Using Dynamic Programming methods in Repair and Replacement Problems

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Abstract

Applications of dynamic programming to problems in reliability, with a specific emphasis on repair and replacement are surveyed. A brief summary of early classic work on such uses of dynamic programming is followed by a review of later approaches adopted by different researchers towards the problem of modeling plausible and pragmatic notions of imperfect repairs. We then provide a synthesis of how such different approaches for the study of repairable systems can be subsumed within a unified setting of a stochastic dynamic programming framework with some illustrative applications of such a formulation.

1 Introduction.

In this article, we focus on the usefulness of dynamic programming ideas and methods in the study of repairable systems with a specific emphasis on replacement and other repair startegies. After an overview of early work in this area, which concentrated on optimal planned replacement of stochastically failing equipment, we briefly survey later development on different formulations of more pragmatic repair schemes that have been proposed over the years, as a prelude to demonstrating how different notions of imperfect repair can be encompassed in a unified stochastic dynamic programming setting.

The name 'dynamic programming' was coined by Richard Bellman as a shorthand description of a body of theory and methods for solving optimization problems in multi-stage decision processes, illustrated in his now classic book (Bellman 1957). Such problems are typically characterized by a system described via proxy varibles (*system states*) that change (undergo deterministic or, stochastic *transitions*) as a result of decisions (*actions*) chosen from a set of available options. Corresponding to each transition is a reward or, penalty (negative reward) which depends on the chosen action associated with the transition as well as the system state before or/and after the transition. This describes a typical stage of the muti-stage process, which is then repeated in the next stage. The *planning horizon* of a

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dynamic programming problem is the finite or, infinite number of stages for which decisions on the choice of actions have to be made. A policy, or *strategy* is any well defined rule to determine the choice of actions based possibly on the system's past history to date. The goal is to maximize the value (expected value, under stochastic regimes) of the stream of rewards over the entire planning horizon, and identify strategies which achieve or, nearly achieve the optimum value function. The 'value' of a reward stream is usually defined as the total cumulative reward with or without discounting. In some problems,in contrast to the total reward, the long run average reward is considered as an alternative value function.

Such optimization problems over a finite or, infinite horizon is conceptually equivalent to finding the global maximum of an objective function with respect to a large number of variables - a task that is often fraught with analytic as well as computational difficulties, referred to collectively as the '*curse of dimensionality*'. Bellman's basic simplifying idea, embodied in his celebrated '*principle of optimality*' was to break down the problem into one of sequential choices, which together with backward induction can routinely solve the optimization problem when the planning horizon and the set of available actions are both finite. Over the years, these methods have correspondingly found extensive use in many application areas in engineering and operations research. Subsequently, in a series of papers, Blackwell (1965, 1966) and Strauch (1966) set the original ideas of Bellman on a firm theoretical foundation, including a general (and not problem specific) proof of Bellman's principle of optimality, that cover both finite and infinite horizon problems. It is this conceptual framework on which rests much of our discussions on dynamic programming formulations of repairable systems in sections 3 and 4. It is important to remark at this point that a dynamic programming formulation to describe the temporal evolution of a system can be a useful vehicle for its modeling and analysis even when there is no choice of actions to be made; i.e., when there is no explicit optimization task at hand, as our discussions in section 4 will illustrate.

The basic premise of the importance of repairs, apart from generally lesser costs compared to replacement, is that it allows the reuse of equipments, eventhough such repaired equipments will typically have altered reliability characteristics compared to the original as a result of repair - as reflected in the distribution of post-repair survival times. Contemporary concerns about finite environmental resources further underscores the importance of such 'recycling' of equipments and systems. Additionally, there may be economic incentives for considering replacement or, less than perfect repair options, since such repairs are typically less costly than replacement. The impact of a repair strategy to maintain an equipment is reflected in the point process of failures as it evolves over time. In applications, the effectiveness of a repair strategy is determined by appropriate objective functions that are statistical functionals of the underlying failure process. Examples of such objective functions are, (i) the expected number of failures / repairs over a given time interval, (ii) the total expected costs / benefits of a repair scheme, etc. It is here in the modeling and analysis of the effectiveness of such repairs that dynamic programming ideas and methods can be fruitfully exploited.

To the best of our knowledge; as of today, there is no standard commercial or, open source general purpose dynamic programming (DP) software that would allow users to conveniently specify and solve arbitrary dynamic programming problems. Instead, the most common and fruitful approach to the computational implementation of dynamic programming solutions to real world problems appears to be development of application specific DP-software. In our search for DP-software on repair and replacement, we came across only one such modular software "CompEcon (toolbox for MATLAB)" which includes solvers for discrete time/discrete or continuous variable dynamic programming problems. It was developed to accompany a book by Miranda and Feckler (2004) and can be found at *http://www4.ncsu.edu/~pfackler/compecon/toolbox.html* website. A similar search for a general purpose DP-software, led us to "dProg", a software package for specifying dynamic programming algorithms that - given a recursive definition of the problem, generates codes for solving the problem using dynamic programming. The "dProg" package, developed by Thomas Mailund, is available as a freeware at *http://www.daimi.au.dk/~mailund/dprog.html*, although it may be of relatively limited use in our context of repairable systems.

Section 2 describes some early work exploiting dynamic programming ideas to investigate problems in reliability, followed by a summary of relevant modeling approaches adopted by different authors to address possible notions of imperfect repairs. It is then shown that these different formulations can be synthesized in a common setting. Section 3 then describes a general dynamic programming formulation for the analysis of a repairable system. Various imperfect repair schemes proposed by different authors, are then seen to be special cases. The usefulness of a dynamic progamming formulation to study repair and replacement problems is illustrated in section 4, drawing on some recent work in the literature.

2 Early work, minimal repairs and other imperfect repair schemes.

2.1 *Early work and repair-replacement.* The early literature on system maintenance aspect of reliability naturally focused on replacements as the analytically tractable option for modeling system repair, since the tools of renewal theory, a well developed theme in probability, are then immidiately applicable. By contrast, attention to modeling other relatively more pragmatic notions of repair had to wait until the 1970s and '80s. Although not directly related to the theme of repair and replacement; a work by Bellman and Dryfus (1958) appears to be the earliest reference on the use of dynamic programming methods for maximizing system reliability under given constraints. Another example of such use of dynamic programming , unrelated to replacemts and repair, is a work by Kettelle (1962) on optimum allocation of redundancy.

Among the early classic illustrations of the application of dynamic programming methods for the analysis of optimal replacement policies are the works by Jorgenson and Radner (1962), Radner and Jorgenson (1963). They consider a series system of $(M+1)$ independent components, $M \geq 1$, when components $i = 1, 2, \cdots, M$ have exponential lifetimes that can be continuously monitored. The remaining other component (component $i = 0$) is assumed only to have a positive density with infinite support, and cannot be inspected except at replacement times. Using a dynamic programming formulation, the authors construct an optimal replacement policy, called as the (n_i, N) policy, that maximizes the total expected discounted value of time that the system is in good operational state less costs, and can be described as follows. For $i = 1, 2, \cdots, M$, (i) replace component-i alone when it fails, if the corresponding age of component-0 is between 0 and n_i, (ii) replace components i and and 0 together, when component-i fails and component-0 has age between n_i and N at that time, and (iii) replace component-0 alone, if all components $1, 2, \cdots, M$ have survived until component-0 has reached age N. The same policy remains optimal for other alternative criterions such as expected total discounted time in good state, and the ratio of expected total discounted good time to expected total dicounted costs.

Work on replacement strategies have continued over the years, with researchers addressing different variations on the assumptions regarding failure and repair time distributions, rewards and costs with or without discounting, finite or infinite planning horizon, and the objective function (suitably defined total or, long run average net reward). Chen and Feldman (1997) consider optimal replacement policies with minimal repair and age-dependent costs. Also notable among works on the optimal repair-replacement theme are : Makis and Jardine (1993), Lam Yeh (1988, 1991), Stadje and Zuckerman (1990), and Boshuizen (1997). In particular Boshuizen's (*ibid.*) analysis derives the optimal rules without imposing some reliability theoretic and monotonicity conditions, as employed in Lam Yeh (*ibid.*) and Stadje and Zuckerman (*ibid.*), by solving an optimal stopping problem first in a finite horizon setting, and then extending it to the infinite horizon case.

Barlow and Proschan (1962) investigated sequential replacement policies for finite time horizons, in which the next scheduled replacement age depends on the time remaining. They show that without loss of generality, it is enough to restrict attention to non-randomized replacement policies, prove the existence of a least cost sequential policy and demonstrate how to construct them. While there is no direct reference to dynamic programming methods in their arguments, since a formal theory of stochastic dynamic programming (Blackwell 1962, 1965; Strauch 1966) was still in its relative infancy; such ideas are clearly recognizable from a careful reading of their arguments for optimality.

2.2 *Minimal repairs and other imperfect repair schemes.* A repair mode is defined by its effect on the equipment as described by the distribution of its post-repair life. If X is the lifelength of a new unit, and L_x denotes the lifelength (time to failure) of an instantaneously repaired unit that most recently failed at age $x \geq 0$; then setting $L_x \stackrel{d}{=} X$ defines a *replacement* by a statistically identical copy ($\stackrel{d}{=}$ denoting equality in distribution), whereas $L_x \stackrel{d}{=} X - x \mid X > x$, the *residual life* after failure at *age* x, defines the so called *minimal repair* (MR) option. A *perfect repair* (PR), by contrast, is defined as a repair that restores an equipment to its original condition, as parametrized by its *age* at the time *when it started working prior to the most recent failure.* Thus, for a currently failed equipment with failure having occured at age x (which implies that it last started functioning at an age t satisfying $0 \leq t < x$); a perfect repair (PR) action is defined by setting the post-repair lfe as $L_x :\stackrel{d}{=} X - t \mid X > t$. Note, a renewal (replacement) is equivalent to perfect repair (PR) only if the currently failed unit was new ($t = 0$) when it started working.

For most equipments that exhibit some from of aging over time; replacement and minimal repair conceptually represent two extreme positions in terms of the extent to which a failed equipment's degradation is arrested and restored by repairs. The need for considering more realistic notions of repair, which are in a broad sense intermediate beween MR and PR, was mainly voiced by the engineering community, which then led to efforts to formulate alternative notions of *imperfect repairs* (IR), The notions of IR, refer to those repair assumptions that are, broadly speaking, more realistic than the traditional renewal (replacement) hypothesis.

The first imperfect repair model was proposed by Brown and Proschan (1983), which underwent a significant and nontrivial extension by Block, Borges and Savits (1985). The Block-Borges-Savits (BBS) model randomizes the choice of repair between a replacement / PR or, MR with a probability that is age-dependent (choosing a replacement / PR or, MR with probabilities $p(x)$ $(1 - p(x)$ respectively) if failure occured at age x. The Brown-Proschan (BP) model is covered by the BBS model as a special case when $p(x)$ is a constant. Bhattacharjee (1987) provided an alternative shock model argument for the distribution of sojourn time between consecutive perfect repairs, possibly interpersed with a random number of minimal repairs in betwen.

Kijima (1989) proposed and investigated two models of imperfect repair, based on the idea that if an equipment failed at a certain age; after repair, its age is not necessarily restored to the age when it failed (as is the case with MR), but may be different depending on how good or, bad the repair is. This effective post-repair age is referred to as the *virtual age*. In Kijima's models, the virtual age process V_k after k^{th} failure is linear in V_{k-1} and the lifetime X_k after $(k-1)^{st}$ failure (the k^{th} functioning lifetime) such that the slope of V_k depends on the 'degree of repair' a assuming values in $[0\,,\,1]$ with $a \;=\; 0\,,1$ implying perfect and minimal repairs respectively, whereas intermediate values of $a \in (0,1)$ reflect varying degrees of repair effectiveness.

Several other virtual age type models, whuch are basically extensions of Kijima's models, have been considered by Finkelstein (1993), Makis and Jardin (1993), and Dagpunar (1998). Virtual age models capture the idea of arresting degradation in performance by syetm age recovery as a result of repairs. Stadje and Zuckerman (1991), (1993) investigate analytical and numerical methods to determine least expected discounted cost age-dependent repair strategies that allow for varying degrees of repair over an infinite planning horizon. Guo, Ascher and Love (2001), in a review article, provide a summary of various virtual age models and provide references to work on optimal repair and replacement policies based on such models.

2.3 *A unified setting for imperfect repair* (IR) *models.* Bhattacharjee (2000) argued that a reasonable way to conceptualize the effect of a given repair mode or, action a is to make the repaired unit have an effective *virtual age* Y after repair, whose distribution A_z^a depends on the repair mode a and the age z at failure. Thus, if $L_{z,a}$ denotes the post-repair life after instantaneous repair using repair mode a on failure at age z, the effect of repair is defined by the post-repair survival probability,

$$P\{L_{z,a} > t\} := E\bar{F}_Y(t) = \int_0^\infty \bar{F}_y(t)\, A_z^a(dy)\,, \tag{2.1}$$

where $\bar{F}_y$ is the tail of the distribution of the *residual life* at age y, given by $\bar{F}_y(x) = \bar{F}(x+y)/\bar{F}(y)$. Two repair modes a, a' such that $L_{z,a} \stackrel{d}{=} L_{z,a'}$, all $z \geq 0$, are equivalent($a \sim a'$), as their effects on equipment degradation cannot be distinguished.

The *virtual age* idea was also considered by Baxter *et. al.* (1996) to study the interfailure time distribution

$$B(t) = \int_0^\infty F_y(t)\, P(dy) = E_P F_Y(t),$$

with P governing the distribution of the virtual age after repairs. Bhattacharjee's (2000) formulation in (2.1) corresponds to parametrizing the virtual age distribution as $P \equiv A_z^a$, leading to the distribution of post-repair life. If $P(Y = z) = A_z^a\{z\} < 1$, we may interpret the value of effective post-repair virtual age Y of the unit which failed at age z, as the difference between the degree of effort needed for a perfect repair vs. the actual repair effort measured in units of operating time.

Starting with an unit (of age $x \geq 0$), which then fails at age $z(\geq x)$, it then follows that the distribution of the post-repair survival times under PR, MR, BP and BBS repair modes are given by,

$$P\{L_{z,a} > t\} = \begin{cases} \bar{F}_x(t), & a := \text{PR} \\ \bar{F}_z(t), & a := \text{MR} \\ p\bar{F}_x(t) + q\bar{F}_z(t), & a := \text{BP} \\ p(z)\bar{F}(t) + q(z)\bar{F}_z(t), & a := \text{BBS} \end{cases} \tag{2.2}$$

where F_x denotes the *residual* life distribution at age x, with $F_0 \equiv F$. When the MR option is chosen in the BP and BBS models, note that the additional age accumulated at the most recent failure is $z - x \geq 0$; setting $S := F_x$, it follows that the post-repair life distribution, conditional on the choice of MR, is

$$\bar{S}_{z-x}(t) = \frac{\bar{S}(z-x+t)}{\bar{S}(z-x)} \equiv \frac{\bar{F}_x(z-x+t)}{\bar{F}_x(z-x)} = \frac{\bar{F}(z+t)}{\bar{F}(z)} = \bar{F}_z(t),$$

justifying (2.2). Note also that, if the currently failed unit began working as a new equipment $(x = 0)$, then a PR in (2.2) corresponds to a replacement. As a general formulation of the notion of *imperfect repair*, (2.1) is sufficiently flexible to allow different variations by suitably choosing the virtual age distribution A_z^a. Such choices can include nonparametric constraints on the post-repair life distribution, such as specifying (i) *maximum allowable improvement in virtual age* : $0 \leq A_z\{0\} < 1$ or/and, $A_z(0\,,\,\epsilon) = 0$, for some $\epsilon > 0$, (ii) *a limit on maximum possible degradation* : $A_z(z+L\,,\,\infty) = 0,\ L > 0$; i.e., virtual age after repair is at most an additional L over the age at failure; or, (iii) *specify a best / worst case scenario for the mean virtual age) the expected virtual age after repair is no more than (at least, respectively) the age at failure* : $\int_0^\infty y\, A_z(dy) \leq (\geq) z$.

3 A dynamic programming framework for repairable systems

The dynamic programming model described below is based on Bhattacharjee (2000), drawing on the foundations of a theory of stochastic dynamic programming developed by Blackwell

(1965),(1966) and Strauch (1966). The *states* s of the system are defined by a pair $s = (t, x)$, where

$$\begin{aligned} t &= \text{remaining clock-time, for a given time horizon,} \\ x &= \textit{virtual age} \text{ of the unit, which starts functioning at } t. \end{aligned}$$

The set $S = \{(t, x) \ : \ x \geq 0\}$ is the state space; and the states of the system correspond to those time epochs, when a unit starts working, possibly after a repair. The subset $S^* = \{(t\,,\, x) \ : \ x \geq 0\,,\, t \leq 0\}$ is the set of *trap-states* in the sense that the process ends on entering S^*. (For an infinite time horizon, it is simpler to define the system's state s as the virtual age x of the unit at those time epochs when the unit starts working.) A repair *action* is a repair mode that one can implement with its effect on post-repair life described by (2.1). The set of available choices of repair actions a is $A = \{a \ : \ a \in A\}$.

Let $T_x \stackrel{d}{=} X - x \mid X > x \sim F_x$ denote the 'residual life' at age x ($T_0 \stackrel{d}{=} X \sim F$). Note also that if we start with an unit of age x (i.e., at a state $(\,.\,,\, x)$), its time to next failure is T_x with d.f. F_x, and *not* F. When we choose a repair action $a \in A$ at a state $s = (t, x) \notin S^*$, our system moves to a new state s', possibly depending on $(s \equiv (t, x),\, a)$, according to the transition scheme :

$$s = (t, x) \ \stackrel{a}{\longrightarrow} \ s' = (\, t - T_x\,,\, V_{x,a}\,), \tag{3.1}$$

where, the repaired unit's virtual age $V_{x,a}$ depends on its age x when it last began functioning, and the executed repair mode a when it subsequently failed. Thus, the conditional probability distribution $q(ds' \mid (s, a))$, governing the transition in (3.1) is determined by the joint distribution of $(T_x, V_{x,a})$. The particular form of the virtual age $V_{x,a}$ in turn depends on the chosen repair action a ; e.g.,

$$V_{x,a} = \begin{cases} 0\,, & \text{for } a = \text{Replacement} \\ x\,, & \text{for } a = \text{PR} \\ x + T_x\,, & \text{for } a = \text{MR} \\ x + aT_x\,, & \text{Model-I (Kijima)}, 0 \leq a \leq 1 \\ a(x + T_x)\,, & \text{Model-II (Kijima)}, 0 \leq a \leq 1 \end{cases} \tag{3.2}$$

where $a \in A = [0, 1]$ reflects the 'degree of repair' in Kijima's (1989) models. Here, increasing values of a represent progressively worse quality of repair. When $a = 1$, both models of Kijima reduce to minimal repair (MR).By contrast, $a = 0$ represents a perfect repair (PR) in Model-I, but a replacement in Model-II.

A *repair strategy* is a sequence $\pi = \{\pi_1, \pi_2, \cdots\}$ that spells out a sequential plan of repair actions to choose after each failure, possibly based on the past history. Formally, the choice of repair after the n-th failure is goverened by π_n, a conditional distribution on the available set A of repair actions, given the past history $(s_1\,, a_1 \cdots s_{n-1}\,, a_{n-1}\,, s_n)$ of failures and repairs. Note that, based on our formulation of the state space, a repair strategy need only specify each π_n when the most recent state $s_n \notin S^*$.

Among the simpler strategies, which are intuitively appealing, are those that are Markovian or / and stationary. A strategy $\pi = \{\pi_n,\ n \geq 1\}$ is Markovian if each π_n is a regular conditional distribution on the set of repair actions A, given the state at the most recent failure. If each π_n is degenerate, i.e., if $\pi_n \equiv f_n : S \mapsto A$; then $\pi = \{f_n,\ n \geq 1\}$ is a *non-randomized Markov* repair strategy. If $f_n \equiv f : S \mapsto A$ is time-homogenious, then $\pi := f^\infty \equiv \{f, f, \cdots\}$ is a *nonrandomized stationary strategy* of repairs.

To illustrate some of the IR-repair modes, described in section 2, and the corresponding repair schemes considered in the literature, as repair strategies in a dynamic programming framework; let 0 and 1 respectively denote a *perfect* (PR) and *minimal* repair (MR) modes. If $A = \{1\}$, then the only available strategy is 'forever minimal repairs' (FMR) : $\pi = 1^\infty$. Similarly, if $\{0, 1\} \subset A$, then available choices include the age-dependent Block-Borges-Savits (BBS) repairs, as the randomized stationary strategy $\pi := \pi_1^\infty$, where

$$\pi_1(\{0\} \mid (t,x)) = p(T_x),\ \pi_1(\{1\} \mid (t,x)) = 1 - p(T_x); \quad (t,x) \notin S^* \tag{3.3}$$

noting that the age of the currently failed unit, which started working as an unit of age $x \geq 0$, is $T_x \stackrel{d}{=} X - x \mid X > x$. The Brown-Proschan (BP) repair scheme corresponds to the special case of the BBS-strategy when $p(\cdot) \equiv p$ with $0 < p \leq 1$. The case $p \equiv 1$ is the strategy of 'forever perfect repairs' (FPR), which is equivalent to a strategy of replacements when the process starts with a new equipment. This equivalence is clear by noting that while replacement by a new copy of the failed equipment is technically an option that is distinct from PR; setting $x = 0$ in the stochastic transition for a PR option :

$$s = (t,x) \quad \stackrel{\text{PR}}{\longrightarrow} \quad s' = (t - T_x\,,\, x),$$

the transition correponding to $x = 0$ (new equipment) is $s = (t, 0) \stackrel{\text{PR}}{\longrightarrow} s' = (t - X, 0)$, and thus equivalent to a replacement (note, $T_0 \stackrel{d}{=} X$).

Similarly, for various 'degree of repair' models considered by Kijima (1989) and others; $A = [0,1]$ parametrizes the degree of repair efforts, as described in (3.2); any repair strategy is a sequence of distributions on the unit interval given the past.

For a given initial state $s = (t,x)$; a repair strategy π determines a probability P_π on $H := ASAS\cdots$, the set of future histories of the system. Rewards between transitions are defined by a bounded real valued function r on SAS such that $r(s,a,s')$ is the reward for choosing repair action a, at state s, used to repair the equipment when it fails. The choice of r is determined by our objective. For example, to compute the expected number of failures (repairs) in a time interval $(0,t)$, take $r(s,a,s') = 1$, for all a, if s and $s' \notin S^*$, and $= 0$ otherwise. A history $h = (s \equiv s_1, a_1, s_2, a_2, \cdots)$, starting at s, generates a sequential stream of random rewards $r_n := r(s_n, a_n, s_{n+1})$ resulting in a total reward $R := \sum_{n=1}^\infty \beta^n r_n$ defined on SH, with a discount factor $\beta \in (0,1]$ (in the undiscounted case $\beta = 1$). The '*return*' (total expected reward) of a repair strategy π is

$$I_\pi(s) := E_{P_\pi} R = \int_H R(s,h) P_\pi(dh), \quad s \in S.$$

I_π is finite under discounting, or other suitable conditions on r. The '*optimal return*' is $I^*(s) := \sup_\pi I_\pi(s)$, $s \in S$. An optimal repair strategy π^*, if it exists, is one for which $I_{\pi^*} = I^*$, all $s \in S$.

4 Illustrative applications.

We give two examples of how the preceeding ideas can be employed to investigate repair and replacement problems. It should be emphasized that even if our interest is confined to the analysis of performance of a specific repair strategy, such as the Barlow-Proschan (BP) repair scheme, and not in pursuing possibile optimality; ideas from dynamic programming can often allow us an elegant way to analyse specific repair schemes.

4.1 *Computing the sojourn time to the next perfect repair.* In the Brown-Proschan (BP) repair scheme, which defines a stationary randomized repair strategy with a constant probabilities p and $(1-p)$ respectively, of choosing a PR or, MR repair option; suppose that there is a *terminal pay-off* =1 on entering the trap states S^*, and that all other pay-offs are zero. Beginning with a unit of age x, let T^*_x be the *sojourn time to the next* PR. If $X_1, X_2, \cdots$ are the post-repair lives under successive minimal repairs, and the next PR occurs at the τ-th failure ($\tau < \infty$, w.p. 1, if $p > 0$); then, with 1_A denoting the indicator function of a set A, the *expected total return* of the BP-repair strategy is clearly,

$$E_{BP}\{\, 1_{\{t-X_1-X_2-\cdots-X_\tau<0\}}\,\} \;=\; P_{BP}(T^*_x > t), \tag{4.1}$$

the tail of the sojourn time distribution to the next perfect repair. To compute the sojourn time tail (4.1) by using dynamic programming ideas, consider an operator L mapping non-negative bounded functions into itself, such that for $(t,x) \notin S^*$,

$$\begin{aligned} Lu(t\,,\,x) \;&:=\; p\,E1_{T_x>t} \;+\; qE\{\,1_{T_x>t} \;+\; 1_{T_x\le t}\,u(t \,-\, T_x\,,\,x \,+\, T_x)\} \\ &=\; \bar{F}_x(t) \;+\; q\,\int_0^t u(t-y\,,\,x+y)\;F_x(dy), \end{aligned} \tag{4.2}$$

and $Lu(t\,,\,x) := 1$, if $(t,x) \in S^*$. Then $Lu(t,x)$ is our expected reward if we use the BP-repair mode at the first failure after (t,x) and then stop with our terminal payoff $u(\cdot,\cdot)$ at next state. Let $\mathbf{0}$ denote the function which identically vanishes on S. Then, it can be shown that,

Theorem. (Bhattacharjee (2000))

(i) *L is a contraction.*

(ii) $L^n\mathbf{0}\,(t,x) \to P_{BP}(T^*_x \,>\, t)$, for $(t,x) \notin S^*$, as $n \to \infty$.

(iii) $u^*(t,x) := 1_{(t,x)\in S^*} + 1_{(t,x)\notin S^*}[\bar{F}_x(t)]^p$, *is the unique fixed point of L.*

(iv) $P_{BP}(T^*_x \,>\, t) = [\bar{F}_x(t)]^p$, *for* $(t,x) \notin S^*$.

Brown and Roschan's (1983) result on the sojourn time distribution follows as a special case of claim (iv) above, with $x = 0$.

4.2 *Bounding the renewal function via dynamic programming.* Blackwell (1964) first pointed out the utility of exploiting dynamic programming ideas to derive sharp probability bounds. Renewal functions, which represent the expected number of failures (repairs) under a 'replacements only' strategy, can be explored using such ideas, to provide sharp bounds on renewal functions under various nonparametric constraints on the failure distribution generating the renewal process.

The renewal function can be viewed as the total expected pay-off, in a game against a passive nature, in which the states are points $t \geq 0$ on the half-line, and stochastic transitions are defined by $t \longrightarrow \max(t - X, 0)$, where $X \geq 0$ with d.f. F. The game stops on hitting the origin. Corresponding to this transition, we earn a 'reward' $1_{\{X<t\}}$. Also, corresponding to successive moves dictated by a i.i.d. sequence $X_1, X_2, \cdots$ with a common d.f F, the total expected pay-off is clearly,

$$E\left(\sum_{n=1}^{\infty} 1_{\{\sum_{j=1}^{n} X_j \leq t\}}\right) = E \max\{n : \sum_{j=1}^{n} X_j \leq t\} := M(t),$$

the renewal function. In the spirit of section 3, and the work of Blackwell (1964), a method to construct sharp upper bounds to the renewal function was developed by Bhattacharjee (1999), by considering an *operator* U mapping the set of real valued, nonnegative nondecreasing functions Q, such that $Q(0) = 0, Q(\infty) = \infty$, into itself; defined by

$$UQ(t) := F(t) + \int_0^t Q(t - x)\, F(dx), \quad t \geq 0$$

to show that,

(i) $U^n\, \mathbf{0}\ (t) \to M(t)$ *as* $n \to \infty$,
(ii) U *is monotone, and* $UQ \leq Q$ *implies* $M \leq Q$ *pointwise,*
(iii) $M(t)$ *is the minimal fixed point of* U.

The well known upper bound $M(t) \leq (t/\mu)$ for NBUE (new better than used in expectation) life distributions with a mean μ can be derived as an application of this result, by choosing $Q(t) := t/\mu$. When F has the stronger NBU (new better than used) property, the same method is used by Bhattacharjee (1999), to show

$$F \text{ is NBU implies } M(t) \leq -\ln\{1 - F(t)\}, \quad t \geq 0, \tag{4.3}$$

i.e., the 'hazard function' is an improved sharp upper bound for the renewal function under the NBU hypothesis.

4.3 One can thus explore the performanec of repair strategies for appropriate pay-off functions corresponding to state transitions that capture a desired objective function, such as the total expected number of failures (repairs) in a time interval $(0, t)$, starting with a unit of age $x \geq 0$. The corresponding technical issues that must be addressed in such analysis often reduce to finding the minimal solution or, tight bounds to the solution of an integral equation. Such an approach can also be used to explore optimal or, near-optimal repair strategies. When we superimpose suitable costs (of repair) and rewards between transitions (incomes generated during equipment functioning cycles), and possibly an infinite time horizon with or without discounting; we then have a corresponding economic optimization problem, for which dynamic programming methods are eminently suitable.

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