. Dynamics with  $r : 1$  nodes.** Consider a network of  $n, 3 : 1$  nodes with the simple loading rule  $SL$  defined in (2.3). The state space of this network consists of  $3^n$  points. A graph  $G \in \mathcal{G}(n, p)$  can support a periodic solution only if it contains a  $k$ -cycle with  $k \geq 3$ . Note the property of a graph containing a periodic solution of a specific length is not monotone. For example, consider a graph consisting of four nodes connected in an minimal 4-cycle. This graph has a period 4 solution which can be found by evolving the initial state  $(1, 0, 0, -1)$ . Next add an edge along a diagonal of the graph so that it now has five edges. The new graph obviously contains the 4-cycle as a subgraph, but it cannot support a period 4 solution. Instead it has only a period 3 solution. Because of this loss of monotonicity, there is no apparent reason that once dynamics are placed on  $\mathcal{G}(n, p)$  that thresholds or phase transitions will exist. However, it is true that with  $SL$ , the property that a graph contains any periodic solution is in fact monotone.

**Proposition 3.1** The property of a graph containing periodic activity is monotone under the  $SL$  rule.

**Proof** - Let  $G$  be a graph on which there exists periodic activity. Let  $G_1$  be a subgraph from the family of subgraphs that contain the smallest length minimal  $k$ -cycles such that  $k \geq r$ . Note that a necessary condition for a periodic solution on a  $k$ -cycle for  $r : 1$  nodes is that  $k \geq r$ . If  $k = r$ , label the nodes of  $G_1$  by  $1, 2, \dots, k$  where nodes 1 and  $k$  are taken to be neighbors. Then the initial condition

$$\sigma_0 = (1, 0, -1, \dots, -(r - 2))$$

lies on a periodic solution. The addition of a diagonal to  $G_1$  will not change the fact that  $\sigma_0$  still lies on the periodic solution. This is because at any time  $t$ , there is exactly one triple of sequentially labeled nodes in states  $(-(r - 2), 1, 0)$  (or  $(0, 1, -(r - 2))$ ). The third node neighboring the active one must necessarily be in the refractory state. A similar argument shows that if an external node, labeled  $k + 1$ , and edge is added to  $G_1$  with neighbor  $j \leq k$ , then the initial condition  $(1, 0, -1, \dots, -(j - 1), \dots, -(r - 2), -(j - 1))$  lies on a periodic solution. If  $k > r$ , the same two arguments apply by considering

$$\sigma_0 = (1, 0, -1, \dots, -(r - 2), -(r - 2), \dots, -(r - 2))$$

or its appended version when an external node is added. These results show that a graph obtained by adding an external node and edge, or a diagonal can support periodic activity if the original subgraph did. Finally, any graph  $G$  that supports activity can be constructed as a monotone sequence of graphs starting from a minimal  $k$ -cycle. Thus the property is monotone. ■

Although the existence of periodic activity with  $SL$  is a monotone property of a graph, that a particular initial condition lies in the basin of attraction of a periodic orbit is not monotone. For example, consider a graph consisting of a minimal 5-cycle with 5:1 dynamics on the nodes. Then the initial condition  $(1, 0, 0, 0, -1)$  lies in the basin of attraction of the periodic orbit obtained from iterating  $(1, 0, -1, -2, -3)$ . However if an edge is introduced between nodes 2 and 5, then

$$(1, 0, 0, 0, -1) \rightarrow (-3, 1, 0, 0, 0) \rightarrow (-2, -3, 1, 0, 1) \rightarrow (-1, -2, -3, 1, -3) \rightarrow (0, -1, -2, -3, -2) \rightarrow \dots \rightarrow (0, 0, 0, 0, 0),$$

and the trajectory tends to the 0 fixed point. On the other hand, it is clear that initial conditions that tend to 0 on a subgraph may in fact lie in the basin of attraction of a

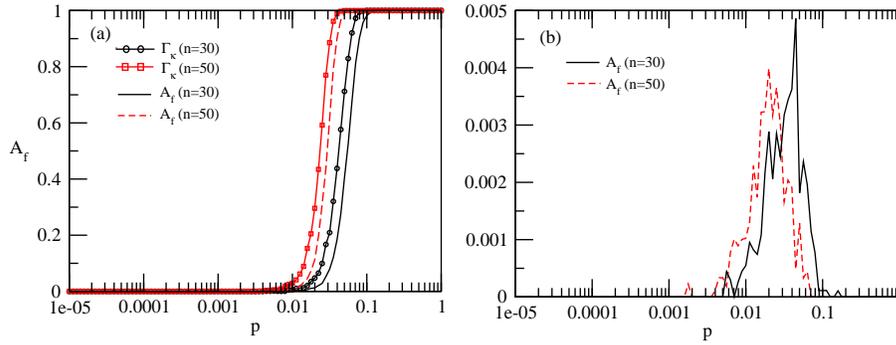


FIG. 3.2. Phase transition for the existence of period solutions. (a) The dynamics of 3:1 nodes for  $SL N_a(t) \geq 1$  are shown [solid (dashed) curve for  $n = 30$  (50)] illustrating the activity emerges with nearly the same threshold function as the emergence of  $k$ -cycles [curve with circle (box) for  $n = 30$  (50) nodes]. (b) The dynamics of 3:1 nodes under MR  $N_a(t) \geq N_I(t)$  showing the emergence and destruction of activity. Solid (dashed) curve for  $n = 30$  (50) nodes. Note the small units on the vertical axis relative to panel (a).

periodic orbit on a larger graph. Since  $A_f$  measures the fraction of initial conditions that lie in the basin of attraction of a periodic orbit, it is not obvious how this quantity will change as the graph grows.

To begin to understand the behavior of  $A_f$  as a function of  $p$ , consider first two extremal cases. When  $p$  is small, a typical graph in  $\mathcal{G}(n, p)$  is sparsely connected and will not contain any  $k$ -cycles. Thus periodic solutions for small  $p$  are unlikely to exist and  $A_f \approx 0$ . Alternatively, in the opposite extreme when  $p = 1$  and the graph is complete, the situation is entirely different.

**Proposition 3.2** Let  $p = 1$ . There exist  $(1 - 3 \cdot (2/3)^n)3^n$  distinct period 3 solutions each of whose basin of attraction is exactly the orbit itself. Further  $A_f \rightarrow 1$  as  $n \rightarrow \infty$ .

**Proof** - Recall from equation (2.5) the definitions for  $A(t)$ ,  $R(t)$  and  $S(t)$ . There exist  $3^n$  distinct initial states in the network each of which leads to an assignment of nodes to  $A(0)$ ,  $R(0)$  and  $S(0)$ . A necessary and sufficient condition for an initial state to lie on a periodic solution is if for that initial state,  $A(0), R(0), S(0) \neq \emptyset$ . There are  $2^n$  initial states leading to  $A(0) = \emptyset$  and similarly for  $R(0) = \emptyset$  and  $S(0) = \emptyset$ . Thus there are  $(1 - 3 \cdot (2/3)^n)3^n$  initial states that satisfy the necessary and sufficient conditions.

Each of the periodic orbits only attracts itself since  $R(t) = S(t+1) = A(t+2) = R(t+3)$ . Thus there are no transients when converging to a periodic state. The remaining initial states that converge to the 0 fixed point may have transients, but these will be no larger than length 3. Finally, as  $n \rightarrow \infty$  the fraction of initial states converging to the fixed point tends to 0 as can be seen by taking the limit of the expression given in the statement of the proposition. Thus  $A_f \rightarrow 1$  as  $n \rightarrow \infty$ . ■

Proposition 3.1 concerning the monotonicity of the existence of periodic activity with  $SL$  suggests that there may exist a threshold in  $p$  for the activity curve  $A_f$ . This curve should be bounded to the left by  $\Gamma_k$ , the threshold for the existence of  $k$ -cycles, at  $p_c \sim 1/n$ . In Fig. 3.2(a), we plot  $A_f$  the activity of the network as a function of edge probability  $p$  (solid line  $n = 30$ , dashed line  $n = 50$ ). As can be seen, there is a rapid transition in  $p$  from where  $A_f \approx 0$  to where  $A_f \approx 1$ . For  $p \in (p_c, 1)$  the vast majority of solutions are period 3, but occurrence of other periodic solutions is possible

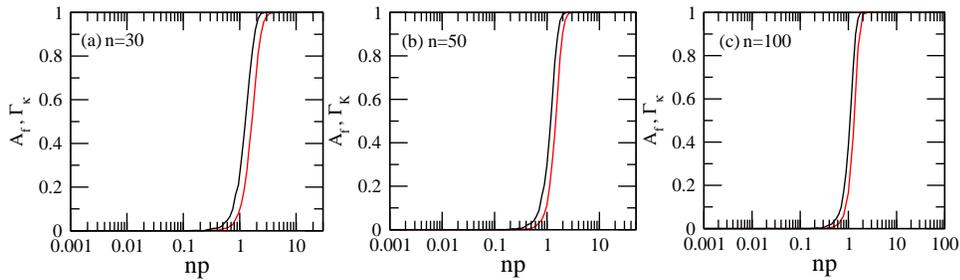


FIG. 3.3.  $A_f$  (red curve) and  $\Gamma_k$  (black curve) plotted as a function of  $np$ . The dynamics of 3:1 nodes for  $SL$  (a)  $n = 30$ , (b)  $n = 50$  and (c)  $n = 100$  indicate that the gap between  $A_f$  and  $\Gamma_k$  is shrinking as  $n \rightarrow \infty$  such that  $np = c$ .

along any  $k$ -cycle. The possible periodic solutions on a minimal  $k$ -cycle consist of all integers  $q$  and  $s$  which satisfy the relation  $qs = k$  where  $q, s \in \mathbb{Z}$  and  $q, s \neq 1, 2$ . As  $p \rightarrow 1$  all periodic solutions have period 3. Note that the phase transition for the almost sure existence of periodic solutions is at  $p_c \sim 1/n$ . As  $n$  increases the threshold moves to the left along with the entire activity curve as seen in the figure. Also shown in Figure 3.2(a) are the threshold functions for the emergence of  $k$ -cycles denoted by  $\Gamma_k$  ( $n = 30$  line with circle,  $n = 50$  line with box).

In Figure 3.2(a),  $\Gamma_k > A_f$  since the emergence of cycles is necessary for there to be activity on the graph.  $\Gamma_k$  is the probability that an element of  $\mathcal{G}(n, p)$  contains at least one  $k$ -cycle, which clearly increases with  $p$ .  $A_f$  can be interpreted as the probability that a randomly chosen initial condition lies in the basin of attraction of a periodic orbit. This quantity will increase as the probability of the existence of cycles within the graph increases with  $p$ . While only a fraction of initial states in the network converge to a periodic solution, meaning  $A_f$  is less than  $\Gamma_k$ , it is an open question about the behavior of  $A_f$  as  $n \rightarrow \infty$  such that  $np = c$  for  $c > 0$ . In Fig. 3.3, we show simulation results of  $n = 30, 50$  and  $100$  plotted versus  $np$ . In this scaling, the simulations indicate that  $A_f$  may be converging to  $\Gamma_k$ . Despite this numerical evidence, it is still not clear if there is a well defined limit for  $A_f$ , and if so, what the limiting curve is. The primary issue is how the size of the state space grows relative to the set of initial conditions that converge to periodic solutions.

Next let us consider the majority rules ( $MR$ ) loading rule given by  $N_a(t) \geq N_i(t)$  (2.4). In this scenario, a node will switch from the 0 state to the 1 state only if a majority of its neighbors is currently in state 1. Let us first establish that the situation will be different than  $SL$  by discussing the  $p = 1$  case. In the fully connected network of  $n$  nodes, consider the cardinality of the sets  $A(t)$ ,  $R(t)$  and  $S(t)$ , which at any  $t$ , satisfy  $|A(t)| + |R(t)| + |S(t)| = n$ . A necessary condition for sustained activity is that none of the sets  $A(0), R(0)$  and  $S(0)$  are empty. Further for nodes in  $S(t)$  to fire,  $A(t) \geq n/2$ . If that is the case, then  $|A(t+1)| = |S(t)| < n/2$  and thus nodes in  $S(t+1)$  do not satisfy  $MR$  and the activity dies out. Thus at  $p = 1$ , in contrast, to  $SL$ , the activity of the network with  $MR$  tends to 0. Of interest then is to determine what occurs for intermediate values of  $p$ .

In Fig. 3.2(b), we plot activity  $A_f$  versus edge probability  $p$  and note that the scale on the vertical axes is very different than in Fig. 3.2(a). In fact very few initial states converge to a sustained activity pattern. Those patterns that do exist are periodic solutions living on minimal  $k$ -cycles. Such minimal cycles are likely to occur in only a small window of  $p$  values centered around  $p = 1/n$ . For example, if

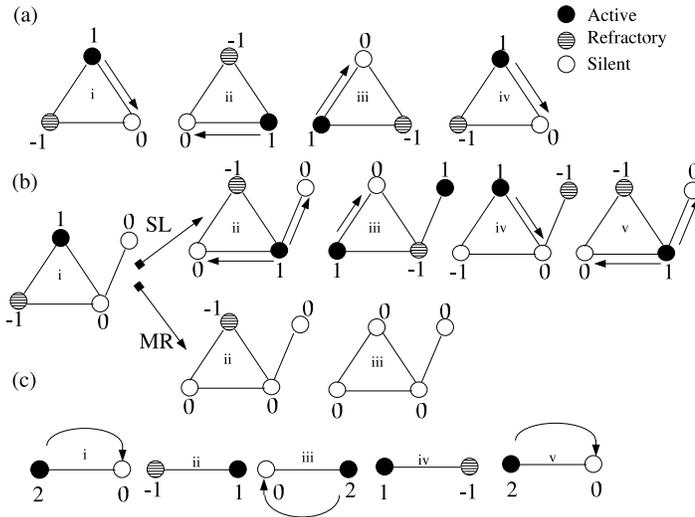


FIG. 3.4. Motifs giving rise to sustained activity: (a) minimal cycle structure give rise to rhythmic activity with both loading rules  $N_a \geq 1$  and  $N_a \geq N_i$  for 3 : 1 nodes. Arrows indicate the direction in which the activity flows. Lower case roman numerals denote discrete time step. (b) Additional links to the isolated, minimal structure destroy activity with loading rule  $N_a \geq N_i$  for 3 : 1 nodes. (c) When there is a longer active state on the nodes compared to the refractory state, a connected pair of nodes is sufficient to generate sustained rhythmic activity in the network. Dynamics of 4 : 2 nodes are shown in this example.

$X$  denotes the number of minimal 3-cycles, then as  $n \rightarrow \infty$  with  $np = c$  fixed, the expected number of minimal 3-cycles is  $E(X) = [c^3 \exp(-3c)]/6$ , which has a local maximum at  $c = 1$ . A similar result with local maximum at  $c = 1$  holds for minimal cycles of other lengths. Thus  $A_f$  will be non-zero, but very small over a small window of  $p$  values around  $1/n$ . To see why activity dies with  $MR$  as  $p$  increases, observe that as with  $SL$ ,  $k$ -cycles are necessary for sustained activity. However for  $MR$ , activity can only be sustained on minimal cycles where each node has degree  $d = 2$ ; see Fig. 3.4 (a)-(b). The addition of a single additional edge between nodes in the loop, or of an edge and node exterior to the loop implies that  $MR$  can not be satisfied by all nodes in the sub-graph. Thus activity begins to emerge as in  $SL$  around  $p = O(1/n)$ , but it very quickly thereafter (in  $p$ ) dies away. The disappearance of activity with  $MR$  is due to the disappearance of minimal cycles in the random network as the value of the edge probability  $p$  increases. While this would seem to be a negative result of sorts, it does suggest something quite general. Namely, that in these networks, there may exist a second threshold which governs the destruction of activity. We shall explore this further in subsequent sections.

We end this section by considering  $r : 1$  nodes for  $r > 3$ . The activity curves for several of these cases are shown in Fig. 3.5. Note that the curves move to the right for increasing  $r$ , and most continue to converge to 1 as  $p \rightarrow 1$ . Proposition 3.2 can be generalized to show this convergence at  $p = 1$ , but with a slight caveat. The fraction  $2/3$  that appears in that proposition would now be replaced by the fraction  $(r - 1)/r$ . Thus the generalization of Proposition 3.2 for any  $r$  fixed and  $n \rightarrow \infty$  implies that the activity curves will approach one as  $p \rightarrow 1$ . However in Fig. 3.5, we are actually plotting the results of taking a different limiting process. In the figure, we are fixing

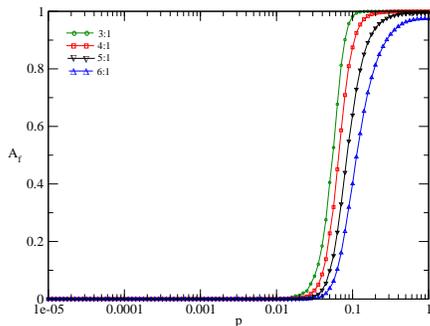


FIG. 3.5. Dynamics of  $r : 1$  nodes ( $n=30$ ) under the  $SL$  rule. Note that as  $r$  increases the curves move to the right as explained in the text.

$n$  and increasing  $r$ . Since  $(r - 1)/r \rightarrow 1$  as  $r \rightarrow \infty$ , the value at  $p = 1$  of the activity curves would decrease toward 0 with increasing  $r$ . That is why the activity curve for the 6 : 1 network of nodes has a value smaller than 1 at  $p = 1$ . The reason that the curves move to the right with increasing  $r$  has to do with the emergence of a sufficient number of cycles of length  $r$  relative to the size of the state space  $r^n$ . A periodic solution for  $r : 1$  nodes requires the existence of a  $r$ -cycle. Although cycles of any length begin to appear at roughly the same time in  $\mathcal{G}(n, p)$ , the average number of cycles of a particular length depends on the cycle length itself. As shown by Diestel [14] if  $X$  denotes the number of  $k$ -cycles within a graph  $G \in \mathcal{G}(n, p)$ , then the expected number of  $k$ -cycles is  $E(X) = (n_k/2k)p^k$  with  $n_k = n(n - 1)(n - 2) \cdots (n - k + 1)$ . Clearly larger numbers of 3-cycles are expected for small  $p$  than  $k$ -cycles for  $k > 3$ . Further, while there is a larger number of expected 4-cycles as  $p \rightarrow 1$ , the size of the state space of a 4:1 network is much larger than that of a 3:1 network. Indeed the size of the state space grows faster than the set of initial conditions that converge to a periodic solution. Thus the activity curves must move to the right as shown. We also computed the activity for  $MR$  for  $r : 1$  nodes but are not showing the results here. The curves are similar to what is shown for 3:1 nodes in Fig. 3.2(b) with the local maximum of these curves having very small value.

**3.3. Dynamics with  $r : b$ ,  $b > 1$  nodes.** We now generalize the nodal dynamics by allowing each node to be active for  $b$  time steps, where  $b$  is a positive integer. The node can also take on negative values from  $-(r - b - 1)$  to  $-1$  while in the refractory state. Finally the node may be silent and in state 0. These type of dynamics can be thought of as bursting dynamics where the node is active for several time steps before becoming refractory for some number of time steps and then silent. In general for a  $r : b$  node, we can classify intrinsic dynamics of nodes into two types with  $r > 2b$  or  $r \leq 2b$ . In situations where the nodal states satisfy the condition  $r \leq 2b$ , a node is in an active state for greater or equal amount of time than it spends in inactive states. If  $r > 2b$ , a node spends less amount of time in active states than in inactive states. In the previous section, we have studied the dynamics of  $r : 1$  nodes (that clearly satisfy  $r > 2b$ ) using different loading rules. As in these cases, in general for  $r > 2b$  nodes, sustained dynamical activity takes place on  $k$ -cycles ( $k \geq r$ ) with  $SL$  and minimal  $k$ -cycles are required with  $MR$ .

For  $r \leq 2b$  node, there is a sharp contrast in the structural unit supporting sustained dynamical activity. The first point to note is that for  $r : b$  nodes with  $r \leq 2b$ ,  $k$ -cycles ( $k \geq 3$ ) are not needed for sustained activity. A simple pair of nodes

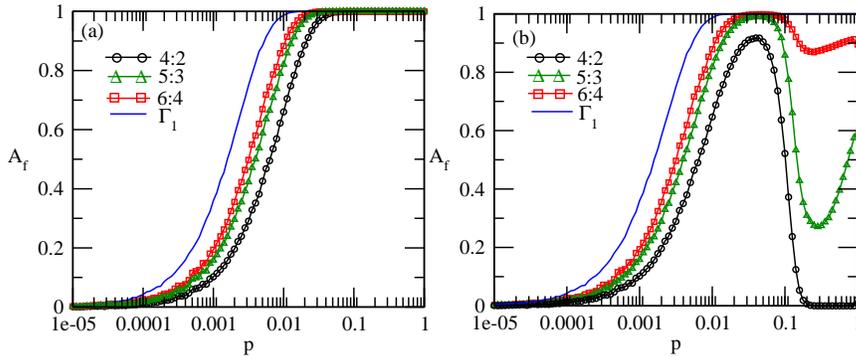


FIG. 3.6. Dynamics of  $r : b$  nodes ( $n=30$ ): (a) Activity curves for various different types of nodes under  $SL$ . The blue curve is the threshold function for the existence of a single edge in  $\mathcal{G}(n, p)$  and serves as the left-most bound for these curves. (b) Activity curves for different types of nodes under  $MR$ . Note that  $A_f$  becomes monotone increasing as  $r = b + 2 \rightarrow \infty$  and approaches its  $SL$  counterpart in this limit.

is sufficient. Consider two  $4 : 2$  nodes ( $n = 2$ ) with one edge. This pair can have sustained activity with both  $SL$  and  $MR$  loading rules. Consider the initial condition  $(2, 0)$  which upon evolution yields the sequence  $(2, 0) \rightarrow (-1, 1) \rightarrow (0, 2) \rightarrow (1, -1) \rightarrow (2, 0)$ ; see Fig. 3.4 (c). The existence of this period 4 solution results from the longer active duration of these nodes relative to their refractory state. More generally, we establish the following result.

**Proposition 3.3:** Consider two  $r : b$  nodes with one edge. Under both loading rules  $SL$  and  $MR$ , there exists a period  $r$  solution if and only if  $r \leq 2b$ .

**Proof:** Consider  $r = 2b$  and the initial condition  $(0, b)$ . Since  $r = 2b$ , the length of the refractory state is exactly  $b - 1$ . Therefore after  $b$  time steps, node 1 is in state  $b$ , while node 2 has transitioned through the refractory state and is now silent with state 0. The new state of the network is  $(b, 0)$  and the nodes have switched orientations, implying that after  $b$  more steps the network returns to its initial value. Thus a period  $r = 2b$  solution exists.

If  $r < 2b$ , let  $h = 2b - r$ . The initial state  $(h, b)$  leads after  $r - b$  steps to the state  $(b, 0)$  since the length of the refractory state is  $r - b - 1$ . After an additional  $b$  steps, the state of the network returns to  $(h, b)$  yielding periodicity.

If  $r > 2b$  consider a state  $(0, j)$ ,  $1 \leq j \leq b$ . After  $b + 1$  steps the node will be in state  $(1 - (r - b), 1 - (r - b) + j)$ . Note  $1 - (r - b) + j \leq 2b + 1 - r \leq 0$  since  $j \leq b$  and  $r > 2b$ . Thus both nodes will be inactive and activity will end. Any other initial state  $(-m, j)$ ,  $1 \leq m \leq r - b + 1$ ,  $1 \leq j < b$  can be handled similarly to show convergence to the  $(0, 0)$  fixed point. ■

Now consider  $r : b$  nodes with simple loading ( $SL$ ). Consider first  $r = b + 2$  meaning that the refractory state consists of just the value  $-1$ . In Fig. 3.6(a) we show the fraction of initial states  $A_f$  that lead to sustained activity for  $4 : 2$ ,  $5 : 3$  and  $6 : 4$  nodes. First observe that as  $p \rightarrow 1$  almost all initial conditions converge to periodic solutions (of periods 4, 5 and 6 respectively). When  $p < 1$ , recall that for a  $3 : 1$  node, a closed cycle was needed for sustained activity and this occurs at  $p \sim O(1/n)$ . But  $4 : 2$ ,  $5 : 3$  or  $6 : 4$  nodes now do not require closed cycles but instead require pairs of nodes connected by edges which emerge at  $O(1/n^2)$  as shown in Fig. 3.1(c). As suggested by Proposition 3.3, graphs with exactly one edge play an important role in sustained activity. Indeed  $\Gamma_1$ , the threshold for the emergence

of the first edge, serves as a left (or upper) bound for any of the activity curves.

**Proposition 3.4:** For  $r = b + 2$  the activity curves for a set of  $n$ ,  $r : b$  nodes under the  $SL$  rule converge to  $\Gamma_1$  as  $r \rightarrow \infty$ .

**Proof:** Consider first a pair of  $4 : 2$  nodes connected by one edge. It is straightforward to check that  $1/4$  of all initial conditions lead to a period 4 solution in this network. Next observe that the addition of a single isolated node continues to imply that for the  $n = 3$  sized network,  $1/4$  of all initial conditions converge to a period 4 solution. This value  $1/4$  holds for the addition of any number of isolated nodes. Further the addition of an edge increases the set of initial conditions that sustain activity. In general, for a network of  $n$   $r : b$  nodes,  $(b - 1)/r = (r - 3)/r$  provides a lower bound for the fraction of initial conditions leading to sustained activity. Thus as  $r$  increases, the activity curves shift to the left. Since  $(r - 3)/r \rightarrow 1$ , the lower bound for the activity curves converges to the curve that gives the probability for the existence of one edge in the network, which by definition is  $\Gamma_1$ . Finally, it is clear that, on average, activity cannot emerge to the left of  $\Gamma_1$  since there, almost surely no graphs have edges. ■

We next turn our attention to the  $MR$  loading rule for  $r : b$  nodes. This case turns out to be more complicated and depends on the ratio  $f = b/r$  of active to total number of states as well as the size  $n$  of the graph.  $A_f$  is non-monotonic for a large set of choices of  $r$ ,  $b$  and  $n$  as described below. It is then an interesting question to address whether non-monotonicity persists as  $n \rightarrow \infty$ . We provide numerical evidence that suggests that as  $n \rightarrow \infty$ , there exists a critical ratio  $f^*$  such that if  $f > f^*$  then  $A_f$  is monotonic for the  $MR$  case and coincides with its  $SL$  counterpart. However, if  $1/2 \leq f < f^*$ , then the activity curves remain non-monotonic as  $n \rightarrow \infty$ .

We start with the  $r = b + 2$  case for  $n$  relatively small ( $n = 30$ ). In Fig. 3.6(b), we have plotted the activity curves  $A_f$  vs  $p$  for  $4 : 2$ ,  $5 : 3$  and  $6 : 4$  nodes. These curves have certain characteristic features. They all contain a local maximum at which the largest fraction of initial states leads to sustained activity. They all contain a local minimum (the one associated with the  $4 : 2$  nodes is difficult to discern). Finally, all tend to non-zero values as  $p \rightarrow 1$  (again  $4 : 2$  case is difficult to discern).

Fig. 3.7 shows the difference in the behavior of dynamical activity with the two loading rules for  $n = 30$ . At lower value of  $p$ , the activity for  $SL$  and  $MR$  overlaps. This is due to the fact that at these lower  $p$  values, activity is largely supported by pairs of connected nodes or connected triplets. For these two structural units, nodes have degree  $d \leq 2$  and initial states which tend to sustained activity see no difference in the loading rules. Silent nodes fire independent of the loading rules and the fraction of initial states converging to sustained activity is directly proportional to the growth of clusters  $C_2$  and  $C_3$ .

With the increase in the value in  $p$ , nodes with higher degree  $d > 2$  begin to appear. Higher degree nodes provide greater chance for initial states to support more activity for the case of  $SL$ . However for  $MR$ , higher degree nodes may initially decrease activity in the network. Consider the range of  $p$  values at which minimal  $k$ -cycles emerge. On these minimal cycles,  $MR$  can be satisfied by a large set of initial conditions. With higher degree, at higher  $p$ , these  $k$ -cycles gain edges either between nodes of the cycle or with exterior nodes. Now certain initial conditions that led to sustained activity when the cycle was minimal no longer do so with the addition of the new edge. This is unlike the  $MR$  case for  $r : 1$  nodes where the introduction of a single edge to a minimal  $k$ -cycle signaled the complete destruction of activity. Instead, here there is simply a decrease in the fraction of initial conditions leading

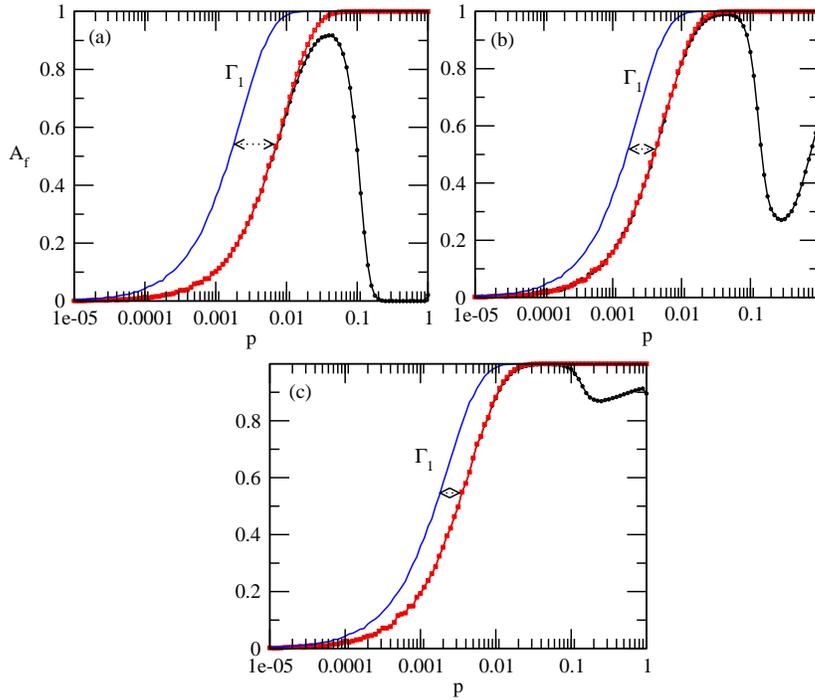


FIG. 3.7. Data from Fig. 3.6 replotted to directly compare activity with loading rules  $SL$  (red curve with square) and  $MR$  (black curve with circle) ( $n = 30$ ). (a) Activity for 4 : 2 nodes (b) 5 : 3 nodes (c) 6 : 4 nodes The black curve  $\Gamma_1$  is the threshold function for the existence of a single edge in  $\mathcal{G}(n, p)$  and serves as the left-most bound for these curves. Note that these curves approach the monotone increasing curve  $\Gamma_1$  as  $r = b + 2 \rightarrow \infty$ . Also note that prior to the local maximum, each of the activity curves is identical indicating that loading plays no role until  $O(1/n)$ . Finally observe that the distance from  $\Gamma_1$  to  $A_f$  is decreasing as  $r$  increases indicating convergence to  $\Gamma_1$ .

to sustained activity. To make this more clear, consider the graph shown in Fig. 3.4(b) with 4 : 2 nodes. Label the nodes in the triangle from the lower left as 1, 2, 3 (node 3 has degree 3) and the exterior node as 4. Suppose first that the node 4 is actually isolated and the 3-cycle is minimal. Then the initial conditions  $(-1, 2, 0, j)$   $j = 1, \dots, 4$  lead to sustained activity. If node 4 does have an edge with node 3, then only  $(-1, 2, 0, 2)$  leads to sustained activity. Thus there is a marked decrease in the set of initial conditions leading to periodic behavior when an extra edge is added to a minimal  $k$ -cycle. Note the activity curve  $A_f$  decreases more sharply for 4 : 2 nodes than for 5 : 3 and 6 : 4 nodes, as shown in Fig. 3.6(b), since these latter two networks provide more possible initial conditions with nodes in the active state. Though the activity vanishes for 4 : 2 nodes in a window of  $p$ , increasing  $b = r - 2$  allows the nodes to have more active states relative to inactive states, thereby supporting more activity in the same network. Further for higher  $b$ , the deviation of the activity curves for  $MR$  from  $SL$  takes place at higher values of  $p$ ; see Fig. 3.7(b)-(c). Thus the emergence of nodes with higher degree as edge probability  $p$  increases, causes the  $MR$  activity curves to become non-monotonic for  $n = 30$ .

As  $b = r - 2$  increases, the activity emerges at lower values of  $p$  which is a consequence of Proposition 3.4. Note that this Proposition applies for small values of  $p$  since here,  $A_f$  behaves the same for both  $SL$  and  $MR$ . This makes the activity

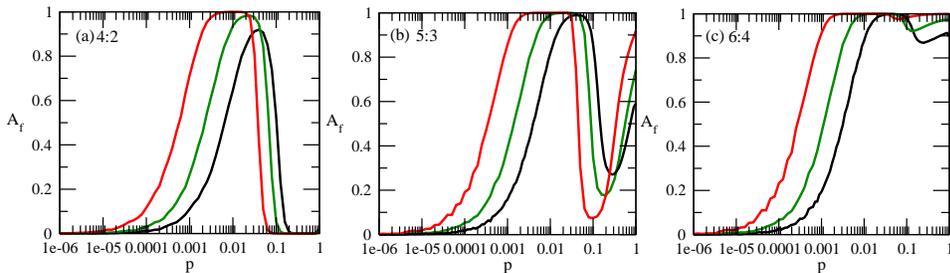


FIG. 3.8. Behavior of  $A_f$  for MR vs.  $p$  with different values of  $n$ :  $n = 30$  (black),  $n = 50$  (Green) and  $n = 100$  (Red) (a) 4:2 node (b) 5 : 3 node (c) 6 : 4 node. There is a qualitative difference how these activity curves change with  $n$  for the two different cases.

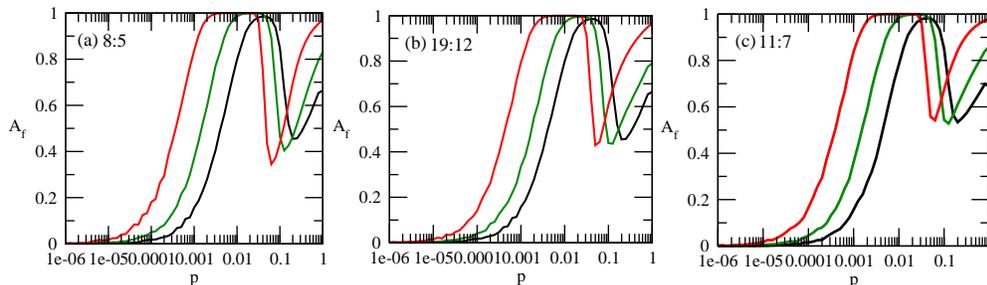


FIG. 3.9. Behavior of  $A_f$  for MR vs.  $p$  with different values of  $n$ :  $n = 30$  (black),  $n = 50$  (Green) and  $n = 100$  (Red) (a) 8 : 5 node (b) 19 : 12 node (c) 11 : 7. In (a),  $f = 5/8 = 0.625 < f^*$ , in (b),  $f = 12/19 \approx 0.6316 < f^*$ , and in (c)  $f = 7/11 \approx 0.6364 > f^*$ . The local minima starts to increase for case (c).

curves for 4 : 2, 5 : 3 and 6 : 4 move upward; see Fig. 3.7(b)-(c). The MR case will converge to the monotone curve  $\Gamma_1$  as  $b \rightarrow \infty$  since, in this limit, individual nodes are almost surely in the active state. Thus a given silent node is very likely to see all of its neighbors in the active state, thus satisfying MR.

We next consider how  $A_f$  depends on the network size  $n$ . In Fig. 3.8, we plot these curves for the 4 : 2, 5 : 3 and 6 : 4 nodes for the case of  $n = 30, 50$  and 100. As can be seen, there is a dramatic difference in how the dynamics of these networks evolve. For the 4 : 2 and 5 : 3 nodes, the local maxima of  $A_f$  broadens and moves to the left with increasing  $n$ . Moreover, the decrease in  $A_f$  near  $p \sim 1/n$  becomes sharper with increasing  $n$ . At  $p = 1$ ,  $A_f$  for the 4 : 2 nodes differs from the 5 : 3 case. In Proposition 3.5, we explain why  $A_f \rightarrow 0$  as  $n \rightarrow \infty$  for 4 : 2 nodes, whereas  $A_f \rightarrow 1$  for 5 : 3 nodes. That  $A_f \rightarrow 1$  for 5 : 3 nodes at  $p = 1$  implies the existence of the local minima, which occurs around  $O((\log n)/n)$  when the last isolated node disappears. These local minima move down and to the left as  $n$  increases. In contrast, for 6 : 4 nodes, as  $n$  increases, the local minima increases, the curves become monotonic and converge toward their corresponding SL counterparts.

These simulations suggest that there is a qualitative difference between how 4 : 2, 5 : 3 and 6 : 4 nodes behave with increasing network size. To further investigate this difference, we computed  $A_f$  for several other cases, three of which, 8 : 5, 19 : 12 and 11 : 7 are shown in Fig. 3.9. The non-monotonic shape of  $A_f$  remains for all, but as can be seen the local minima for the 8 : 5 case ( $f = 0.625$ ) is clearly decreasing more rapidly with  $n$  than in the 19 : 12 case ( $f \approx .6316$ ). For 11 : 7 ( $f = 0.6363$ ), the local

minima of  $A_f$  slowly increases with  $n$ . These numerical evidence suggest that the non-monotonic behavior of  $A_f$  depends on  $f = b/r$ . Namely, we conjecture that there exists a critical fraction  $f^*$ , such that if  $1/2 \leq f < f^*$ , then the  $MR$  activity curves are non-monotonic as  $n \rightarrow \infty$ . However, if  $f > f^*$  then the activity curves become monotone and coincide with their  $SL$  counterparts. Based on numerical results, we estimate  $12/19 < f^* < 7/11$ .

Before discussing why  $f^*$  should exist, let us first document the behavior of the fully connected network at  $p = 1$  in the limit as  $n \rightarrow \infty$ . This will help to explain the difference between the  $5 : 3$  and  $4 : 2$  cases, and more generally, shed light on the behavior for any ratio  $f = b/r$ .

**Proposition 3.5<sup>1</sup>:** Consider a network of  $n r : b$  nodes with  $p = 1$ . If  $f \leq 1/2$  then  $A_f \rightarrow 0$  as  $n \rightarrow \infty$ , if  $f > 1/2$  then  $A_f \rightarrow 1$  as  $n \rightarrow \infty$ .

**Proof:** In the fully connected network, as noted in Proposition 3.2, any initial condition not in the basin of attraction of the 0 fixed point must lie on a periodic solution. Thus  $A_f$  can be determined by understanding the distribution of initial conditions. For an  $r : b$  node, there are  $r$  total states,  $b$  active states,  $r-b-1$  refractory states and one silent state. Let  $m = r - b$ . Denote by  $N_b, N_{b-1}, \dots, N_0, N_{-1}, \dots, N_{-(m-1)}$  the number of nodes in state  $b, b-1, \dots, 0, -1, \dots, -(m-1)$  at  $t = 0$ . In order for activity to propagate through a particular node,  $MR$  must be satisfied meaning that the number of active neighbors must be greater or equal to the number of inactive ones. This implies that the following inequality must hold.

$$(3.1) \quad \sum_{j=1}^b N_j \geq [\sum_{k=0}^{m-1} N_{-k}] - 1$$

If (3.1) holds, then the state of each node is advanced by one with all nodes in state  $b$  switching to state  $-(m-1)$ . For  $MR$  to continue to hold,

$$(3.2) \quad \sum_{j=0}^{b-1} N_j \geq [\sum_{k=1}^{m-1} N_{-k}] + N_b - 1,$$

must be satisfied. Continuing in this manner, we obtain a series of  $r$  inequalities that must hold in order for activity to persist. These inequalities involve only the number of nodes in a particular state given by the initial distribution at  $t = 0$ . Each  $N_i$  obeys a binomial distribution with mean  $n/r$  and variance  $n(r-1)/r^2$ . For large  $n$ , by the Central Limit Theorem, the values of each  $N_i$  obey a normal distribution also with mean  $n/r$  and variance given by  $n(r-1)/r^2$ . In this case, the set of  $r$  inequalities can be satisfied when  $f > 1/2$ , cannot be satisfied when  $f < 1/2$  and can be satisfied for a limited number of initial conditions when  $f = 1/2$ .

To see this note that when  $n$  is large, (3.1) or any of the other inequalities becomes

$$(3.3) \quad b \frac{n}{r} \geq m \frac{n}{r} - 1.$$

This can be rearranged to the inequality  $f = b/r \geq 1/2 - 1/2n$ , showing that as  $n \rightarrow \infty$ ,  $f \geq 1/2$  for activity to be sustained. Therefore if  $f < 1/2$  then  $A_f \rightarrow 0$  as  $n \rightarrow \infty$ . When  $f > 1/2$  the situation is the opposite and  $A_f \rightarrow 1$ .

The case  $f = 1/2$  is a little more subtle however. In this case, while equation (3.1) and its permutations can always be satisfied, there are restrictions on how the various  $N_i$  can be distributed. For  $r : b$  nodes, the active nodes at  $t = 0$  are precisely the ones that are inactive at  $t = b$  and vice versa. In fact, at every time step, the

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<sup>1</sup>We thank one of the anonymous referees for a suggestion to include this result and for outlining the part of its proof related to the Central Limit Theorem.

current set of active nodes, will all be inactive  $b$  time steps later. This leads to a set of  $r$  inequalities one of which has the form

$$(3.4) \quad [\sum_{k=0}^{m-1} N_{-k}] - 1 \leq \sum_{j=1}^b N_j \leq [\sum_{k=0}^{m-1} N_{-k}] + 1,$$

and the others of which are obtained by cycling through the indices appropriately. For large enough  $n$ , these further reduce to  $b = r/2$  equalities  $N_{b-i} = N_{-i}$  for  $i = 0, \dots, b-1$ . Thus only a small fraction of initial conditions satisfying this last set of equalities leads to periodic activity.

The  $f = 1/2$  case is also equivalent to a classical problem of having  $r$  cyclically arranged baskets into which  $n$  balls are randomly distributed. One then wants to know the probability that no set of  $r/2$  adjacent baskets have greater than  $n/2$  balls. For example consider the situation for  $4 : 2$  nodes and  $n$  even. Each basket corresponds to one of the states  $-1, 0, 1$  or  $2$ . We first require that no more than  $n/2$  of the balls in total can land in any two baskets. The four baskets can be divided into subsets of two baskets each subset of which contain a total of  $n/2$  balls ( $n$  is even) in  $4!/2!2!$  different ways. Of this number, one third correspond to cases where adjacent baskets do not contain more than  $n/2$  balls in total for any admissible distribution of balls. To see which distributions are admissible, let  $j = \lfloor n/4 \rfloor$  be the greatest integer function. Consider the number of ways  $(n/2) - i$  balls can land in each of two baskets and  $i$  in each of the others. Ranging over  $i = 1, \dots, j$  we find that there are  $\sum_{i=0}^j n!/[(n/2-i)!i!]^2$  ways in which this can be done. If we denote by  $E$  the event that no two adjacent baskets contain more than  $n/2$  balls, and simplify by noting that  $1/3 \cdot 4!/(2!2!) = 2$  then

$$(3.5) \quad P(E) = (2/4^n) \sum_{i=0}^j n!/[(n/2-i)!i!]^2,$$

where the term  $4^n$  in the denominator represents the total number of ways in which  $n$  balls can be distributed in 4 baskets. Note that  $\lim_{n \rightarrow \infty} P(E) = 0$ . We calculated (3.5) for  $n = 30$  and  $n = 50$  and compared it with our numerical simulations over the 2000 realizations of graphs with the respective number of nodes. For  $n = 30$ , (3.5) yields  $P(E) = 0.02087$  and the numerical results of activity yield 0.02084. For  $n = 50$ , the former yields  $P(E) = 0.01261$ , the numerics yield 0.01243. As can be seen, the numerical values are very close to the theoretically determined values. ■

The conjectured existence of  $f^*$  is based on several observations. First when  $f = 1/2$  as shown above in Proposition 3.5,  $A_f \rightarrow 0$  as  $n \rightarrow \infty$  at  $p = 1$ . In these cases,  $A_f$  is clearly non-monotonic. However as  $f \rightarrow 1$ , then  $A_f \rightarrow 1$  at  $p = 1$  as also illustrated by Proposition 3.5. Thus it is clear that the value of  $f$  does play a role in determining monotonicity. The non-monotonic behavior of  $A_f$  begins around  $p \sim O(1/n)$  with the emergence of the giant component. At this stage, the graph consists of isolated trees and minimal cycles. The average degree of nodes is around one. Earlier we discussed how the addition of an external node and edge to a minimal cycle can decrease  $A_f$ . We can obtain a further idea on how  $A_f$  will behave near this range of  $p$  values by quantifying how  $A_f$  varies on certain trees. Consider a connected subgraph consisting of one node of degree  $l$  such that each of its  $l$  neighbors has degree 1. When  $l = 1$ , for  $0.5 \leq f < 1$  where  $f = b/r$ ,  $A_f = 2f - 1 + 1/r$ . For the cases  $l = 2, 3$ , the respective  $A_f$  values are  $(2f - 1 + 1/r)(3 - 2f - 1/r)$  and  $(2f - 1 + 1/r)^2(5 - 4f - 2/r)$ . As  $l$  increases, the value  $A_f$  on these subgraphs alternatively increases and decreases. The decrease happens when  $l$  increases from

even to odd and increases in the opposite case. By comparing the  $A_f$  values for the cases  $l = 1$  and  $l = 3$ , we find that  $A_f(l = 1) \geq A_f(l = 3)$  when the inequality

$$(3.6) \quad 0 \geq -8f^2 + 14f - 6 - 8\frac{f}{r} + \frac{7}{r} - \frac{2}{r^2}$$

is satisfied. As  $r \rightarrow \infty$ , the right hand side of (3.6) factors to  $2(4f - 3)(1 - f)$ . Thus in this limit if  $f \leq 0.75$  then  $A_f(l = 1) \geq A_f(l = 3)$  and activity decreases. For particular choices of  $r : b$  nodes, such as  $6 : 4$  where  $f = 2/3$ ,  $A_f(l = 1) = A_f(l = 3)$ , while for smaller values of  $f$  such as  $7/11$  or  $12/19$ , the right hand of (3.6) is negative and  $A_f$  decreases. This is one piece of evidence that as  $p$  increases and the graph grows, the fraction  $f$  plays a role in determining whether  $A_f$  increases or decreases. While the above calculation does not yield a definitive bound for  $f^*$  near our numerically computed one, it does provide an indication that a critical ratio may in fact exist. Although  $A_f$  is not determined solely by its behavior on these types of subgraphs, it certainly receives significant contribution from them in the range of  $p$  values between  $O(1/n)$  and  $O((\ln n)/n)$ . It remains an open problem to prove the conjectured existence of  $f^*$ .

**4. Discussion.** Understanding dynamics on complex networks has commanded a large interest in the scientific community. In many cases, the specifics of the network and of the underlying application are critical in formulating a proper model. In other cases, deriving general properties of networks is of interest. While we were motivated by a problem posed by Gansert *et al.* [19] in the context of neuronal networks, our study falls closer toward the latter set of studies. We used a combination of numerical simulations and analysis to investigate properties and dynamics that are generic within random graphs.

The main finding of this paper is that in random graphs, certain architectural features of the graph  $G(n, p)$  combine with intrinsic properties of the nodes and the rules for interaction between them to determine which kinds of dynamic activities are possible. For  $r > 2b$  nodes, when the rules for interaction are simple, as in the *SL* (simple loading) case, the emergence of  $k$ -cycles in the random graph is the important structural feature that determines rhythmicity. The ensuing activity curve,  $A_f$ , is then a monotone increasing sigmoidal shaped curve. Alternatively, when the interaction rule becomes more complicated as in the *MR* (majority rules) case, there are several architectural features that determine the shape of the activity curve. The emergence of minimal  $k$ -cycles continues to signal the emergence of activity for  $r : 1$  types of nodes. The emergence of the giant component signals the end of activity when additional links are added to these structural units.

When the intrinsic dynamics of the nodes are switched to more of bursting type with  $r \leq 2b$ , then the emergence of the first edge at  $O(1/n^2)$  in  $G(n, p)$  determines when activity begins. The formation of  $k$ -cycles at  $O(1/n)$  and the giant cluster shortly thereafter signals the decrease in the activity curve for  $0.5 \leq f < f^*$ . The disappearance of the last isolated node at  $O((\log n)/n)$  is closely related to the local minimum of  $A_f$  for  $0.5 < f < f^*$ . When  $f > f^*$ , the activity curve  $A_f$  for the *MR* case becomes monotone increasing and converges to the activity curve for the *SL* case for the corresponding set of nodes as  $n \rightarrow \infty$ .

Several studies have also focused on how architecture and dynamics may be related. DeVille *et al.* [13] have studied how the size of the giant component of a random graph is related to the number of cells (nodes) that fire together in a synchronous burst. In particular, they showed that in the limit as  $n \rightarrow \infty$  with  $pn \rightarrow \beta$ ,

for  $\beta \in \mathfrak{R}$ , while the size of the giant component grows continuously with  $\beta$ , the size of the burst undergoes a phase transition as  $\beta$  is increased. This is somewhat related to our conjectured existence of the critical ratio  $f^*$  in that both phenomenon show phase transitions of different sorts as the giant component of the graph forms. Other authors have used very general models to describe the intrinsic dynamics of the network elements. For example, Pogromsky *et al.* [36] use a model free approach that relies on the underlying symmetry of nodal dynamics and network architecture to derive conditions for partially and fully synchronized solutions. Josic and Rubin [25] also use a model free approach, but are interested in an inverse problem of sorts. Namely, given that a particular activity pattern exists within a network, they ask what kinds of network architecture allow such activity to occur. They search for network motifs that they call robust in that these networks are able to sustain activity independent of the intrinsic dynamics of individual nodes. Thus their conclusions seem to imply that architecture alone can be a determining factor in the sustainment of activity. However, our results suggest that the interplay between architecture and dynamics can be somewhat subtle, sometimes allowing architecture to have the dominant role in whether activity persists and sometimes allowing the intrinsic dynamics of the nodes to play the dominant role.

Müller-Linow *et al.* [32] show that the distribution of excitation patterns strongly depends on the network connectivity. They show that when  $p$  is of  $O(1/n)$  the average density of active states in their network undergoes a phase transition as the giant component forms. Our work is consistent with their findings as the average density,  $|A(t)|/n$  averaged over time, initial data and network realizations in our model, would have a similar profile to  $A_f$ . Further Müller-Linow *et al.* suggest that  $k$ -cycles are necessary for their model to exhibit what they call bursts of activity, which is equivalent to sustained activity of our model. However, their model allows for silent states to stochastically switch to becoming active. Thus a specific requirement of periodic activity is not required to obtain a non-zero value of the density.

Kinouchi and Copelli [26] considered a random network of  $r : 1$  nodes connected by probabilistic synapses subject to Poisson distributed external inputs. They showed that the dynamic range of the network which codes for changes in the average activity can be maximized when a certain branching process is at a critical value. This branching process determines how many silent nodes an active node is able to fire at the next time step. We note, however, they were not interested in how changes in structure affect the density of activity in the network, as they fixed the number of edges in their random graphs. The relation between dynamic range and criticality has been recently generalized by Larremore *et al.* [27] to consider the effect of network structure. They show that the largest eigenvalue of a weighted adjacency matrix governs the dynamic range of the network thereby demonstrating the effect of topology.

The paradigm of self-organized criticality (SOC) [2] has also been invoked in several earlier studies of neuronal networks following the observation that “avalanches” of neuronal activities show power-law distributions [5, 6, 39]. Since threshold effects are important in SOC as well as in neuronal dynamics, a number of studies have examined evolving networks of coupled neuron models [9, 10, 28]. Both simple neuronal dynamics that merely incorporate integrate-and-fire rules as well as more realistic modeling [31] show that a network of evolving neurons can self-organize to a critical state that is a robust attractor of the dynamics. While our study does not allow for the network to adapt on its own, it does suggest what rules may be placed on any

such adaptation to maximize (or even minimize, if relevant) the chances for sustained activity. Namely, adaptation towards  $p$  values close to relevant thresholds will lead to different capabilities within the network.

Our study brings up several interesting questions for future research. Most of the findings here relied on a combination of simulation and analysis. We raised several open questions in the text related to the  $r : 1$  case for  $A_f$  in the limit as  $n \rightarrow \infty$  that need resolution. Proving the existence of  $f^*$  for the  $MR$  case is still an open question. Although our numerical studies do not suggest so, it is possible as  $n \rightarrow \infty$  with  $0.5 < f < f^*$ , that  $A_f$  becomes monotonic. Even if this were true, it would still be an interesting, and perhaps quite challenging question, to understand why the dependence on network size is so strange. More generally, one would like to analytically address the existence of thresholds related to dynamics. For example, is the decrease in  $A_f$  for the  $MR$  case when  $f < f^*$  the result of some threshold? It is of importance to see what, if any, dynamic properties in a graph are monotone. If, in fact, few monotone dynamic properties exist, one would like to know what would be the appropriate marker for proving the existence of these thresholds. Another interesting question is related to the loading rules that we have imposed. While we have chosen two extremes,  $SL$  and  $MR$ , future work could explore what changes to the activity curves would arise with an intermediate rule. Even more interesting, perhaps would be to modify the strength of the input a node receives based on the the number of outputs that the particular input has. For example, if a node is active and has edges to five other nodes, then one can envision a rule in which each of the output nodes receives only  $1/5$  of the output instead of each receiving the whole. A node could then add up all of its inputs to see if these push the node beyond some prescribed threshold. We conjecture that in this case the level of activity in the graph would drop, but what would be of interest is to see if there is a phase transition that occurs as the prescribed threshold is made progressively smaller.

Many of the above mentioned works consider the effect of noise or external perturbations on average activity. To some extent our study considers the effect of external perturbation if one were to envision the perturbation as acting on a network where each node is silent. The perturbation would then cause some subset of nodes to become active and one would then check whether or not sustained activity ensued. In the context of our set up, this could be achieved by limiting the state space of initial conditions from size  $r^n$  to size  $n$  where each node is set to the zero state. This may have a quite dramatic effect on  $A_f$  in the  $SL$  case for  $r : 1$  nodes. For example, on minimal cycles, a sequence of three neighboring nodes leads to sustained activity if an active node is surrounded by a silent and refractory node. Thus excitatory perturbations would tend to reduce  $A_f$  and make the threshold for  $k$ -cycles irrelevant. However if perturbations from the silent state could make nodes either active or refractory then  $A_f$  may not be so different. The effect of noise on our network also merits consideration. We explicitly chose not to study it because there are several neuronal systems, especially central pattern generating networks associated with motor movements [30], in which the effects of noise are minimal. Nonetheless, in order to generalize our findings, noise would be a next logical step to consider.

Other extensions are motivated by the Gansert *et al.* work [19]. They allow each node to have three daughter nodes, where the edges between individual nodes only occur at the level of daughters. Daughters of the same node are forbidden from having edges. These are graphs that have a zero block matrix along the diagonal of the adjacency matrix  $A_{ij}$ . Investigating graphs with this kind of structure and the ensuing

dynamics on them has to our knowledge not been studied. It would be interesting to study graphs in which the nodes are heterogeneous. From our study, an obvious question would be to investigate the dynamics of a family of  $n, r : 1$  and  $m, r : b$  nodes as the ratio  $n/(n+m)$  is varied under the  $MR$  rule. Clearly as  $n/(n+m) \rightarrow 0$  or 1, we recover what is already known from our work. However one would like to determine if there is a phase transition that occurs at some particular value of this fraction. This study could then be extended the case of  $n r_1 : b_1$  and  $m r_2 : b_2$  nodes. Another area of study involves graphs that exhibit preferential attachment such as scale-free graphs [4] or those obtained through an Achlioptas process [1] to see if structures arising from these graph processes constrain the dynamical outputs of networks.

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## REFERENCES

- [1] D. ACHLIOPTAS, R. D'SOUZA AND J. SPENCER, *Explosive percolation in random networks*, Science, 323 (2009) 1453-1455.
- [2] P. BAK, C. TANG AND K. WIESENFELD, *Self-organized criticality: An explanation of the  $1/f$  noise*, Phys. Rev. Lett., 59 (1987), pp. 381-384.
- [3] J. BALTHROP, S. FORREST, M. E. J. NEWMAN, AND M. M. WILLIAMSON, *Technological networks and the spread of computer viruses*, Science, 304 (2004), pp. 527-529.
- [4] A. BARABASI AND R. ALBERT, *Emergence of scaling in random networks*, Science, 286 (1999), pp. 509-512.
- [5] J. BEGGS AND D. PLENZ, *Neuronal avalanches in neocortical circuits*, J. Neurosci., 23 (2003), pp. 11167-11177;
- [6] J. BEGGS AND D. PLENZ, *Neuronal avalanches are diverse and precise activity patterns that are stable for many hours in cortical slice cultures*, J. Neurosci., 24 (2004), pp. 5216-5229.
- [7] B. BOLLOBÁS, *The evolution of random graphs*, Trans. Amer. Math. Soc., 286 (1984), pp. 257-274.
- [8] C. BORGERS AND N. KOPELL, *Synchronization in networks of excitatory and inhibitory neurons with sparse random connectivity*, Neural Comput., 15 (2006), pp. 509-538.
- [9] S. BORNHOLDT AND T. ROHLF, *Topological evolution of dynamical networks: Global criticality from local dynamics*, Phys. Rev. Lett., 84 (2000), pp. 6114-6117.
- [10] S. BORNHOLDT AND T. ROHLF, *Self-organized critical neural networks*, Phys. Rev. E, 67 (2003), pp. 066118-066123.
- [11] A. R. CARVUNIS, M. LATAPY, A. LESNE, C. MAGNIEN, AND L. PEZARD, *Dynamics of three-state excitable units on Poisson vs. power-law random networks*, Physica A, 367 (2006), pp. 595-612.
- [12] A. CHATTERJEE, S. SINHA, AND B. K. CHAKRABARTI, *Economic inequality: Is it natural?* Curr. Sci., 92 (2007), pp. 1383-1389.
- [13] L. DEVILLE, C. PESKIN AND J. SPENCER, *Dynamics of stochastic neuronal networks and the connections to random graph theory*, Math. Model. Nat. Phenom. 5 (2010) pp. 26-66.
- [14] R. DIESTEL, *Graph Theory*, 2nd. ed. Springer-Verlag, New York, 2000.
- [15] S. N. DOROGOVTSSEV AND J. F. F. MENDES, *Evolution of networks*, Advan. Phys., 51 (2002), pp. 1079-1187.
- [16] R. C. ELSON, A. SELVERSTON, R. HUERTA, N. RUKOV, M. RABINOVICH AND H. ABARBANE, A. SELVERSTON, R. HUERTA, N. RUKOV, M. RABINOVICH AND H. ABARBANELL, *Synchronous Behavior of Two Coupled Biological Neurons*, Phys. Rev. Lett., 81 (1998), pp. 5692-5695.
- [17] P. ERDŐS AND A. RÉNYI, *On the evolution of random graphs*, Publ. Math. Inst. Hung. Acad. Sci., 5 (1960), pp. 17-61.
- [18] D. EYTAN AND S. MAROM, *Dynamics and effective topology underlying synchronization in networks of cortical neurons*, J. Neurosci., 26 (2006), pp. 8465-8476.
- [19] J. GANSERT, J. GOLOWASCH, AND F. NADIM, *Sustained rhythmic activity in gap-junctionally coupled networks of model neurons depends on the diameter of coupled dendrites*, J. Neurophysiol., 98 (2007), pp. 3450-3460.

- [20] L. GLASS, *Synchronization and rhythmic processes in physiology*, Nature, 410 (2001), pp. 277-284.
- [21] M. T. HÜTT, AND A. LESNE, *Interplay between topology and dynamics in excitation patterns on hierarchical graphs*, Front. Neuroinf., 3 (2009) pp. 1-10.
- [22] S. JAIN, AND S. KRISHNA, *A model for the emergence of cooperation, interdependence and structure in evolving networks*, PNAS, 98 (2001), pp. 543-547.
- [23] S. JANSON, T. LUCZAK, AND A. RUCINSKI, *Random graphs*, Wiley, New York, 2000.
- [24] H. JEONG, B. TOMBOR, R. ALBERT, Z. N. OLTVAI, AND A. BARABSI, *The large-scale organization of metabolic networks*, Nature, 407 (2000), pp. 651-654.
- [25] K. JOSIC AND J. RUBIN, *Deriving information about architecture from activity patterns in coupled cell systems*, SIADS, 4 (2005), pp. 53-77.
- [26] O. KINOCHI AND M. COPELLI, *Optimal dynamical range of excitable networks at criticality*, Nature Physics, 2 (2006), pp. 348-351.
- [27] D. LARREMORE, W. L. SHEW, J. G. RESTREPO, *Predicting criticality and dynamic range in complex networks: effects of topology*, Phys. Rev. Lett. 106 (2011), 058101-1-4.
- [28] A. LEVINA, J. M. HERRMANN, AND T. GEISEL, *Dynamical synapses causing self-organized criticality in neural networks*, Nature Physics, 3 (2007), pp. 857-60; A. LEVINA, J. M. HERRMANN, AND T. GEISEL, *Phase transition towards criticality in a neural system with adaptive interactions*, Phys. Rev. Lett., 102 (2009), pp. 118110-1-4.
- [29] T. LEWIS AND J. RINZEL, *Self-organized synchronous oscillations in a network of excitable cells coupled by gap junctions*, Network: Comp. Neur. Syst., 11 (2000), pp. 299-320.
- [30] E. MARDER AND R. CALABRESE, *Principles of rhythmic motor pattern generation*, Physiol. Rev. 76 (1996), pp. 687-717.
- [31] C. MEISEL AND T. GROSS, *Adaptive self-organization in a realistic neural network model*, Phys. Rev. E 80 (2009), pp. 061917- 061922.
- [32] M. MÜLLER-LINOW, C. MARR, AND M. T. HÜTT, *Topology regulates the distribution pattern of excitations in excitable dynamics on graphs*, Phys. Rev. E, 74 (2006), pp. 016112-1-7.
- [33] M. E. J. NEWMAN, *Spread of epidemic disease on networks*, Phys. Rev. E, 66 (2002), pp. 016128-016139.
- [34] M. E. J. NEWMAN AND J. PARK, *Why social networks are different from other types of networks*, Phys. Rev. E, 68 (2003), pp. 036122-036130.
- [35] A. PHADKE AND J. THORPE, *Computer relaying for power systems*, Wiley, New York, 1988.
- [36] A. POGROMSKY, G. SANTOBONI, AND H. NIJMEIJER, *Partial synchronization: From symmetry towards stability*, Phys. D, 172 (2002), pp. 65-87.
- [37] N. F. RULKOV, *Modeling of spiking-bursting neural behavior using two-dimensional map*, Phys. Rev. E, 65 (2002), pp. 041922-1-9.
- [38] R. P. SATORRAS AND A. VESPIGNANI, *Epidemic spreading in scale-free networks*, Phys. Rev. Lett., 86 (2001), pp. 3200-3203.
- [39] W. L. SHEW, H. YANG, T. PETERMANN, R. ROY, AND D. PLENZ, *Neuronal avalanches imply maximum dynamic range in cortical networks at criticality*, J. Neurosci., 23 (2003), pp. 11167-11177.
- [40] D. TERMAN AND D. WANG, *Global competition and local cooperation in a network of neural oscillators*, Physica D, 81 (1995), pp. 148-176.
- [41] W. X. WANG, B. H. WANG, B. HU, G. YAN, AND Q. OU, *General dynamics of topology and traffic on weighted technological networks*, Phys. Rev. Lett., 94 (2005), pp. 188702-1-4.
- [42] D. WATTS AND S. STROGATZ, *Collective dynamics of 'small-world' networks*, Nature, 393 (1998), pp. 440-442.
- [43] D. YANG, Y. LI, AND A. KUZNETSOV, *Characterization and merger of oscillatory mechanisms in an artificial genetic regulatory network*, Chaos, 19 (2009), pp. 033115-033124.