Adaptive Economic Choices under Recurrent Disasters : a Bayesian Perspective

M.C. Bhattacharjee

Department of Mathematical Sciences and
Center for Applied Mathematics and Statistics
New Jersey Institute of Technology, Newark, NJ 07102 and

Indian Statistical Institute

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Manish C. Bhattacharjee*

Center for Applied Mathematics and Statistics
Department of Mathematical Sciences
New Jersey Institute of Technology
Newark, New Jersey 07102

I. Introduction

Consider a geographical region susceptible to some natural disaster that occurs randomly over time. Assume that the economic impact of such disasters is limited to the destruction of capital and corresponding investments in place, which consequently interrupts the stream of economic returns, and that after a disaster, there is no delay to reinvest in a feasible economic activity from a set of available alternatives.

Given the profiles of economic activities one may choose from and a parametrically specified model for the stochastic occurrences of the natural disaster over time, the basic problem of choosing an economic strategy which maximizes the expected total discounted stream of returns is conceptually simple enough. If the length of time between

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consecutive disasters depends *neither* on the economic activities undertaken, *nor* on the past history of such disasters; it is not hard to show that an activity which maximizes the expected return between consecutive disasters is optimal for the infinite horizon.

The situation is not so simple however, if the parameters governing the process of disasters are unknown. In such a case, it is natural to assume a prior on the parameter space and then proceed to determine the best economic choices which are typically adaptive to our evolving beliefs about the parameters through successive posterior distributions. The Bayesian paradigm thus provides a natural framework for the evaluation of our economic options, whether one is a Bayesian by choice or, forced to be one, to model our relative ignorance about the parameters.

With the advent of Markov Chain Monte Carlo (MCMC) methods and algorithms, a computationally intensive approach to our problem is feasible. Complementary to such methods are Bayesian models that are analytically tractable, and when appropriate as a model for the natural disaster, provides a deeper insight into the dynamic nature of sequential investment choices to be made and their asymptotics. It should be remarked here that the specific physical character of the natural disaster is, in a broad sense, not fundamental to our analysis, so long as the distributional assumptions about the disaster event (e.g., floods, earthquakes) are appropriate. The work summarized here, based on an unpublished technical report (1968), focuses on such a Bayesian model and its analysis, in which the family of *inverted Beta distributions* play an important role.

II. The Model

A. Economic Activities

An economic activity is an ordered triple $(a, b(\cdot), m)$ such that $a > 0$, $0 \leq b(\cdot)$ on $[0, \infty)$ and $0 < m \leq \infty$. Here $a > 0$ is the set up cost representing the capital needed to build the corresponding physical infrastructure and technology. The function $b(t)$ describes the net rate of benefits at time t accruing from the operating activity,

conditional on no disaster since its inception. The parameter m denotes the (possibly infinite) value of the activity's operating time, if reached without interruption by an intervening disaster, when a planned replacement by rebuilding the same activity is scheduled.

The case $m = \infty$ corresponds to these economic activities which have no scheduled replacement, and will be referred to as activities of "type-I". Such activities continue to operate until destroyed by a disaster. On the other hand, an economic activity is of "type-II" if the benefit rate $b(t)$ is such that it is either necessary, or considered desirable to rebuild the corresponding technology after it has continuously operated for a finite time $m > 0$. A "one-hass shay" benefit function $b(t) = b1_{\{t \leq m\}}$ where $b > 0$ and $m > 0$ are given constants, and 1_A denotes the indicator function of a set A is an example of the former; while the exponentially decaying benefit $b(t) = b\exp(-et)$, $b > 0$, $c > 0$ is an example of the latter. For an economic activity of type-II, there is typically a positive probability of any number of planned replacements between two consecutive disasters.

At time $t > 0$, the present value of a continuously operating activity, with set up cost $a > 0$ and benefit rate $b(\cdot)$, is

$$W(t) = -a + \int_0^t b(x)\exp(-\rho x)dx, \quad \epsilon > 0, \tag{1}$$

where $\rho > 0$ is the discount rate. To guarantee finiteness of $W(t)$ for all $t > 0$, assuming $\sup_{t>0} b(t) < \infty$ will suffice. Assuming $b(0+) < \infty$ (a finite benefit rate at inception of the activity) will often be sufficient in practice; since in many situations, the benefit rate is typically nonincreasing ($b(t) \downarrow$). Further, since there is no point in engaging in an economic activity from the investor's point of view unless $W(t) > 0$ for same t onward; we additionally assume without loss of generality that all activities are '*productive*' in the sense that the corresponding present value function $W(t) = 0$ has a finite solution $t \in (0, \infty)$; equivalently if $W(\infty) := \lim_{t\to\infty} W(t) > 0$, since $W(t)$ is monotone $\uparrow$ on $(0, \infty)$ with

$$-a = W(0+) \leq W(t) \uparrow W(\infty) \leq -a + \rho^{-1} \sup_{t>0} b(t) < \infty$$

as $t \uparrow \infty$.

Consider an activity of type-II with a replacement time m. The *net present value* (NPV) $W^*(t)$ of income from such an activity continuously operating in $(0,t)$ is

$$W^*(t) = \begin{cases} W(t), & \text{if } 0 \le t < m \\ \left(\sum_{k=0}^{j-1} \exp(-k\rho m)\right) W(m) & \\ \quad + \exp(-j\rho m) W(t - jm), & \text{if } jm \le t < (j+1)m, \end{cases}$$

where $j = 1, 2, \ldots$. Hence, for a cycle of random length T between two consecutive disasters, this income is

$$W^*(T) = \sum_{j=0}^{\infty} 1_{B_j} \left[\left(\frac{1 - \exp(-j\rho m)}{1 - \exp -(\rho m)} \right) W(m) \right.$$
$$\left. + \exp(-j\rho m) W(T - jm) \right] \tag{2}$$

where 1_{B_j} are indicators of the disjoint events

$$B_j = \{jm \le T < (j+1)m\}.$$

Note that if $m = \infty$, the above reduces to $W^*(T) \equiv W(T)$ so that (2) subsumes activities of type-I.

The financial impact of an activity $(a, b(\cdot), m)$ described by its contribution to the income stream, subject to renewals between disasters is fully captured by $EW^*(T)$, where T denotes a typical inter-occurrence time of disasters. Thus, the physical basis of an economic activity as parameterized via $(a, b(\cdot), m)$ is, in a mathematical sense, essentially redundant for computing the expected value of an income stream so long as we can specify $EW^*(T)$. Note $EW^*(T)$ is a function of the underlying parameters (hyper-parameters, in the Bayesian case) governing the law of the disaster process.

Assume that there are N *productive* economic activities, one must choose from to invest in after each disaster. The ith activity

$$(a_i, b_i(\cdot), m_i), \; m_i \le \infty \quad i = 1, 2, \ldots, N$$

is simply denoted by i, for brevity. The set of available choices is $A = \{1, 2, \ldots, N\}$, where each activity $i \in A$ is either of type-I or, type-II.

To avoid trivialities, we may assume that all available activities in A are *essential* and *relevant*, properties which we now define.

Call two activities i, i' to be *equivalent* if $EW_i^*(T) = EW_{i'}^*(T)$. Clearly, there is no reason to prefer an activity $i \in A$ over others, if any, in A which are equivalent. An activity $i \in A$ is *essential* if there is no other activity $i' \in A$ such that i and i' are equivalent.

An activity $i \in A$ is *relevant* if there does *not* exist an activity $i' \in A$ which dominates i in the sense $EW_{i'}^*(T) \geq EW_i^*(T)$ with strict inequality for some value of the underlying parameter(s) of the disaster process.

A given activity $(a, b(\cdot), m)$ can have many additive decompositions $(\oplus)$,

$$(a, b(\cdot), m) := (a', b'(\cdot), m) \oplus (a'', b''(\cdot), m) \,,$$

where $a = a' + a''$, $b(\cdot) = b'(\cdot) + b''(\cdot)$ and $m \leq \infty$. Note that the decomposed constituents on the right hand side may be *virtual activities* in that they may not be physically available choices. An example of such a decomposition, which we will find useful in our analysis (Section III) is,

$$(a, b(\cdot), m) = (a, 0, m) \oplus (0, b(\cdot), m) \,.$$

Finally, note that while we assume that a disaster destroys an ongoing economic activity; our framework nevertheless can allow for restarting the same after repairs subsequent to a disaster. If the impact of a disaster on an activity $(a, b(\cdot), m) \in A$ is limited to partial destruction; it may be possible to salvage and restart the same physical activity, usually with a benefit rate $b'(t)$ typically smaller than the old $b(t)$, but also usually with a smaller setup cost $a' < a$. If $b(t)$ is $\downarrow$, then a typical example of such a relatively diminished return function after repair is $b'(t) = b(t + l), t > 0$, where $l > 0$ is a parameter that reflects how effective the repaired technology is. Such a salvaged activity may be included among the available choices in A, so long as it is still *productive, essential* and *relevant.*

B. The Disaster Process, States and Transitions

Let $T_1, T_2, \ldots$ be the inter-occurrence times between disasters. Assume $\{T_n, n \geq 1\}$ to be conditionally i.i.d. exponential with a rate $\lambda > 0$, given λ. In other words, we assume that there is an environmental variable Λ which describes the proneness of disasters such that $\{T_n, n \geq 1\}$ are conditionally i.i.d exponential with rate λ, given $\Lambda = \lambda$. Since the realized value of Λ is unknown, we adopt a Bayesian posture based on past disasters to assign a suitable prior distribution to Λ which is then revised by successive posterior distributions, as we learn more and more about Λ with subsequent disasters as they are observed.

To do this, we track the number (r) of disasters to date and the calendar time (t) when disasters strike. The ordered pair (r, t) summarizes our experience of disasters and describes the set of possible states, which are

$$\mathcal{S} = \{(r, t) : t \geq 0;\ r = 0, 1, 2, \ldots\}.$$

If $\{T_n, n \geq 1\}$ is a sequence of inter-disaster times beginning at the state (r, t), then T_j is the time spent in the jth cycle between the $(r + j - 1)$st and $(r + j)$th disasters. The corresponding state transition is from the state $(r + j - 1, t + S_{j-1})$ to $(r + j, t + S_j)$, where $S_j = \sum_{k=1}^{j} T_k$, $j \geq 1$, $S_0 = 0$. To describe the probability law of the sequence of inter-disaster times, beginning at (r, t), which are conditionally i.i.d. exponential given Λ; we assign to Λ a gamma-prior $G(\cdot|r_0 + r, t_0 + t) \in \mathcal{G}$ for some $r_0 = 0, 1, 2, \ldots$, and $t_0 \geq 0$, where

$$\mathcal{G} = \{G : G = G(.|r, t);\ r > 0;\ t \geq 0\}$$

is the family of *gamma* distributions

$$G(\lambda|r, t) = \frac{t^r}{\Gamma(r)} \int_0^\lambda \exp(-tx) x^{r-1} dx, \quad \lambda > 0.$$

Let

$$\mathcal{F} = \{F : F = F(\cdot|p, q, b);\ p > 0,\ q > 0,\ b > 0\}$$

be the family of *inverted Beta* distributions,

$$F(y|p,q,b) = \frac{\Gamma(p+q)b^q}{\Gamma(p)\Gamma(q)} \int_0^y \frac{x^{p-1}}{(b+x)^{p+q}} dx, \quad y > 0 \tag{3}$$

which correspond to the familiar Beta distributions (McGuire et al., 1972) on the unit interval in the following way. If X is a r.v. with a standard beta density

$$\text{Const. } x^{p-1}(1-x)^{q-1}, \ 0 < x < 1, \ p > 0, \ q > 0$$

then the r.v. $Y = bX/(1-X)$, $b > 0$ has the inverted Beta distribution $F(\cdot|p,q,b)$ in (3. The following facts about the probability law $(\mathcal{L})$ of the sequence $\{T_n, n \geq 1\}$ are now standard.

Lemma 1 *For the inter-occurrence times $\{T_1, T_2, \ldots\}$ between disasters beginning at the state (r,t), as specified above,*

(i) $\mathcal{L}(T_1) = F(\cdot|1, r_0 + r, t_0 + t)$

$$\mathcal{L}\left(T_{n+1}|T_1,\ldots,T_n\right) = F\left(\cdot|1, r_0 + r + n, \ t_0 + t + S_n\right), \ n \geq 1$$

from which all finite dimensional distributions can be computed.

(ii) *The posterior of Λ at the state $(r+n, t+S_n)$ is*

$$\mathcal{L}\left(\Lambda|T_1,\ldots,T_n\right) = G\left(\cdot|r_0 + r + n, \ t_0 + t + S_n\right), \ n \geq 1.$$

(iii) *The unconditional distribution of the total time up to n disasters, is $\mathcal{L}(S_n) = F(\cdot|n, r_0 + r, t_0 + t)$ $n \geq 1$.*

Our activities and corresponding incomes evolve as follows. A typical transition is $(r,t) \to (r+1, t+T)$ where T is the cycle time to the next disaster. At a state (r,t), we can choose any economic activity $i \in A$, the set of available choices, which then operates until the next disaster at $(r+1, t+T)$ when we collect an economic return that has a present value $\exp(-\rho t)W_i^*(T)$. The transition probability $q(\cdot|r,t)$ which specifies the distribution of the next state $(r+1, t+T)$ is a probability measure on the states $\mathcal{S}$ such that for every bounded measurable $h : \mathcal{S} \to (-\infty, \infty)$,

$$\int_{\mathcal{S}} h(\cdot,\cdot) dq(\cdot,\cdot|r,t)$$

$$= \int_0^\infty h(r+1, t+T) dF\left(y|1, r_0+r, t_0+t\right)$$
$$= \left(r_0+r\right)\left(t_0+t\right)^{r_0+r} \int_0^\infty \frac{h(r+1, t+y)}{\left(t_0+t+y\right)^{r_0+r+1}} dy.$$

An (adaptive) investment program π is a policy in the sense of Blackwell (1962, 1965) that chooses economic activities in A, or more generally a family of conditional distributions (defining a randomized choice mechanism) on the set A of available activities, possibly depending on our past experience of the system's history.

A (non-randomized) *stationary* investment program is specified by a function $J : \mathcal{S} \to A$ such at any state $(r, t) \in \mathcal{S}$, we choose the economic activity $J(r, t) \in A$. Such stationary policies are conceptually the simplest and intuitively appealing, as they require only a memory of the current state to choose an economic activity to invest in. Since $A = \{1, 2, \ldots, N\}$ is finite, a stationary investment program is equivalent to specifying a disjoint partition $\{A_1, \ldots, A_N\}$ of $\mathcal{S}$ such that for states in A_j, we invest in the economic activity $j \in A$ until the next disaster.

III. Optimal Bayesian Policies

In a series of papers, Blackwell (1962, 1965) and Strauch (1965) articulated a rigorous theoretical framework to accommodate and extend Richard Bellman's (2003) ideas and methods for dynamic programming, in three broad categories: the positive, negative and discounted cases. These respectively correspond to the stepwise income at each transition being either positive (profit maximization problems) or, negative (cost minimization problems) or, discounted by a deterministic factor γ^n at the n-th transition, $0 < \gamma < 1$. Since the income $W^*(T)$ between disasters can be negative or positive and the discount factors $\exp(-\rho S_n)$ are stochastic, the model formulated in Section II is thus a *non-standard* dynamic programming problem. To put our approach to the solution of the economic choice problem on a rigorous footing, there is thus a need to show that expected incomes of all policies in our setup is finite and then find appropriate

technical means to identify an optimal investment policy.

A. Income of Adaptive Programs

The NPV of the income $i(h)$ associated with any history $h = ((r,t), i_1, (r+1, t+S_1), i_2, (r+2, t+S_2), \ldots)$ of the system, is

$$i(h) = \sum_{n=0}^{\infty} \exp\{\rho(t+\mathcal{S}_n)\} W_{i_n}^*(T_{N+1}) := \exp(-\rho t) i^*(h),$$

where $i^*(h)$ is the *spot-value* of the income stream at the initial state (r,t). A policy π in the sense defined in Section II - specifying the activity choices, induces a probability on the set of histories. The expected income of a policy π starting at (r,t) is clearly of the form

$$U_\pi(r,t) = \exp(-\rho t) U_\pi^*(r,t), \tag{4}$$

where

$$U_\pi^*(r,t) = E_\pi i^*(h) = E_\pi \left\{ \sum_{n=0}^{\infty} \exp(-\rho S_n) W_{i_n}^*(T_{n+1}) \right\} \tag{5}$$

is the spot-value of π at (r,t). The optimal return function is

$$U(r,t) := \sup_\pi U_\pi(r,t) = \exp(-\rho t) U^*(r,t),$$

using (4), where $U^*(r,t) := \sup_\pi U_\pi^*(r,t)$ is the *spot-value of the optimal return.* To show $|U_\pi^*(r,t)| < \infty$ for all π and all $(r,t) \in \mathcal{S}$; set

$$\begin{aligned} m_0 &:= \min_{i \in A} m_i \leq \infty, \\ w_0 &:= \max_{i \in A} \sup_{t>0} |W_i(t)| < \infty, \\ w^* &:= \left[1 + \{1 - \exp(-\rho m_0)\}^{-1}\right] w_0 < \infty. \end{aligned} \tag{6}$$

m_0 is thus the smallest replacement time among the available activities, with $m_0 = \infty$ iff all activities in A are type-I. In virtue of (2); for any activity $i \in A$, we have

$$|W_i^*(T)| \leq \sum_{j=0}^{\infty} 1_{B_j} \left(\frac{1 - \exp(-j\rho_{m_i})}{1 - \exp(-\rho m_i)} + \exp(-j\rho m_i) \right) w_0$$

$$\leq \left\{ \frac{1}{1 - \exp(-\rho m_i)} + 1 \right\} w_0$$
$$\leq w^*.$$

Accordingly, from (5),

$$\begin{aligned} |U_\pi^*(r,t)| &\leq E_\pi \sum_{n=0}^{\infty} \exp(-\rho S_n) \left| W_{i_n}^*(T_{n+1}) \right| \\ &\leq w^* \sum_{n=0}^{\infty} E\left\{\exp(-\rho S_n)\right\} < \infty, \end{aligned} \tag{7}$$

provided the sum converges to a finite value for all $(r,t) \in \mathcal{S}$. Note, the expected values in the last step no longer depend on π. The terms

$$\begin{aligned} \psi_n(\rho; r, t) &:= E\left\{\exp(\rho S_n)\right\} \\ &= E_\Lambda \left\{E(\exp(-\rho(T_1 + \ldots + T_n)) \,|\Lambda)\right\}, \ n \geq 1 \\ &= E_\Lambda \left[E^n \left\{\exp(-\rho T_1) \,|\Lambda\right\}\right], \end{aligned}$$

since $T_1, T_2, \ldots$ are conditionally i.i.d., given Λ. The random variable

$$\alpha(\Lambda) := E\left\{\exp(-\rho T_1) \,|\Lambda\right\} = \frac{\Lambda}{\Lambda + \rho} \in (0,1),$$

since $T_1|\Lambda$ has distribution $G(\cdot|1,\Lambda)$, i.e. exponential with rate Λ. Hence,

$$\begin{aligned} \sum_{n=0}^{\infty} E\left\{\exp(-\rho S_n)\right\} &= 1 + \sum_{n=1}^{\infty} \psi_n(\rho; r, t) \\ &= \sum_{n=0}^{\infty} E_\Lambda \left(\Lambda + \rho^{-1}\right)^n \\ &= E_\Lambda \left(1 + \rho^{-1}\Lambda\right) \\ &= \int_0^\infty \left(1 + \rho^{-1}\lambda\right) dF\left(\lambda|1, r_0 + t, t_0 + t\right) \\ &= 1 + \frac{(r_0 + r)}{\rho\,(t_0 + t)} < \infty. \end{aligned}$$

B. Rebuilding Costs and Static Policies

In our quest for an optimal policy, it will be useful for technical reasons, to first consider the rebuilding cost associated with a *static policy* that always chooses a fixed activity $(a, b(\cdot), m) \in A$, irrespective of our current state and the corresponding posterior distribution of the environmental variable Λ. Conceptually we may think of such an activity decomposed as the "*sum*" of two imaginary activities

$$(a, b(\cdot), m) = (a, 0, m) \oplus (0, b(\cdot), m)$$

so that the spot-value of a static rebuilding policy always using the fixed activity at left is the sum of the corresponding spot values of the two hypothetical static policies defined by $(a, 0, m)$ and $(0, b(\cdot), m)$. The latter gives the expected spot value of the stream of benefits, while the negative of the spot value of the former represents the expected rebuilding costs. Let $K(r, t; a, m)$ denote the spot value of total expected rebuilding costs of the static policy using $(a, b(\cdot), m)$, beginning at (r, t), i.e. the spot value of the total expected costs associated with $(a, 0, m)$. By (7), $0 < K(r, t; a, m) < \infty$.

To describe $K(r, t; a, m)$, first note that the spot value $c(r, t; a, m)$ of the expected rebuilding costs in a typical cycle (r, t) to $(r + 1, t + T)$ between two disasters, is

$$c(r, t; a, m) = a \int_0^\infty \{1 - \exp(-\lambda m)\}^{-1} dG(\lambda | r_0 + r, t_0 + t),$$

since, if $N(T)$ is the number of planned replacements in $[t, t + T)$; then,

$$\begin{aligned} E(N(T)|\Lambda) &= \sum_{j=0}^{\infty} j \Pr(jm \leq T < (j+1)m | \Lambda) \\ &= \exp(-\Lambda m) \{1 - \exp(-\Lambda m)\}^{-1}, \end{aligned}$$

so that,

$$\begin{aligned} c(r, t; a, m) &= E[a\{1 + N(T)\}] \\ &= a[1 + E_\Lambda \{E(N(T)|\Lambda)\}] \\ &= a\left[1 + \int_0^\infty \frac{\exp(-\lambda m)}{1 - \exp(-\lambda m)} dG(\lambda | r_0 + r, t_0 + t)\right], \end{aligned}$$

which leads to (B). Writing $N'(\cdot) \equiv 1 + N(\cdot)$ for brevity; the spot value of the total stream of random rebuilding costs is

$$\begin{aligned} & a \sum_{n=0}^{\infty} \exp\left(-\rho S_n\right) N'\left(T_{n+1}\right) \\ = & \, a N'\left(T_1\right) + \exp\left(-\rho T_1\right) \\ & \times \left[a \sum_{n=2}^{\infty} \exp\left\{-\rho\left(T_2 + \ldots + T_n\right)\right\} N'\left(T_{n+1}\right) \right]. \end{aligned}$$

The second term within the braces is simply the actual rebuilding costs starting at $(r+1, t+T_1)$ for the same static policy. Thus averaging both sides,

$$\begin{aligned} & K(r,t;a,m) \\ = & \, c(r,t;a,m) + E\left\{\exp(-\rho T)\right\} K(r+1, t+T; a, m) \\ = & \, c(r,t;a,m) \\ & + \int_0^{\infty} \exp(-\rho y) K(r+1, t+y; a, m) dF\left(y | 1, r_0 + r, t_0 + t\right). \end{aligned}$$

For a type-I activity, $m = \infty$ and $c(r,t;a,\infty) \equiv a$, as is also clear by letting $m \to \infty$ in (B). The corresponding total expected rebuilding costs $K(r,t,a) := \lim_{m\to\infty} K(r,t;a,m)$ satisfy the simpler equation

$$K(r,t;a) = a + E\left\{\exp(-\rho T)\right\} K(r+1, t+T; a).$$

C. Finding An Optimal Policy

Upper bounds for the spot value. The spot value of the expected rebuilding costs associated with any policy π is bounded above by

$$k^*(r,t) = \begin{array}{ll} K\left(r,t;a^*,m_0\right), & \text{if } m_0 < \infty \\ K\left(r,t;a^*\right), & \text{if } m_0 = \infty, \end{array}$$

where $m_0 = \min_{i\in A} m_i \leq \infty$ is defined in (6) and $a^* = \max_{i\in A} a_i$. Hence, the spot value of total expected income satisfies $U_\pi^* \geq -k^*$ pointwise, for all π. Accordingly,

$$U^*(r,t) := \sup_\pi U_\pi^*(r,t) \geq -k^*(r,t), \quad \text{all } (r,t).$$

To search for reasonable upper bounds for U^*, we may therefore

restrict ourselves to

$$\mathcal{Q} = \{Q|Q : \mathcal{S} \to (-\infty, \infty),\ Q \geq -k^* \text{ on } \mathcal{S}\}.$$

Consider the linear operator L carrying measurable functions on $\mathcal{Q}$ into itself, such that

$$\begin{aligned} LQ(r,t) &:= \max_{i \in A} \int_0^\infty \{W_i^*(y) + \exp(-\rho y) Q(r+1, t+y)\} \\ &\qquad \times dF(y|1, r_0 + r, t_0 + t) \\ &\equiv \max_{i \in A} E\{W_i^*(T) + \exp(-\rho T) Q(r+1, t+T)\}. \end{aligned}$$

To check that $Q \in \mathcal{Q}$ implies $LQ \in \mathcal{Q}$, we argue as follows. Corresponding to the activity $i \in A$ defined by $(a_i, b_i(\cdot), m_i)$, let $(a_i, 0, m_i)$ define a *virtual activity* i' (which need not be in A). Then,

$$EW_i^*(T) \geq EW_{i'}^*(T) = -c(r, t; a_i, m_i) \geq -c(r, t; a^*, m_0)$$

where, the function c is defined in (B). Together with $Q \geq -k^*$ pointwise, since $Q \in \mathcal{Q}$; the above implies

$$\begin{aligned} & E\{W_i^*(T) + \exp(-\rho T) Q(r+1, t+T)\} \\ \geq & -[c(r,t;a^*,m_0) + E\{\exp(-\rho T)\} k^*(r+1, t+T)] \\ = & -k^*(r,t) \end{aligned}$$

for all $i \in A$, the last being true in view of (B)–(C). Hence $LQ \geq -k^*$.

Lemma 2 *$Q \in \mathcal{Q}$, $LQ \leq Q \Rightarrow U^* \leq Q$ pointwise.*

Proof. Consider a hypothetical problem with the same states and transition probabilities except that each economic activity $i \in A = \{1, 2, \ldots, N\}$ is replaced by an hypothetical *action* i^* such that choosing i^* at (r, t) yields an income with present value $\exp(-\rho t)\widehat{W}_{i^*}(T)$ realized at the next state $(r+1, t+T)$, where

$$\widehat{W}_{i^*}(T) = W_i^*(T) + a^*\{1 + N(T, m_0)\}, \tag{8}$$

where $N(t, m_0)$ is the number of planned replacements between disasters, of an activity with scheduled replacement time m_0; i.e. $N(T, m_0) = n$ on $\{nm_0 \leq T < (n+1)m_0\}$, $n = 0, 1, 2, \ldots$ if $m_0 < \infty$, and $N(T, m_0) = 0$ w.p. 1 if $m_0 = \infty$ (all activities are type - I). In any

case, $m_0 \leq m_i \leq \infty$ and $a_i \leq a^* < \infty$, implies

$$\widehat{W}_{i^*}(T) \geq W_i^*(T) + a_i \{1 + N(T, m_i)\} \geq 0,$$

since the second term, being the actual rebuilding costs of $i \in A$ in the time interval $[t, t+T)$, represents the negative part of $W_i^*(T)$. The hypothetical problem so defined, with A replaced by the set of *virtual activities* $A^* = \{1^*, 2^*, \ldots, N^*\}$, is a *positive dynamic programming problem* in the sense of Blackwell (1965). Clearly, the optimal income's spot value for the "positive" problem has the form

$$\widehat{U}(r,t) = U^*(r,t) + k^*(r,t)$$

where U^* is the optimal spot value function of our original problem and the second term k^* represents the expected contribution of the stream of second terms in (8) over entire histories. If $\widehat{L}$ is the operator

$$\widehat{L}h(r,t) := \max_{i^* \in A^*} E\left\{\widehat{W}_{i^*}(T) + \exp(-\rho T) h(r+1, t+T)\right\}$$

carrying non-negative functions h on the (r,t)-plane into itself; then Blackwell's (1965) results imply that any function $h \geq 0$ that satisfies $\widehat{L}h \leq h$ must itself be an upper bound on $\widehat{U}$. If Q satisfies the hypothesis in the statement of Lemma 3.1, then, $\widehat{Q} := Q + k \geq 0$ is such a function; *viz.*

$$\begin{aligned}
\widehat{L}\widehat{Q}(r,t) &= \max_{i^* \in A^*} E\Big[\widehat{W}_{i^*}(T) + \exp(-\rho T)\widehat{Q}(r+1, t+T)\Big] \\
&= \max_{i \in A} E\Big[W_i^*(T) + a^* \{1 + N(T, m_0)\} \\
&\qquad + \exp(-\rho T)\{Q(r+1, t+T) + k^*(r+1, t+T)\}\Big] \\
&= LQ(r,t) + E\Big[a^* \{1 + N(T, m_0)\} \\
&\qquad + \exp(-\rho T) k^*(r+1, t+T)\Big].
\end{aligned}$$

In view of (B)–(C), the second term equals $k^*(r,t)$. Hence,

$$\widehat{L}\widehat{Q}(r,t) = LQ(r,t) + k^*(r,t) \leq Q(r,t) + k^*(r,t) \equiv \widehat{Q}(r,t).$$

This in term implies $U^* + k^* = \widehat{U} \leq \widehat{Q} = Q + k^*$, or $U^* \leq Q$ pointwise. □

Set,

$$A_j = \left\{(r,t) : EW_j^*(T) = L0(r,t)\right\}, \ j \in A, \tag{9}$$

where

$$\begin{aligned} L0(r,t) &= \max_{i\in A} EW_j^*(T) \\ &= \max_{i\in A} \int_0^\infty W_i^*(y) dF\left(y|1, r_0 + r, t_0 + t\right) \\ &= \max_{i\in A} \int_0^\infty E\left(W_i^*(T)|\lambda\right) dG\left(\lambda|r_0 + r, t_0 + t\right), \end{aligned}$$

is the spot value of the maximal one-cycle income. Without loss of generality, $\{A_1, \ldots, A_N\}$ may be taken as a disjoint partition of $\mathcal{S}$. Let $g : \mathcal{S} \to A = \{1, 2, \ldots, N\}$ such that

$$g(r,t) = j, \quad \text{if } (r,t) \in A_j.$$

Theorem 1 *The stationary adaptive investment policy defined by g is optimal.*

Proof. The spot value function U_g of the stationary policy defined by g in (C), obviously satisfies

$$\begin{aligned} U_g(r,t) &= E\left\{W_{g(r,t)}^*(T) + \exp(-\rho T) U_g(r+1, t+T)\right\} \\ &= L0(r,t) + E\left\{\exp(-\rho T)\right\} U_g(r+1, t+T) \\ &= LU_g(r,t). \end{aligned}$$

Further, $U_g \geq -k^*$ as argued in the remark following (C). Hence the optimal spot value $U^* \leq U_g$ by Lemma 2; while $U_g \leq U^*$ necessarily. Thus $U_g(r,t) = U^*(r,t)$ which proves the optimality of g. □

The optimal Bayesian spot value is

$$\begin{aligned} U^*(r,t) &= U_g(r,t) \\ &= \textstyle\sum_{n=0}^{\infty} E\left\{\exp\left(-\rho S_n\right) W^*_{g(r+n,t+S_n)}\left(T_{n+1}\right)\right\} \\ &= \sum_{n=0}^{\infty} E\left\{\exp\left(-\rho S_n\right) \sum_{j=1}^{N} W_j^*\left(T_{n+1}\right) 1_{A_j}\left(r+n, t+S_n\right)\right\} \\ &= \sum_{n=0}^{\infty}\sum_{j=1}^{N} \int_0^{\infty}\int_0^{\infty} 1_{A_j}(r+n,t+x)\exp(-\rho x) \\ &\qquad \times W_j^*(y)dH_1(y|x)dH_2(x) \end{aligned}$$

where the integrating measures are the *inverted beta* distributions

$$\begin{aligned} H_1(y|x) &= F\left(y|1, r_0+r+n, t_0+t+x\right), \\ H_2(x) &= F\left(x|n, r_0+r, t_0+t\right) \end{aligned}$$

defined in (3). Clearly (C) shows that U^* cannot in general be evaluated in a closed form. For a given set of economic activities with specified set up costs and benefit rates, first the optimal partition $\{A_j : j = 1, 2, \ldots, N\}$ and then U^* can be numerically evaluated using (9)–(C).

Static rebuilding programs. These correspond to stationary policies that repeatedly choose a fixed activity in a A, and are typically not optimal, although they are conceptually appealing and among the simplest. Another reason they are useful is that such policies play an important role in the asymptotic behavior of the optimal return $U^*(r,t)$, defined in (C), under fairly reasonable conditions on the benefit rates, as we show in Section (IV.

The Bayesian spot value $U_0(r,t)$ of a static policy has a simple representation

$$U_0(r,t) = \int_0^{\infty} V(\lambda)dG\left(\lambda|r_0+r, t_0+t\right),$$

where $V(\lambda)$ is the corresponding spot value in the non-Bayesian case, i.e. when the environmental variable Λ is fixed at a value $\lambda > 0$. To justify (C), suppose the fixed economic activity defining the static policy has a planned replacement time $m \leq \infty$. Since given $\Lambda = \lambda$, the time T between disasters is exponential; the memoryless property

of exponential distributions imply that the fixed activity defining the stationary policy is *renewed* after operating for time $\min(T, m)$, so that V satisfies the renewal type equation

$$V = E_\lambda \left[W(\min(T, m)) + \exp\{-\rho \min(T, m)\} V \right],$$

which gives

$$V(\lambda) = \frac{E_\lambda W(\min(T, m))}{E_\lambda \left[1 - \exp\{-\rho \min(T, m)\}\right]}, \quad \lambda > 0.$$

For a type-I activity, this takes a simpler form

$$V(\lambda) = \left(1 + \frac{\lambda}{\rho}\right) E_\lambda W(T), \quad \lambda > 0.$$

While (C) is the preferred way to compute V for a type -II activity, it can also be expressed in a form analogues to (C), resulting from the equation

$$V = E_\lambda \{W^*(T) + \exp(-\rho T) V\}$$

instead of (C), where $W^*(T)$ given in (2) is the income between consecutive disasters, allowing for possibly multiple planned replacements in between. Thus implies

$$V(\lambda) = \frac{E_\lambda W^*(T)}{1 - \alpha(\lambda)} = \left(1 + \frac{\lambda}{\rho}\right) E_\lambda W^*(T),$$

where

$$\alpha(\lambda) := E_\lambda \exp(-\rho T) = \lambda/(\lambda + \rho) \in (0, 1).$$

Equation (C) of course corresponds to (C), when $m = \infty$. The claim (C) is now immediate. By (5), a static policy's spot value function is

$$\begin{aligned} U_0(r, t) &= E \sum_{n=0}^{\infty} \exp(-\rho S_n) W^*(T_{n+1}) \\ &= \sum_{n=0}^{\infty} E \left[E \left\{ \exp(-\rho S_n) W^*(T_{n+1}) | T_1, \ldots, T_n, \Lambda \right\} \right] \\ &= \sum_{n=0}^{\infty} E \left[E \left(\exp(-\rho S_n) | \Lambda \right) \right] \end{aligned}$$

$$\times E\left\{W^*\left(T_{n+1}\right)|\, T_1, \ldots, T_n, \Lambda\right\}\Big].$$

Since T_n are conditionally i.i.d. given Λ; the nth term above, conditional $\Lambda = \lambda$, is

$$E_\lambda^n \left\{\exp(-\rho T)\right\} E_\lambda W^*(T) \equiv \alpha^n(\lambda) E_\lambda W^*(T), \quad n \geq 0$$

which implies

$$\begin{aligned} U_0(r,t) &= \sum_{n=0}^{\infty} E\left[\alpha^n(\Lambda) E\left(W^*(T)|\Lambda\right)\right] \\ &= E\left\{\frac{E\left(W^*(T)|\Lambda\right)}{1-\alpha(\Lambda)}\right\} \\ &= \int_0^\infty \frac{E_\lambda W^*(T)}{1-\alpha(\lambda)} dG\left(\lambda | r_0 + r, t_0 + t\right). \end{aligned}$$

Together with (C), this proves (C).

As remarked in Section I, in the non-Bayesian case, an optimal policy is always static. McGuire et al. (1972) consider some illustrative examples of benefit rates and the spot values $V(\lambda)$ for the corresponding static policies when the disasters are assumed to be a Poisson process with a known rate λ. The following is a summary of their results, which we then use to evaluate the respective Bayesian spot value functions, by exploiting the representation (C).

<u>Activities</u>	
No decay	$(a, b(t) \equiv b, \infty),\ b > 0$
Delayed benefits	$(a, b(t) = b1_{\{t>l\}}, \infty),\ b > 0$
One-hoss shay	$(a, b(t) = b1_{\{t<m\}}, m)\ b > 0, \delta > 0$
Exponential decay	$(a, b(t) = b\exp(-\delta t), m)\ b > 0,\ \delta > 0$

The spot value functions $U_0(r,t)$, using (C), (C)–(C) are
No decay:

$$V(\lambda) = -a(1 + \rho^{-1}\lambda) + \rho^{-1} b,$$

$$U_0(r,t) = EV(\lambda) = \frac{b}{\rho} - a\left(1 + \frac{r_0 + r}{\rho(t_0 + t)}\right).$$

Delayed benefits:

$$V(\lambda) = \rho^{-1} b \exp\{(\lambda+\rho)l\} - a\left(1+\rho^{-1}\lambda\right),$$

$$U_0(r,t) = EV(\lambda) = \frac{b}{\rho}\left(\frac{t_0+t}{t_0+t+l}\right)^{r_0+r}\exp(-\rho l) - a\left(1+\frac{r_0+r}{\rho(t_0+t)}\right).$$

As $l \to 0$, this reduces to 'no decay'.
One-hoss shay:

$$V(\lambda) = \frac{b}{\rho} - \frac{a(\lambda+\rho)}{\rho\left[1-\exp\{-(\lambda+\rho)m\}\right]}.$$

With the prior $G(\cdot|k,c)$ of λ, where $k \equiv r_0 + r,\ c \equiv t_0 + t$; we have

$$\begin{aligned}
&E_\lambda\left[\frac{\Lambda+\rho}{1-\exp\{-(\Lambda+\rho)m\}}\right] \\
&= \frac{c^k}{\Gamma(k)}\sum_{j=0}^{\infty}\int_0^\infty (\lambda+\rho)\exp\{-j(\lambda+\rho)m\}\exp(-c\lambda)\lambda^{k-1}d\lambda \\
&= \frac{c^k}{\Gamma(k)}\sum_{j=0}^{\infty}\exp(-j\rho m)\int_0^\infty (\lambda+\rho)\exp\{-(c+jm)\lambda\}\,\lambda^{k-1}d\lambda \\
&= c^k\sum_{j=0}^{\infty}\exp(-j\rho m)\left\{\frac{k}{(c+jm)^{k+1}} + \frac{\rho}{(k+jm)^k}\right\} \\
&= c^k\sum_{j=0}^{\infty}\frac{\exp(-j\rho m)}{(c+jm)^k}\left(\rho + \frac{k}{c+jm}\right).
\end{aligned}$$

Accordingly, the Bayesian spot-value is

$$U_0(r,t) = \frac{b}{\rho} - \frac{a}{\rho}(t_0+t)^{r_0+r}\sum_{j=0}^{\infty}\frac{\exp(-j\rho m)}{(t_0+t+jm)^{r_0+r}} \times\left(\rho + \frac{r_0+r}{t_0+t+jm}\right).$$

As $m \to \infty$, we get

$$U_0(r,t) \to \frac{b}{\rho} - \frac{a}{\rho}\left(\rho + \frac{r_0+r}{t_0+t}\right) = -a + \frac{1}{\rho}\left(b + \frac{r_0+r}{t_0+t}\right),$$

the case of 'no-decay'.

For the case of exponential decay, we have

$$V(\lambda) = \left(1 + \frac{\lambda}{\rho}\right)\left[-a + \frac{b\left[1 - \exp\{-(\lambda+\rho+\delta)m\}\right]}{\lambda+\rho+\delta}\right] \Big/ \left[1 - \exp\{-(\lambda+\rho)m\}\right].$$

The Bayesian spot value $EV(\Lambda)$ is rather messy. For exponential decay with no planned replacement ($m = \infty$),

$$U_0(r,t) = E\left\{\left(1 + \frac{\Lambda}{\rho}\right)\left(-a + \frac{b}{\Lambda+\rho+\delta}\right)\right\}$$

can be evaluated by computing integrals along the lines of (C).

IV. Asymptotic Behavior

As we have seen, whenever the set of activities contains two or more choices, the return of the optimal Bayesian policy is usually not readily computable in a simple way. An investigation of the asymptotic behavior of the optimal spot value shows that with type-I activities, an investor, who pretends to behave as if Λ is known by setting the value of Λ as the Bayes' estimator with squared error loss will be close to optimal, for initial states which are suitably "large". This nearly Bayesian optimal behavior is a static rebuilding policy.

Let $V_j(\lambda)$ be the non-Bayesian spot value of the static policy using activity $j \in A$; $j = 1, 2, \ldots, N$, and let

$$\widetilde{V}(\lambda) = \max_{j \le N} V_j(\lambda), \quad \lambda > 0$$

be their envelope.

Theorem 2 *If all available activities are of type-I with set up costs and benefit rates $(a_j, b_j(\cdot))$, such that each $b_j(t)$ is non-increasing and differentiable with $\sup_{t>0} b_j(t) < \infty$, $j = 1, 2, \ldots, N$; then*

$$\left|U^*(r,t) - \widetilde{V}(\beta)\right| \to 0$$

as $r, t \to \infty$ along the paths $r \sim \beta t$ (i.e. along any locals on the (r,t)-plane, such that $r/t \to \beta > 0$).

To prove our claim, we will exploit the following technical results. Recall, $W(t)$ of (1) is the cumulative return at time t from an uninterrupted activity in $(0,t)$ with expected value $g(\lambda) := E_\lambda W(T)$. Let $H(\theta) := g(\theta^{-1})$ be this expected value as a function of the mean time θ between two disasters.

Lemma 3 *(a) If $b(t)$ is differentiable a.e., $\sup_{t>0} b(t) < \infty$, then*
i) $g(\lambda)$ is convex and nonincreasing.
ii) W and H both satisfy the Lipschitz condition of order 1.
(b) If $b(t)$ is nonincreasing, then $W(t)$ is concave.

Lemma 4 *The Bayesian optimal spot value U^* satisfies*

$$U^*(r,t) \leq \int_0^\infty \widetilde{V}(\lambda) dG\left(\lambda | r_0 + r,\ t_0 + t\right).$$

Proof of Lemma 3. (a). By (1) and interchanging the order of integration

$$\begin{aligned} g(\lambda) = E_\lambda W(T) &= a + \int_0^\infty \int_0^t \exp(-\rho y) b(y) \lambda \exp(-\lambda y) dy dt \\ &= -a + \int_0^\infty \exp\left\{-(\lambda+\rho)y\right\} b(y) dy \end{aligned}$$

which gives,

$$\begin{aligned} \dot{g}(\lambda) &= -\int_0^\infty \exp\left\{-(\lambda+\rho)y\right\} y b(y) dy \leq 0, \\ \ddot{g}(\lambda) &= \int_0^\infty \exp\left\{-(\lambda+\rho)y\right\} y^2 b(y) dy \ \geq 0, \end{aligned}$$

for all $\lambda > 0$. For any $t > t' > 0$, again from (1),

$$\begin{aligned} \left|W(t) - W(t')\right| &= \left|\int_{t'}^t b(y) \exp(-\rho y) dy\right| \\ &= \left|t - t'\right| b(\xi) \exp(-\rho\xi), \text{ some } \xi \in (t', t) \\ &\leq b_0 \left|t - t'\right|, \end{aligned}$$

where $b_0 = \sup_{t>0} b(t) < \infty$. To show H also satisfies the Lipschitz condition, write

$$g(\lambda) = \lambda \int_0^\infty W(t) \exp(-\lambda t) dt = \int_0^\infty W(y/\lambda) \exp(-y) dy,$$

so that for any positive θ, θ'

$$\begin{aligned}
\left|H(\theta) - H(\theta')\right| &\equiv \left|g\left(\tfrac{1}{\theta}\right) - g\left(\tfrac{1}{\theta'}\right)\right| \\
&\leq \int_0^\infty \left|W(\theta y) - W(\theta' y)\right| \exp(-y) dy \\
&\leq b_0 \int_0^\infty \left|\theta - \theta'\right| y \exp(-y) dy \\
&= b_0 \left|\theta - \theta'\right|
\end{aligned}$$

(b) Finally note, if $0 \leq b(t)$ is decreasing then (1) implies

$$\dot{W}(t) = \exp(-\rho t) b(t) \geq 0$$

is decreasing, since $b(t)$ is. Hence W is concave. Note that $0 \leq b(t) \downarrow$ is sufficient for W to be concave, and $b(t)$ need not be differentiable.

□

Proof of Lemma 4. We show, the function $\widetilde{Q}(r,t) := E\widetilde{V}(\Lambda)$ defined in the righthand side of (IV), satisfies the hypothesis of Lemma 2, which will prove our claim. If $U_j(r,t) := EV_j(\Lambda)$ denotes the spot value at (r,t) of the static policy that always uses activity $j \in A$, then

$$\widetilde{Q}(r,t) \geq E\widetilde{V}(\Lambda) = EV_j(\Lambda) = U_j(r,t) \geq -k^*(r,t),$$

where the last inequality holds in virtue of the argument preceding (C). To prove $L\widetilde{Q} \leq \widetilde{Q}$, consider $E\{W_j^*(T) + \exp(-\rho T)\widetilde{Q}(r+1, t+T)\}$. The second term, by appealing to Lemma 1, and writing $k = r_0 + r, c = t_0 + t$ for brevity, is

$$\begin{aligned}
& E\left\{\exp(-\rho T) \int_0^\infty \widetilde{V}(\lambda) dG\left(\lambda | r_0 + r + 1, t_0 + t + T\right)\right\} \\
= & \int_0^\infty \int_0^\infty \exp(-\rho y) \widetilde{V}(\lambda) dG\left(\lambda | r_0 + r + 1, t_0 + t + y\right) \\
& \qquad \times dF\left(y | 1, r_0 + r, t_0 + t\right) \\
= & \int_0^\infty \int_0^\infty \exp(-\rho y) \widetilde{V}(\lambda) \frac{(c+y)^{k+1}}{\Gamma(k+1)} \exp\left\{-(c+y)\lambda\right\} \\
& \qquad \times \frac{k(\lambda c)^k}{(c+y)^{k+1}} d\lambda
\end{aligned}$$

$$= \int_0^\infty \left(\int_0^\infty \exp\{-(\lambda+\rho)y\}\, dy \right) \widetilde{V}(\lambda) \frac{(c\lambda)^k}{\Gamma(k)} \exp(-c\lambda)\, d\lambda$$
$$= \int_0^\infty \alpha(\lambda) \widetilde{V}(\lambda) dG(\lambda|k,c),$$

where $\alpha(\lambda) = E_\lambda \exp(-\rho T) = \lambda/(\lambda+\rho)$. Hence, for any $j \in A$, the above and an appeal to (C) gives,

$$\begin{aligned}
& E\left\{ W_j^*(T) + \exp(-\rho T)\, \widetilde{Q}(r+1, t+T) \right\} \\
&= \int_0^\infty E_\lambda W_j^*(T) dG(\lambda|r_0+r, t_0+t) \\
&\qquad + E\left\{ \exp(-\rho T) \int_0^\infty \widetilde{V}(\lambda) dG(\lambda|r_0+r+1, t_0+t+T) \right\} \\
&= \left[\{1-\alpha(\lambda)\} V_j(\lambda) + \alpha(\lambda) \widetilde{V}(\lambda) \right] dG(\lambda|r_0+r, t_0+t) \\
&\le \int_0^\infty \widetilde{V}(\lambda) dG(\lambda|r_0+r, t_0+t) \equiv \widetilde{Q}(r,t)
\end{aligned}$$

since $0 < \alpha(\lambda) < 1$ and $V_j(\lambda) \le \widetilde{V}(\lambda)$ all $\lambda > 0$, and all $j \in A$. Thus $L\widetilde{Q} \le \widetilde{Q}$. □

Proof of Theorem 2. Let $U_j(r,t)$ denote the Bayesian spot value of the static policy using activity $j \in A$. For fixed $\lambda > 0$, the expected income $g_j(\lambda)$ of type-I activity j between two disasters, satisfies

$$g_j(\lambda) \ge g_j(\beta) + (\lambda - \beta)\dot{g}_j(\beta), \quad \text{all } \lambda > 0,\ \beta > 0$$

where the slope $\dot{g}_j \le 0$ since $g_j(\lambda)$ is convex and nonincreasing in λ (lemma 3) under the stated assumptions. Hence, for the corresponding static policy,

$$V_j(\lambda) \ge \left(1 + \rho^{-1}\lambda\right) g_j(\beta) + \left\{ \rho^{-1}\lambda^2 + \lambda\left(1 - \rho^{-1}\beta\right) - \beta \right\} \dot{g}_j(\beta)$$

by a reference to (C), so that (C) then implies

$$\begin{aligned}
U_j(r,t) &= \int_0^\infty V_j(\lambda) dG(\lambda|r_0+r, t_0+t) \\
&\ge \left(1 + \frac{r_0+r}{\rho(t_0+t)}\right) g_j(\beta) + \dot{g}_j(\beta) I(r,t;\rho,\beta),
\end{aligned}$$

where

$$\begin{aligned} I(r,t;\rho,\beta) &:= E_\lambda\left\{\rho^{-1}\Lambda^2 + \Lambda(1-\rho^{-1}\beta) - \beta\right\} \\ &= \frac{(r_0+r)\,(r_0+r+1)}{\rho\,(t_0+t)^2} + \frac{r_0+r}{t_0+t}\left(1-\frac{\beta}{\rho}\right) - \beta. \end{aligned}$$

Note, since each benefit rate $b_j(t)$ is nonincreasing, we have

$$0 \le -\dot{g}_j(\beta) = \int_0^\infty \exp\{-(\beta+\rho)t\}\, t b_j(t) dt \le \frac{b^*}{(\beta+\rho)^2} < \infty,$$

for all $j \le N$, where

$$b^* = \max_{j\le N} \sup_{t>0} b_j(t) = \max_{j\le N} b_j(0+) < \infty.$$

Consider any locus on the (r,t)-plane such that $r \sim \beta t$, i.e. such that $r = \beta t + o(t)$. Denote by $\liminf_{r\sim\beta t}$($\limsup_{r\sim\beta t}$, respectively) the operation $\liminf_{t\to\infty}$($\limsup_{t\to\infty}$, respectively), as $t \to \infty$, along any path on the (r,t)-plane for which $r \sim \beta t$. Then from (IV)–(IV),

$$\begin{aligned} \liminf_{r\sim\beta t} U_j(r,t) &= \left(1+\rho^{-1}\beta\right) g_j(\beta) + \dot{g}_j(\beta) \lim_{t\to\infty} I(\beta t, t; \rho, \beta) \\ &= V_j(\beta) + \dot{g}_j(\beta)\left\{\rho^{-1}\beta^2 + \beta\left(1-\rho^{-1}\beta\right) - \beta\right\} \\ &= V_j(\beta). \end{aligned}$$

As the optimal spot value U^* clearly satisfies $U^*(r,t) \ge U_j(r,t)$, for all $j \in A$, we get,

$$\liminf_{r\sim\beta t} U^*(r,t) \ge \liminf_{r\sim\beta t} U_j(r,t) \ge V_j(\beta), \qquad \text{all } j \in A$$

so that,

$$\liminf_{r\sim\beta t} U^*(r,t) \ge \max_{j\le N} V_j(\beta) = \widetilde{V}(\beta).$$

On the other hand, by Lemma 3(a) -ii), viz. (IV),

$$|g_j(\lambda) - g_j(\beta)| \le b^* \left|\lambda^{-1} - \beta^{-1}\right|, \quad \text{all } \lambda > 0, \beta > 0$$

for each activity j, where b^* is given by (IV). Thus,

$$\begin{aligned} \widetilde{V}(\lambda) &= \left(1+\rho^{-1}\lambda\right) \max_j g_j(\lambda) \\ &\le \left(1+\rho^{-1}\lambda\right) \max_j g_j(\beta) + b^*\left(1+\rho^{-1}\lambda\right)\left|\lambda^{-1} - \beta^{-1}\right|. \end{aligned}$$

Hence, an appeal to Lemma 4 yields,

$$U^*(r,t) \leq \left(1 + \frac{r_0 + r}{\rho\,(t_0 + t)}\right) \max_j g_j(\beta) + b^* \Delta(r,t),$$

where

$$0 < \Delta(r,t) = \int_0^\infty \left(1 + \rho^{-1}\lambda\right) \left|\lambda^{-1} - \beta^{-1}\right| dG\left(\lambda | r_0 + r, t_0 + t\right) \to 0,$$

as $r, t \to \infty$ such that $r \sim \beta t$, since

$$\begin{aligned}
&\Delta^2(r,t) \\
&= E^2\left\{(1 + \rho^{-1}\Lambda)\left|\Lambda^{-1} - \beta^{-1}\right|\right\} \\
&\leq E\left\{(1 + \rho^{-1}\Lambda)^2\right\} E\left\{\left(\Lambda^{-1} - \beta^{-1}\right)^2\right\} \\
&= \left\{1 + \frac{2}{\rho}\frac{r_0 + r}{t_0 + t} + \frac{1}{\rho^2}\frac{(r_0 + r)\,(r_0 + r + 1)}{(t_0 + t)\,(t_0 + t + 1)}\right\} \\
&\qquad \times \left\{\frac{1}{\beta^2} - \frac{2}{\beta}\frac{t_0 + t}{r_0 + r - 1} + \frac{(t_0 + t)^2}{(r_0 + r - 1)\,(r_0 + r - 2)}\right\}
\end{aligned}$$

implies

$$\begin{aligned}
\lim_{r \sim \beta t} \Delta^2(r,t) &= \lim_{t \to \infty} \Delta^2(\beta t, t) \\
&= \left(1 + \rho^{-1}\beta)^2(\beta^{-2} - 2\beta^{-2} + \beta^{-2}\right) \\
&= 0.
\end{aligned}$$

By (IV), we then have

$$\begin{aligned}
\limsup_{r \sim \beta t} U^*(r,t) &\leq \left(1 + \rho^{-1}\beta\right) \max_j g_j(\beta) + b^* \lim_{t \to \infty} \Delta(\beta t, t) \\
&= \widetilde{V}(\beta).
\end{aligned}$$

Together with (IV), this yields the desired conclusion. □

When available choices include activities of type-II, the asymptotic behavior of the optimal Bayesian policy in Theorem 4.1, remains an open question as of this writing.

References

1. Bellman, R.E. *Dynamic Programming*; Dover Publications, 2003.
2. Bhattacharjee, M.C. Optimal investments in the face of floods, Working Paper # 259, Center for Research in Management Science, University of California, Berkeley, 1968.
3. Blackwell, D. Discounted dynamic programming, *Annals of Mathematical Statistics* **1962**, *36*, 226-235.
4. Blackwell, D. Positive dynamic programming, Proceedings of the Fifth Berkeley Symposium on Mathematical Statics and Probability **1965**, *1*, 415-418.
5. Johnson, N.L.; Kotz, S. *Distribution in Statistics: Continuous Univariate Distribution* (volume 2); John Wiley and Sons: New York, 1970.
6. McGuire, C.B.; Brown, J.P.; Contini, B. An economic model of flood plain use, *Water Resources Research* **1972**.
7. Strauch, R.E. Negative dynamic programming, *Annals of Mathematical Statistics* **1965**, *37*, 871-890.