Systems of Coupled Diffusion Equations with Degenerate Nonlinear Source Terms: Linear Stability and Traveling Waves

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Abstract

Diffusion equations with degenerate nonlinear source terms arise in many different applications, e.g., in the theory of epidemics, in models of cortical spreading depression, and in models of evaporation and condensation in porous media. In this paper, we consider a generalization of these models to a system of n coupled diffusion equations with identical nonlinear source terms. We determine simple conditions that ensure the linear stability of uniform rest states and show that traveling wave trajectories connecting two stable rest states can exist generically only for discrete wave speeds. Furthermore, we show that families of traveling waves with a continuum of wave speeds cannot exist.

1 Introduction

Nonlinear diffusion equations arise in a diversity of applications, including chemical reactions with diffusion, epidemics, fluid mechanics, population dynamics, action potential propagation, and flow in porous media [2, 3, 5, 6, 10]. (For brevity, we will use the terms "nonlinear diffusion equations" or "reaction-diffusion equations" in place of "diffusion equations with nonlinear source terms", even when the source terms do not correspond to chemical reactions and are not polynomial functions in the dependent variables.)

There are a number of applications involving two or more "populations" where the reaction terms for each pair of species are proportional to one another. These usually arise as the result of a simple physical conservation law. For example, systems with evaporation and condensation must locally conserve the total mass of fluid. Thus, the mass of material that is removed from the liquid phase by evaporation must go into the gaseous phase and vice versa. This implies that the reaction terms in such reaction-diffusion models are of opposite signs and are proportional to each other, cf. [6]. A second example occurs in a mathematical formulation that describes the propagation of non-fatal diseases, known as the SIS model [2]. In this model, births and deaths are ignored, and thus, there is a conservation of the population. Moreover, the model assumes that there is no long-term immunity from re-infection. Therefore, the total population can be described by two sub-populations, namely, those individuals who are infected and those individuals who are susceptible to the disease. The reaction terms in this case describe the recovery and infection of individuals. Individuals that are removed from the infected population by the recovery process go into the susceptible population, and individuals that are removed from the susceptible population by infection go into the infected population. For this model, the reaction terms in the reaction-diffusion equations are equal and opposite. Another example is when atoms can be converted from one state to another state by a chemical reaction that must locally conserve the total number of atoms.

There are a number of applications that exhibit these local conservation laws. Despite the obvious importance of such systems, there has been surprisingly little analysis of them. Most work on such systems has focussed on the situation where there are two species and the reaction terms are proportional to the product of the two species [1, 7, 8, 9, 11]. Moreover, the focus has been on the case where the diffusion coefficients are equal or when one of the diffusion coefficients is zero. Wylie and Miura [13] considered systems of two reaction-diffusion equations with non-equal diffusion coefficients and general reaction terms. Huang et al. [5] applied linear stability analysis to a system of three equations with non-equal diffusion coefficients in the context of bread-baking.

When these systems are subjected to large amplitude disturbances, they can generate traveling waves, which move in both directions. These traveling waves may propagate and replace the initial state by a final state. The initial state may be stable or unstable, but the final state must be stable if it is to replace the initial state after the traveling wave has passed. Wylie and Miura [13] considered stable systems that are perturbed by a large-amplitude, highly-localized disturbance and determined conditions under which traveling waves could be triggered.

In general, there are two types of traveling wave solutions that can occur. Firstly, there are infinite families that have continuous ranges of wave propagation speeds. In this case, the system typically selects one of the waves based on a frontal condition [10]. Secondly, there are waves that only exist for discrete values of the wave speed for which no frontal selection is necessary. In this paper, we show that the types of traveling waves that occur in reaction-diffusion systems with local conservation laws are determined by the linear stability of the uniform states that represent the initial and final states of the traveling wave. Moreover, we show that if the initial and final states are both stable, then continuous families of traveling waves cannot exist and the only generic traveling waves have discrete wave speeds.

In this paper, we generalize the results obtained in [5] and [13]. We specify the general model equations in Section 2. In Section 3, the uniform rest states of the system are examined and linear stability of these rest states is studied. Traveling wave solutions are derived in Section 4 and the results are presented. In Section 5, we conclude the paper with a brief discussion of future directions.

2 General Model Equations

We consider a system of n nonlinear diffusion equations specified in vector form as

$$\frac{\partial \vec{u}}{\partial t} = \frac{\partial}{\partial x}\left(\mathbf{D}\frac{\partial \vec{u}}{\partial x}\right) + f(\vec{u})\vec{1}, \quad t > 0,\ -\infty < x < \infty, \tag{1}$$

where t is time, x is a one-dimensional spatial coordinate, $\vec{u} \equiv [u_1, u_2, \ldots, u_n]^T$ is the transpose of the row vector, $\mathbf{D} \equiv [D_{ij}]$ is a diagonalizable diffusion tensor, f is a nonlinear real-valued function of the dependent variables $u_1,\ u_2,\ \ldots,\ u_n$, and $\vec{1}$ is the unit column vector. Without loss of generality, we will assume that $\mathbf{D}$ is diagonal with

$$1 = D_1 \geq D_2 \geq D_3 \geq \cdots \geq D_{n-1} \geq D_n > 0.$$

(For notational simplicity, we will not always set $D_1 = 1$.) The scalar forcing term f that occurs in all of the equations corresponds to a degenerate nonlinear source term, i.e., all of the source terms are proportional to the scalar-valued function f. Without loss of generality, we have set all of the source terms equal to f.

Such a system of nonlinear diffusion equations is a generalization of model equations that arise in a number of different applications, including the epidemiological model referred to as the SIS system (Edelstein-Keshet [2]), the model equations for cortical spreading depression (Tuckwell and Miura [12]), and the model equations for bread baking (Huang et al. [5]).

In the following sections, we will derive simple conditions that determine the linear stability of uniform rest states. We also consider the equations for traveling waves with solutions corresponding to heteroclinic orbits that connect two uniform rest states. We derive simple conditions that determine the dimensionalities of the manifold on which trajectories of the traveling wave equations asymptotically tend to as they approach a uniform rest state and of the manifold on which the trajectories diverge from a uniform rest state. If both of the uniform rest states are stable with respect to time, then surprisingly, we show that the required conditions for stability also uniquely determine the dimensionalities of these manifolds. We use this result to show that continuous families of traveling waves that replace one stable uniform state by another stable uniform state cannot exist.

3 Linear Stability of Uniform Rest States

Let $\vec{R}$ be a constant uniform rest state of the reaction-diffusion equation system (1). Therefore, it must satisfy the equation $f(\vec{R}) = 0$. In order to determine the linear stability of this rest state with respect to time, we introduce perturbations of the form

$$\vec{u} = \vec{R} + \vec{p}e^{\lambda t + ikx} \tag{2}$$

where λ is the growth rate and k in the wavenumber. After substituting into the reaction-diffusion equation (1) and linearizing in $\vec{p}$, we obtain

$$\lambda\vec{p} = -k^2\mathbf{D}\vec{p} + \nabla_u f(\vec{R})\vec{p} \tag{3}$$

where $\nabla_u f(\vec{R})$ is the gradient of f with respect to u evaluated at the rest state $\vec{u} = \vec{R}$.

In order to obtain non-trivial solutions, λ must satisfy

$$\det[\lambda\mathbf{I} + k^2\mathbf{D} - \nabla_u f(\vec{R})\vec{1}^T] = 0 \tag{4}$$

where $\mathbf{I}$ is the identity matrix and $\vec{1}^T$ is the transpose of the unit vector.

If all of the components of $\nabla_u f(\vec{R})$ are non-zero, then it is easy to see that $\lambda = -k^2 D_i < 0$ is not a solution of (4). In this case, the matrix $\lambda\mathbf{I}+k^2\mathbf{D}$ is invertible and by direct application of the matrix determinant lemma [4] , we obtain

$$\det[\lambda\mathbf{I} + k^2\mathbf{D}]\left(1 - \vec{1}^T[\lambda\mathbf{I} + k^2\mathbf{D}]^{-1}\nabla_u f(\vec{R})\right) = 0. \tag{5}$$

Since $\mathbf{D}$ is a diagonal matrix, we can rewrite (5) as

$$\prod_{j=1}^{n}(\lambda + k^2 D_j)\left(1 - \sum_{i=1}^{n}\frac{\partial_i f}{\lambda + k^2 D_i}\right) = 0 \tag{6}$$

where we use the notation $\partial_i f$ to represent the derivative of f with respect to the i^{th} component of u evaluated at the rest state $\vec{u} = \vec{R}$. Equation (6) represents the dispersion relation for the rest states, and it is easy to show that it is valid even if some of the components of $\nabla_u f(\vec{R})$ are zero.

In order for the rest state $\vec{u} = \vec{R}$ to be stable, all roots of (6) must have the property $\mathrm{Re}(\lambda) < 0$ for all $k > 0$. In the following analysis, we derive conditions that ensure stability of the rest state. At $k = 0$, we obtain the root

$$\lambda = \sum_{i=1}^{n}\partial_i f \tag{7}$$

with multiplicity one and the root $\lambda = 0$ with multiplicity $n - 1$. Since λ is a continuous function of k, stability of the rest state at small k requires

$$\sum_{i=1}^{n}\partial_i f < 0. \tag{8}$$

We also need to consider whether the $(n-1)$-zero roots at $k=0$ become positive or negative for small values of k. We do this by putting $\bar{\lambda} = k^{-2}\lambda$ and isolating the leading order part in k to obtain

$$\prod_{j=1}^{n}(\bar{\lambda} + D_j)\sum_{i=1}^{n}\frac{\partial_i f}{\bar{\lambda} + D_i} = 0. \tag{9}$$

Stability requires that $\mathrm{Re}(\bar{\lambda}) < 0$, and hence, all the roots of this polynomial must have negative real part. A necessary and sufficient condition for this to occur is that the equation

$$\sum_{i=1}^{n}\frac{\partial_i f}{\bar{\lambda} + D_i} = 0 \tag{10}$$

only has solutions with $\mathrm{Re}(\bar{\lambda}) < 0$. This condition along with (8) ensures stability at small k. For large k, the situation is simpler since the roots are given by $\lambda = -k^2 D_j$ which are necessarily stable.

All that remains is to derive conditions to ensure that the real part of λ does not pass through zero for any value of k. We will separate $\mathrm{Re}(\lambda) = 0$ into the two cases, $\lambda = 0$ and $\lambda = i\omega,\ \omega \neq 0$. If $\lambda = 0$, then (6) reduces to

$$\sum_{j=1}^{n}\frac{\partial_j f}{D_j} = k^2. \tag{11}$$

Therefore, a necessary and sufficient condition that there are no real values of k for which $\lambda = 0$ is given by

$$\sum_{i=1}^{n}\frac{\partial_i f}{D_i} < 0. \tag{12}$$

If $\lambda = i\omega$, then (6) reduces to

$$\sum_{j=1}^{n}\frac{(k^2 D_j - i\omega)\partial_j f}{k^4 D_j^2 + \omega^2} = 1. \tag{13}$$

Equating real and imaginary parts, we obtain

$$\sum_{j=1}^{n}\frac{D_j \partial_j f}{k^2 D_j^2 + \omega^2 k^{-2}} = 1, \qquad \sum_{j=1}^{n}\frac{\omega \partial_j f}{k^2 D_j^2 + \omega^2 k^{-2}} = 0. \tag{14}$$

To ensure stability, we require that there are no real values of k and ω for which the above equations are satisfied.

In summary, linear stability of a rest state requires four conditions to be met. Firstly, (8) must hold. Secondly, all of the solutions of (10) must have $\mathrm{Re}(\bar{\lambda}) < 0$. Thirdly, (12) must hold. Fourthly, there must be no real values of k and ω for which (14) is satisfied.

4 Traveling Waves

Traveling waves consist of either traveling wavefronts or solitary waves and can be instigated by initial data or applied disturbances. Here we will consider the situation where a disturbance triggers a traveling wave that replaces a uniform rest state $\vec{R}^{(+)}$ ahead of the wave by another uniform rest state $\vec{R}^{(-)}$ that is left behind the wave. The rest state $\vec{R}^{(+)}$ is given by the initial condition, but the final rest state $\vec{R}^{(-)}$ is unknown and must be determined. A uniform rest state of (1) only requires that the scalar equation $f(\vec{R}^{(-)}) = 0$ is satisfied. Since there is only one equation for the n unknowns $\vec{R}^{(-)}$, it appears that the rest state $\vec{R}^{(-)}$ can be any value on an $(n-1)$-dimensional manifold given by $f(\vec{R}^{(-)}) = 0$ and that there is a continuous family of rest states. However, as we show below, this is not the case.

We leave the first equation in (1) unchanged and eliminate all of the terms containing the nonlinear source terms f from the remaining equations by subtracting the $(i+1)^{st}$ equation from the i^{th} equation for $i = 1 \ldots n-1$, to obtain

$$\frac{\partial}{\partial x}\left(D_1 \frac{\partial u_1}{\partial x}\right) - \frac{\partial u_1}{\partial t} + f(\vec{u}) \quad = \quad 0, \tag{15}$$

$$\frac{\partial}{\partial x}\left(D_i \frac{\partial u_i}{\partial x}\right) - \frac{\partial u_i}{\partial t} - \frac{\partial}{\partial x}\left(D_{i+1} \frac{\partial u_{i+1}}{\partial x}\right) + \frac{\partial u_{i+1}}{\partial t} \quad = \quad 0 \quad \text{for} \quad i = 1 \ldots n-1. \tag{16}$$

To further examine the traveling wave solutions, we adopt a uniformly moving reference frame in which the traveling wave is stationary, i.e.,

$$\vec{u}(x,t) = \vec{U}(\xi) \tag{17}$$

where $\xi \equiv x - ct$ and c is the constant traveling wave speed. Substitution into (15) and (16) yields the system of ordinary differential equations

$$\frac{d}{d\xi}\left(D_1 \frac{dU_1}{d\xi}\right) + c\frac{dU_1}{d\xi} + f(\vec{U}) \quad = \quad 0, \tag{18}$$

$$\frac{d}{d\xi}\left(D_i \frac{dU_i}{d\xi}\right) + c\frac{dU_i}{d\xi} - \frac{d}{d\xi}\left(D_{i+1} \frac{dU_{i+1}}{d\xi}\right) - c\frac{dU_{i+1}}{d\xi} \quad = \quad 0 \quad \text{for} \quad i = 1 \ldots n-1. \tag{19}$$

Integrating (19) and applying the boundary conditions $dU_i/d\xi = 0$ and $U_i = R_i^{(+)}$ far ahead of the wavefront, we obtain

$$\frac{d}{d\xi}\left(D_1 \frac{dU_1}{d\xi}\right) + c\frac{dU_1}{d\xi} + f(\vec{U}) \quad = \quad 0, \tag{20}$$

$$D_i \frac{dU_i}{d\xi} + cU_i - D_{i+1}\frac{dU_{i+1}}{d\xi} - cU_{i+1} \quad = \quad c\left(R_i^{(+)} - R_{i+1}^{(+)}\right) \quad \text{for} \quad i = 1 \ldots n-1. \tag{21}$$

The equations (20)-(21) represent an $(n+1)^{st}$-order nonlinear dynamical system with variables $(U_1,\ dU_1/d\xi,\ U_2,\ U_3, \ldots, U_n)$. We note that the initial and final rest states of the reaction-diffusion equation (1), namely

$$\vec{\mathcal{U}} \equiv \left(U_1, \frac{dU_1}{d\xi}, U_2, U_3, \ldots, U_n\right)^T = \left(R_1^{(\pm)}, 0, R_2^{(\pm)}, R_3^{(\pm)}, \ldots, R_n^{(\pm)}\right)^T \equiv \vec{\mathcal{R}}^{(\pm)}, \tag{22}$$

are both singular points of (20)-(21). Traveling waves of the original system (1) correspond to heteroclinic orbits of (20)-(21) that connect the two singular points.

Up to this point, the only requirement in selecting the n-dimensional vector $\vec{\mathcal{R}}^{(-)}$, that represents the rest state that is left behind by the traveling wave, is that it must satisfy the single equation $f(\vec{\mathcal{R}}^{(-)}) = 0$. However, far behind the wavefront, we also have $dU_i/d\xi = 0$ and $U_i = R_i^{(-)}$. Substituting these into (21), we see that

$$c\left(R_i^{(+)} - R_{i+1}^{(+)}\right) = c\left(R_i^{(-)} - R_{i+1}^{(-)}\right) \quad \text{for} \quad i = 1 \ldots n-1. \tag{23}$$

For $c \neq 0$, these additional $(n-1)$ equations, together with the equation $f(\vec{\mathcal{R}}^{(-)}) = 0$, imply that, generically, there will only be a finite set of rest states that can represent the rest state that is left behind by the traveling wave. The same is also true for $c = 0$; however, in this case, (21) can be integrated again and after using the conditions far ahead and far behind the travelling wave, we also obtain $(n-1)$ equations

$$D_i\left(R_i^{(+)} - R_i^{(-)}\right) = D_{i+1}\left(R_{i+1}^{(+)} - R_{i+1}^{(-)}\right) \quad \text{for} \quad i = 1 \ldots n-1. \tag{24}$$

In order to determine the type of heteroclinic orbits that can exist, we need to examine the dimensionality of the manifolds on which trajectories of (21) tend to each of the singular points asymptotically as $\xi \to \infty$ and as $\xi \to -\infty$. We achieve this by introducing small deviations from the singular points in the form

$$\vec{\mathcal{U}} = \vec{\mathcal{R}} + \vec{d}e^{\nu\xi}. \tag{25}$$

After substitution into (20)-(21) and linearizing, we see that nontrivial solutions exist only if

$$\det\begin{bmatrix} D_1\nu^2 + c\nu + \partial_1 f & \partial_2 f & \partial_3 f & \cdots & \partial_{n-1} f & \partial_n f \\ -(D_1\nu + c) & (D_2\nu + c) & 0 & \cdots & 0 & 0 \\ 0 & -(D_2\nu + c) & (D_3\nu + c) & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & -(D_{n-1}\nu + c) & -(D_n\nu + c) \end{bmatrix} = 0. \tag{26}$$

Expanding the determinant, we obtain

$$\prod_{i=1}^{n}(D_i\nu + c)\left(\frac{D_1\nu^2 + c\nu + \partial_1 f}{D_1\nu + c} + \frac{\partial_2 f}{D_2\nu + c} + \cdots + \frac{\partial_n f}{D_n\nu + c}\right) = 0, \tag{27}$$

which can be rewritten as

$$\prod_{i=1}^{n}(D_i\nu + c)\left(\nu + \sum_{j=1}^{n}\frac{\partial_j f}{D_j\nu + c}\right) = 0. \tag{28}$$

This equation is invariant under the transformation $(\nu, c) \to (-\nu, -c)$, and so we need only consider $c \geq 0$. We note that this equation is similar in form to the dispersion relation for the linear stability of the uniform rest states (6). However, this polynomial is of degree $(n+1)$ in ν, whereas the dispersion relationship is of degree n in λ, and there is no simple relation between the two. Nevertheless, we will show that conditions that are required for the stability of uniform rest states uniquely determine the dimensionalities of the manifolds on which trajectories of (21) tend to each of the singular points asymptotically as $\xi \to \infty$ and as $\xi \to -\infty$. We determine this by finding the number of roots that the polynomial (28) in ν has with positive real parts and negative real parts .

When $c = 0$, (28) becomes

$$\nu^{n-1} \left(\nu^2 + \sum_{j=1}^{n} \frac{\partial_j f}{D_j} \right) = 0. \tag{29}$$

Linear stability of the uniform rest states requires that $\sum_{j=1}^{n} \partial_j f / D_j < 0$ and so (28) has a root $\nu = 0$ with multiplicity $n-1$ and one positive and one negative real root.

For small positive c, there will still be one positive real root and one negative real root, but to determine whether the $(n-1)$ roots that are zero when $c = 0$ have positive or negative real parts, we perform an asymptotic expansion for small c. We define $\bar{\nu} = c^{-1}\nu$ and substitute into (28) to get

$$c^n \prod_{i=1}^{n} (D_i \bar{\nu} + 1) \left(c\bar{\nu} + \frac{1}{c} \sum_{j=1}^{n} \frac{\partial_j f}{D_j \bar{\nu} + 1} \right) = 0. \tag{30}$$

Since $c \ll 1$, at leading order, this reduces to

$$\sum_{j=1}^{n} \frac{\partial_j f}{D_j \bar{\nu} + 1} = \bar{\nu}^{-1} \sum_{j=1}^{n} \frac{\partial_j f}{D_j + \bar{\nu}^{-1}} = 0. \tag{31}$$

However, (31) is effectively the same as (10) with λ replaced by $\bar{\nu}^{-1}$. Therefore, linear stability of the uniform rest states requires that $\sum_{j=1}^{n} \partial_j f / (D_j + \bar{\nu}^{-1}) = 0$ only has roots with $\mathrm{Re}(\bar{\nu}^{-1}) < 0$. Using elementary properties of complex numbers, we deduce that $\mathrm{Re}(\nu < 0)$. Thus, for small c, we have one root with positive real part and n roots with negative real parts.

As $c \to \infty$, (28) has n roots given by

$$\nu = -\frac{c}{D_i} < 0 \quad \text{for} \quad i = 1 \ldots n, \tag{32}$$

and these roots must be real and negative. The other root is given by

$$\nu = -\frac{1}{c} \sum_{j=1}^{n} \partial_j f. \tag{33}$$

Linear stability of the uniform rest states requires that $\sum_{j=1}^{n} \partial_j f < 0$, which implies that this root must be positive. So at large values of c, we also must have one root with positive real part and n roots with negative real part.

It only remains to show that there are no values of c for which $\mathrm{Re}(\nu)$ changes sign. We will separate $\mathrm{Re}(\nu) = 0$ into the two cases, $\nu = 0$ and $\nu = i\alpha$, $\alpha \neq 0$. If $\nu = 0$, then (28) yields

$$c^{n-1} \sum_{j=1}^{n} \partial_j f = 0. \tag{34}$$

Linear stability of the uniform rest states requires that $\sum_{j=1}^{n} \partial_j f < 0$, which implies that $c = 0$ is the only value of c for which $\mathrm{Re}(\nu)$ could change sign.

On the other hand, for $\nu = i\alpha$, substitution into (28) and equating the real and imaginary parts, we obtain

$$\sum_{j=1}^{n} \frac{\partial_j f}{\alpha^2 D_j^2 + c^2} = 0, \qquad \sum_{j=1}^{n} \frac{D_j \partial_j f}{\alpha^2 D_j^2 + c^2} = 1. \tag{35}$$

However, these equations are identical to (14) with $\alpha = k$ and $c = \omega/k$ and so linear stability of the uniform rest states requires that there are no real values of α and c for which $\mathrm{Re}(\nu)$ changes sign. Hence, we conclude that linear stability of the uniform rest states ensures that for $c > 0$, ν has one root with positive real part and n roots with negative real part. Since both uniform rest states are linearly stable, this applies to both singular points of (20)-(21). For $c < 0$, there is one root with negative real part and n roots with positive real part for both singular points of (20)-(21).

Hence, for both $c > 0$ and $c < 0$, the dimensioniality of the unstable manifold for one singular point plus the dimensionality of the stable manifold for the other singular point equals the dimensionality of the system. Therefore, for an arbitrary value of c, the unstable manifold of one singular point will not generically coincide with the stable manifold of the other singular point. However, we may expect generically that the stable and unstable manifolds may coincide at particular discrete values of c. Therefore, heteroclinic orbits, and hence traveling waves, can only exist for discrete values of c.

5 Conclusion

In this paper, we have presented a general framework for analyzing the stability of a system of coupled diffusion equations with degenerate nonlinear source terms. Such systems arise naturally in many applications, including the classical model of epidemics and population dynamics, chemical reactions, action potential propagation, and fluid flow with phase change. We have analyzed the linear stability of the uniform rest states and have shown that the dimensionalities of the stable and unstable manifolds for the traveling wave solutions are determined by the stability criteria for the uniform rest states. We also have shown that the traveling wave trajectory that connects two stable uniform rest states can generically exist only for discrete wave speeds.

Since the behavior of the system is largely determined by the specific form of the nonlinearity, it would be of practical interest to apply the general results obtained in this paper and compare them with numerical predictions. Furthermore, it would be desirable to study the stability of traveling waves in such systems since in higher spatial dimensions the wavefront may be unstable to fingering instabilities [5].

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