WARRANTY SERVICING WITH A BROWN-PROSCHAN REPAIR OPTION

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Reducing warranty servicing costs are of great interest to product manufacturers or, sellers who are contractually bound to provide post-sales support, up to a specified warranty period, usually in the form of some remedial action that restores a failed item to a functioning condition. Here, in the spirit of Jack, Iskandar and Murthy (2009) strategy based on partitioning the effective warranty period into three intervals, we consider and analyze the cost of a new two-dimensional warranty servicing strategy, that probabilistically exercises a choice between a replacement and a minimal repair to rectify the first failure if any, in the middle interval. A numerical illustration of our analysis with Weibull failure model is included.

1. Introduction and summary

A warranty is a contractual agreement that requires a manufacturer or, seller of a product provide an assurance of satisfactory product performance during a specified length of time (called, the warranty period) after purchase, by replacement or, repair. It serves as a promotional tool, and is thus an important component of the product's marketing strategy to attract potential buyers. On the other hand, warranties when adequately designed not only serve as an instrument that address the protectional needs of the buyer/consumer (against unwelcome and unforeseen disruption of service due to product failures), but also that of the producer by controlling costs through appropriate warranty servicing strategies. In recent times, contemporary research on warranty modeling is focused on both the lifetime characteristics of the product as well as product usage intensity as important determinants of the time to product failure. Warranty policies of this

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type are referred to as two-dimensional (2-D) warranties. For relevant background and related work see [1], [2] and [3]. Research reported in this paper considers a new servicing strategy in the 2-D warranty set up, in the spirit of Jack et.al.[4], extending their work in a different direction, and includes their servicing strategy as well as the minimal repairs only strategy as special cases. We demonstrate the modeling and analysis of total expected costs of our strategy along with a numerical illustration.

2. The 2-D Warranty Modeling

We assumes that for a given customer the usage rate Y is constant. Conditional on $Y = y$, the total usage u of an unit at age x is thus

$$u = yx, \qquad 0 \leq u < \infty \tag{1}$$

2.1. *Modeling failures*

The distribution of failure time conditional on a customer's usage rate y is the appropriate distribution to model an unit's failures, with corresponding conditional hazard rate $h(x; y)$ at age x.

2.1.1. *Modeling first failure*

We use an 'Accelerated Failure Time (AFT) model' ([5],[6]) to describe the impact of a given usage rate y on the unit's time to failure. If y_0 (y, respectively) represent the *nominal* (typical, resp.)usage rate with corresponding time to failure T_0 (T_y, resp.); then the standard AFT model postulates,

$$\frac{T_y}{T_0} = \left(\frac{y_0}{y}\right)^{\gamma},$$

where $\gamma \geq 1$ is the acceleration parameter. If $F(\cdot; \alpha_0)$ with a scale parameter α_0 denote the baseline cdf of T_0, then the accelerated failure time T_y has cdf $F(\cdot; \alpha(y))$ with scale parameter given by

$$\alpha(y) = \left(\frac{y_0}{y}\right)^{\gamma} \alpha_0,$$

and conditional hazard rate $h(\cdot; \alpha(y))$.

2.1.2. *Modeling subsequent failures*

For a repairable product, the subsequent failures depend on the type of rectification action carried out. Under minimal repairs, failures over time

occur according to a non-homogeneous Poisson process(NHPP) with intensity function having the same form as the hazard function for time to first failure [7].
We further assume (i) $h(x;\alpha(y))$ is a non-decreasing function of age x and usage rate y, (ii) no preventive maintenance is carried out on the item during the warranty period, (iii) all item failures are detected immediately and result in immediate claims by the consumer, (iv) all claims are valid and must be rectified by the manufacturer immediately, (v) repair and replacement times are small relative to the mean time between item failures and therefore can be ignored.

2.1.3. *Warranty policy and coverage*

Consider a repairable item sold with a 2-D non-renewing free replacement warranty of period W and maximum usage limit U. Thus the 2-D warranty region is the rectangle $[0, W) \times [0, U)$. Given y, the usage sensitive warranty expires when the item currently in use reaches an age $W_y = min(W, \frac{U}{y})$.

3. Servicing strategies for 2-D warranties

Jack *et.al.* [4] have considered a 2-D warranty servicing strategy using minimal repairs and one replacement. Such a strategy is described via three disjoint intervals $[0, K_y)$, $[K_y, L_y)$ and $[L_y, W_y)$ with $0 \leq K_y \leq L_y \leq W_y$, along the age (time) scale where failures in $[0, K_y)$ undergo only minimal repairs; the first failure in $[K_y, L_y)$ rectified by a replacement and all subsequent failures therein, and in $[L_y, W_y)$ are repaired minimally. For a given usage rate y, the selected optimal values of the parameters K_y^* and L_y^* minimize the expected warranty servicing cost. As y varies the set of points (K_y^*, L_y^*) define a closed curve as indicated in fig. 1, Jack *et.al.* [4].
Our work described here, is an attempt to extend the model and analysis of 2-D warranties by allowing a warranty servicing action, henceforth referred to as 'Brown-Proschan repair', which randomizes the choice of restoration between a replacement or, minimal repair with a probability p and $(1-p)$ respectively, that was first introduced by Brown and Proschan [8], although not in the cost of warranty servicing context. The servicing strategy we consider and analyze, can be described as follows:
With the warranty period partitioned into three intervals as described at the beginning of this section; the first failure, (if any) in the middle interval $[K_y; L_y)$ *undergoes a Brown-Proschan repair; all other failures undergo minimal repair.*

It is clear that our strategy reduces to the strategy of Jack *et.al.* [4] when $p = 1$, and to the strategy of minimal repairs only when $p = 0$. Intuition suggests that higher the chance of choosing a replacement, smaller should be the total expected cost of servicing the warranty under reasonable degradation profiles of the unit's failure time and replacement vs minimal repair cost ratio (see e.g. Table 1, sec. 5.1 where the expected cost is decreasing with increasing p).
Pragmatically however, there may be reasons to choose a 'Brown-Proschan repair' with $0 < p < 1$. Choosing between them probabilistically will result in a total expected servicing cost bracketed between the corresponding costs of the strategy of minimal repairs only and that of Jack *et.al.* [4].
We may note that in the frame work of Yun *et.al.* [9] which rectifies the first failure in the middle interval with an 'imperfect repair' defined by any degree of repair $\delta \in [0, 1]$; $\delta = 0$ corresponds to minimal repair and $\delta = 1$ corresponds to replacement. Hence a Brown-Proschan repair is equivalent to randomizing the choice between the degree of repairs $\delta = 0$ and $\delta = 1$. Our strategy is distinct from Yun *et.al.* [9] who tacitly assumed that it is feasible to accomplish any degree of repair $\delta \in [0, 1]$.

3.1. *Model formulation*

Our objective here is to model the expected warranty servicing cost denoted by $J(K_y, L_y, p)$ given usage rate y and find the optimal values of the parameters such that the cost is minimum. Let C_m denote the cost of a minimal repair and C_r $(> C_m)$ denote the cost of a replacement.
For a given usage rate y, let T_{1y} denote the time of the first failure under usage rate y after age K_y. The conditional cdf of T_{1y} is given by

$$F_1(t; \alpha(y)) = \frac{F(t; \alpha(y)) - F(K_y; \alpha(y))}{\overline{F}(K_y; \alpha(y))}. \tag{2}$$

All failures over $[0, K_y)$ are minimally repaired, so the failures occur according to an NHPP process with conditional intensity function $h(x; \alpha(y))$ and the conditional expected warranty servicing cost for this interval is given by

$$C_m \int_0^{K_y} h(x; \alpha(y))dx = C_m H(K_y; \alpha(y)), \tag{3}$$

where $H(x; \alpha(y))$ is the cumulative hazard function at age x. For failures occurring after age K_y we need to consider two cases:

(i) $K_y \leq T_1 = t \leq L_y$ and (ii) $T_1 = t > L_y$.
The conditional expected cost, conditional on $K_y \leq T_1 = t \leq L_y$, is obtained as follows. The first failure in $[K_y, L_y)$ occurs at age t and is either replaced with probability p or, minimally repaired with probability $(1-p)$. All failures over the remaining interval $(t, W_y]$ are minimally repaired. The expected cost function of servicing failures over $(t, W_y]$ is given by

$$p[C_r + C_m \int_t^{W_y} h(x-t;\alpha(y))dx] + (1-p)C_m \int_t^{W_y} h(x;\alpha(y))dx$$
$$= p[C_r + C_m H(W_y - t;\alpha(y))] + (1-p)C_m[H(W_y;\alpha(y))) - H(t;\alpha(y))] \quad (4)$$

As a result, the expected warranty cost over the intervals $[K_y, L_y)$ and $[L_y, W_y]$ for usage rate y conditioned on $K_y \leq T_1 = t \leq L_y$ is given by

$$J(K_y, L_y, p|K_y \leq t \leq L_y) = \int_{K_y}^{L_y} \Big[p\{C_r + C_m H(W_y - t;\alpha(y))\}$$
$$+(1-p)C_m\{H(W_y;\alpha(y)) - (H(t;\alpha(y)))\}\Big] \frac{f(t;\alpha(y))}{\bar{F}(K_y;\alpha(y))} dt. \quad (5)$$

The expected cost, conditioned on $T_1 = t > L_y$, is obtained as follows. Note that there is no failure in $[K_y, L_y]$ and failures over the remaining interval $(L_y, W_y]$ occur according to an NHPP with intensity function $h(t;\alpha(y))$, $L_y \leq t \leq W_y$. Therefore, the conditional expected warranty cost is given by

$$J(K_y, L_y, p|t > L_y) = C_m[H(W_y;\alpha(y)) - H(L_y;\alpha(y))]. \quad (6)$$

For a given usage rate y the expected warranty cost is obtained by removing the conditioning on T_1 and is given by

$$J(K_y, L_y, p) = C_m H(K_y;\alpha(y)) + \int_{K_y}^{L_y} \Big[p\{C_r + C_m H(W_y - t;\alpha(y))\}$$
$$+(1-p)C_m[H(W_y;\alpha(y)) - H(t;\alpha(y))]\Big] \frac{f(t;\alpha(y))}{\bar{F}(K_y;\alpha(y))} dt$$
$$+C_m[H(W_y;\alpha(y)) - H(L_y;\alpha(y))] \frac{\bar{F}(L_y;\alpha(y))}{\bar{F}(K_y;\alpha(y))}. \quad (7)$$

4. Model Analysis and Optimization

Let us assume the probability $p \in [0,1]$ is known, then the optimization problem is $\min_{K_y, L_y} J(K_y, L_y)$. Note that this involves selecting optimally the two parameters K_y and L_y for a given y (subject to the constraints

$0 \leq K_y \leq L_y \leq W_y$). Let K_y^* and L_y^* denote the optimal solution. We obtain this using a two-stage approach. In stage 1, for a fixed K_y, obtain the optimal $L_y^*(K_y)$ that minimizes $J(K_y, L_y)$. Then, in stage 2, we obtain the optimal K_y^* by minimizing $J(K_y, L_y^*(K_y))$. Thus for a fixed K_y, the optimal $L_y^*(K_y)$ can be obtained from the first order condition, $\frac{\partial}{\partial L_y} J(K_y, L_y) = 0$, i.e.,

$$C_m \Big[p\xi_y(L_y) - H(W_y; \alpha(y)) + H(L_y; \alpha(y)) - 1 \Big] \frac{f(L_y; \alpha(y))}{\bar{F}(K_y; \alpha(y))} = 0,$$

$$\text{or,} \qquad p\xi_y(L_y) - H(W_y; \alpha(y)) + H(L_y; \alpha(y)) - 1 = 0. \tag{8}$$

since $\frac{f(L_y;\alpha(y))}{\bar{F}(K_y;\alpha(y))} > 0$ w.l.o.g. and $0 < C_m < \infty$; and

$$\xi_y(L_y) = \frac{C_r}{C_m} + H(W_y - L_y; \alpha(y)) - H(W_y; \alpha(y)) + H(L_y; \alpha(y)). \tag{9}$$

Finally the optimum K_y^* is obtained by solving $\frac{\partial}{\partial K_y} J(K_y, L_y^*(K_y)) = 0$. We have used computational approach to find the optimal values of K_y^* and L_y^*.

5. Special case: Weibull failure distribution

Let the cdf of T_0 is Weibull with scale parameter $\alpha_0 > 0$ and shape parameter $\beta > 1$. Then the survival function of T_0 is

$$\overline{F}(x; \alpha_0) = 1 - F(x; \alpha_0) = \exp{(-\frac{x}{\alpha_0})^{\beta}}.$$

Thus using the AFT formulation the survival, hazard and cumulative hazard functions for T_y are respectively,

$$\overline{F}(x; \alpha(y)) = 1 - F(x; \alpha(y)) = \exp{(-(\frac{y}{y_0})^{\gamma} \frac{x}{\alpha_0})^{\beta}},$$

$$h(x; \alpha(y)) = \beta(\frac{y}{y_0})^{\gamma\beta}(\frac{x^{\beta-1}}{\alpha_0^{\beta}}), \qquad H(x; \alpha(y)) = (\frac{y}{y_0})^{\gamma\beta}(\frac{x^{\beta}}{\alpha_0^{\beta}}).$$

Using the Weibull survival function above, one can derive the special forms of equations (7)-(9) for any given $p \in [0, 1]$. Since these equations are too complex to be solved analytically, one needs to use a suitable computational scheme to find the optimal values of K_y, L_y.

5.1. *Numerical example*

We normalize costs so that cost of minimal repair, $C_m = 1$ and cost of replacement(perfect repair) $C_r = 2$. We assume the nominal values warranty period, $W = 2$, total usage limit, $U = 2$, Weibull scale(baseline) parameter, $\alpha_0 = 1$, Weibull shape parameter, $\beta = 2$, nominal usage rate, $y_0 = 1$ and the AFT model parameter, $\gamma = 2$. We assign values 0, 0.2, 0.4, 0.6, 0.8 and 1 to the probability of replacement p and observe the cost behavior for different p's. Table 1 shows the cost $J(K_y^*, L_y^*)$ behavior for different combinations of (y, p). Table 2 displays the corresponding optimal values of the decision parameters (K_y^*, L_y^*).

Table 1. Expected servicing costs for $C_r = 2$ and $\beta = 2$

y	$p = 0$	$p = 0.2$	$p = 0.4$	$p = 0.6$	$p = 0.8$	$p = 1$
0.90	2.62	2.61 (1)	2.56 (2)	2.54 (3)	2.49 (5)	2.46 (6)
1.00	4.10	3.79 (8)	3.52 (14)	3.41 (17)	3.32 (19)	3.23 (21)
1.20	5.76	5.29 (8)	4.96 (14)	4.68 (19)	4.41 (23)	4.10 (29)
1.40	7.84	7.19 (8)	6.66 (15)	6.32 (19)	5.53 (29)	5.10 (35)
1.60	10.24	9.38 (8)	8.56 (16)	7.83 (24)	7.01 (32)	6.28 (39)
1.80	12.96	11.89 (8)	10.91 (16)	9.54 (26)	8.34 (36)	7.62 (41)
2.00	16.00	14.59 (9)	12.98 (19)	11.66 (27)	10.09 (37)	9.12 (43)
2.50	25.00	22.66 (9)	20.17 (19)	17.49 (30)	15.20 (39)	13.80 (45)
3.00	36.00	32.24 (10)	28.21 (22)	24.17 (33)	21.35 (41)	19.79 (45)
3.50	49.00	43.53 (11)	37.77 (23)	33.62 (31)	29.75 (39)	27.67 (44)
4.00	64.00	57.22 (11)	48.64 (24)	44.55 (30)	39.95 (38)	37.44 (41)
4.50	81.00	71.43 (12)	60.44 (25)	57.53 (29)	50.40 (38)	48.26 (40)
5.00	100.00	88.30 (12)	75.07 (25)	71.04 (29)	62.58 (37)	61.67 (38)

In Table 1, the figures in brackets are the percentage cost savings relative to the strategy of always minimal repair (i.e., $p = 0$). Note as expected the percentage cost savings increase monotonically in p for fixed usage rate y.

6. Conclusions and future research

Our proposed servicing strategy extends the work of Jack *et.al.* [4] by introducing a randomized choice between replacement and minimal repair in the middle interval. Since a replacement is costlier than a minimal repair ($C_r > C_m$); the manufacturer/warranty provider has a natural incentive to do minimal repairs rather than a replacement. However, allowing a randomized choice between minimal repairs and replacement will have an impact on the reliability of the item in use at the end of warranty and, under reasonable assumptions on the aging profile of the item's life distribution,

Table 2. Optimal (K_y^*, L_y^*) for $C_r = 2$ and $\beta = 2$

y	$p = 0.2$		$p = 0.4$		$p = 0.6$		$p = 0.8$		$p = 1$	
	K_y^*	L_y^*	K_y^*	L_y^*	K_y^*	L_y^*	K_y^*	L_y^*	K_y^*	L_y^*
0.90	0.79	1.60	0.67	1.54	0.67	1.64	0.68	1.33	0.66	1.49
1.00	0.66	1.23	0.66	1.54	0.67	1.55	0.68	1.69	0.66	1.71
1.20	0.59	1.30	0.65	1.21	0.61	1.35	0.60	1.50	0.60	1.51
1.40	0.55	0.95	0.62	1.15	0.59	1.23	0.57	1.28	0.56	1.33
1.60	0.56	0.94	0.53	1.00	0.51	1.19	0.54	1.09	0.51	1.19
1.80	0.51	0.93	0.46	0.59	0.43	1.06	0.48	0.99	0.48	1.06
2.00	0.50	0.74	0.40	0.56	0.39	0.78	0.45	0.87	0.44	0.97
2.50	0.43	0.53	0.34	0.54	0.36	0.70	0.42	0.75	0.34	0.78
3.00	0.31	0.52	0.30	0.55	0.30	0.65	0.34	0.53	0.26	0.66
3.50	0.27	0.50	0.23	0.47	0.27	0.49	0.29	0.49	0.20	0.57
4.00	0.10	0.46	0.20	0.44	0.24	0.43	0.25	0.43	0.16	0.50
4.50	0.22	0.39	0.16	0.40	0.13	0.38	0.21	0.30	0.13	0.44
5.00	0.14	0.21	0.15	0.31	0.11	0.38	0.15	0.37	0.11	0.40

will typically be increasing in p, and hence higher than the corresponding reliability with minimal repairs only ($p = 0$). The corresponding analysis of resulting final reliability at the end of warranty is the subject of a future work.

Also as remarked in section 3; under plausible assumptions such as an increasing failure rate (IFR) property of the unit's failure time and the relative cost ratio of the replacement vs minimal repair we may intuitively expect the total average cost of warranty to be decreasing in p, since a replacement in the middle interval is likely to result in less degradation compared to minimal repairs only and correspondingly to less number of expected failures in the remaining time to end of warranty. Exploring such conditions would also be a topic of future research.

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